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STABILIZABILITY FOR NONLINEAR DIFFERENCE CONTROLS SYSTEMS WITH MULTIPLE DELAYS

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Abstract. In this paper, we investigate the problem of stabilizability for nonlinear difference controls systems with multiple delays. We present a generalized discrete Gronwall's inequality for stabilizability analysis of this systems. Based on a new Gronwall's inequality, sufficient conditions for stabilizability of this systems are obtained. Numerical examples illustrate the results are given.

1. Introduction and preliminaries

Difference equations are often used to model an approximation of differential equations, an approach which underlies the development of many numerical methods. However, there are many situation, for example, recurrence relations and the modelling of discrete process such as traffic flow with finite number of entrances and etc. [1]–[7]. In recent years, the various condition for stability and stabilizability of nonlinear control difference equations systems with multiple delays has been extensively studied in many methods. Some criteria on stability are presented by employing a Lyapunov function [5], [7]. The Gronwall's inequality is an important tool in the study of stability and stabilizability of this system. In 2000, P. Niamsup and V. N. Phat [2] studied the asymptotic stabilizability of nonlinear control system described by difference equation with multiple delays of the form

$$x(k+1) = L_{p,q}(x_k,u_k) + f_{p,q}(k,x_k,u_k), \quad k \in \mathbb{Z}^+, \tag{1}$$

where

$$L_{p,q}(x_k,u_k) = \sum_{j=1}^{p} A_j(k)x(k-p_j) + \sum_{i=1}^{q} B_i(k)u(k-q_i),$$

$$f_{p,q}(k,x_k,u_k) = f(k,x(k-p_1),\ldots,x(k-p_p),u(k-q_1),\ldots,u(k-q_q)),$$

where $x(k) \in \mathbb{R}^n$; $u(k) \in \mathbb{R}^m$, $n \geqslant m$; $A_j(k)$ and $B_i(k)$ are $n \times m$ matrices with $k \in \mathbb{Z}^+$, $j = 1,2,\ldots,p$, $i = 1,2,\ldots,q$; $f(k,.) : \mathbb{Z}^+ \times \mathbb{R}^{pn} \times \mathbb{R}^{qm} \longrightarrow \mathbb{R}^n$ with $p,q \geqslant 1$, $q_q \leqslant p_p$,

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$0 = p_1 < p_2 < \cdots < p_p$, $0 = q_1 < q_2 < \cdots < q_q$. They consider system (1) with the initial delay condition

$$x(k) = x_0, \quad k = -p_p, \ldots, 0. \tag{2}$$

Unlike differential equation, discrete controls system (1) with initial condition (2) always has solution for every control sequence $u(k)$, $k = -q_q, \ldots, 0, 1, \ldots$. They assume that $f(k, 0, \ldots, 0) = 0$, $k \in \mathbb{Z}^+$. They also consider the delay system without controls of the form

$$x(k+1) = \sum_{j=1}^{p} A_j(k)x(k-p_j) + g(k, x(k-p_1), x(k-p_2), \ldots, x(k-p_p)) \tag{3}$$

where $A_j(k)$ is an $n \times m$ matrix, $k \in \mathbb{Z}^+$, $j = 1,2,...,p$; $p \geqslant 1$, $0 = p_1 < p_2 < \cdots < p_p$; $f(k,.) : \mathbb{Z}^+ \times \mathbb{R}^{pn} \to \mathbb{R}^n$ is a given vector function satisfying $g(k, 0, \ldots, 0) = 0$, $k \in \mathbb{Z}^+$, $\forall x_0 \in \mathbb{R}^n$, $k_0 \in \mathbb{Z}^+$. The solution of (3) with initial condition $x(k) = x_0$, $k = k_0 - q_q, \ldots, k_0$ is given by

$$x(k) = P_k x_0 + \sum_{s=k_0}^{k-1} G_{s+1}^k g(s, x(s-p_1), x(s-p_2), \ldots, x(s-p_p)) \tag{4}$$

where the transition matrix G_{s+1}^k, $s \geqslant k_0$ satisfy

$$G_s^{k+1} = \sum_{i=1}^{p} A_i(k) G_s^{k-p_i}, \quad G_k^k = I, \quad G_s^k = 0, \quad k < s,$$

and

$$P_k = G_0^k + \sum_{i=2}^{p} \sum_{s=0}^{p_i - 1} G_{s+1}^k A_i(s). \tag{5}$$

We introduce some notations and definitions that will be used throughout the paper.

$\mathbb{R}^n -$ the n dimensional Euclidean vector space,
$\mathbb{R}^+ -$ the set of all non-negative real number,
$\mathbb{Z}^+ -$ the set of all non-negative integers,
$\|x\| -$ the Euclidean norm vector $x \in \mathbb{R}^n$,
$\|A\| -$ the norm of matrix A, $\|A\| = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$.

DEFINITION 1. The zero solution of system (3) is stable if for every $\varepsilon > 0$ and for every $k_0 \in \mathbb{Z}^+$ there is $\delta > 0$ (depending on ε and k_0) such that $\|x(k)\| < \varepsilon$, $k \geqslant k_0$, whenever $\|x_0\| < \delta$. The zero solution is asymptotically stable if it is stable and there is $\delta > 0$ such that $\lim_{k \to \infty} \|x_k\| = 0$, whenever $\|x_0\| < \delta$.

DEFINITION 2. The zero solution of system (3) is weakly asymptotically stable if there is a number $\delta > 0$ such that every solution of the system satisfies $\lim_{k \to \infty} \|x_k\| = 0$, whenever $\|x_0\| < \delta$.

DEFINITION 3. The control system (1) is stabilizable if there are matrices $D(k)$, $k \geqslant -q_q$, such that the system (1) with $u(k) = D(k)x(k)$ is asymptotically stable. The control $u(k) = D(k)x(k)$ is feedback control of the system.

DEFINITION 4. The control system (1) is weakly stabilizable if there exist control $u(k)$, $k \geqslant -q_q$, and number $\delta > 0$ such that the solution $x(k)$ according to these control of system (1) satisfies $\lim_{k\to\infty} \|x_k\| = 0$, whenever $\|x_0\| < \delta$.

LEMMA 1. [2] *Assume that there exist numbers $K > 0$, $w \in (0,1)$ such that*

$$\|G_s^k\| \leqslant K w^{k-s}, \quad \forall k > s \geqslant 0. \tag{6}$$

Then there is a number $K_1 > 0$ such that $\|P_k\| \leqslant K_1 w^k$, $k \in \mathbb{Z}^+$ where

$$K_1 = K + \frac{MK(p-1)}{w^{p_p}(1-w)}, \quad M = \max\{\|A_j(k)\|,\ k = 0,\dots,p_j - 1,\ j = 2,\dots,p\}.$$

2. Generalized discrete Gronwall's inequality

In this section, we present some discrete versions of the Gronwall-type inequality that will be used in studying the stabilizability properties of nonlinear difference controls systems with multiple delays.

THEOREM 1. *Let $z(k) : \mathbb{Z}^+ \to \mathbb{R}^+$. Assume that*

$$z(k) \leqslant C_k + \sum_{s=0}^{k-1} \sum_{j=1}^{p} a_j(s) z(s-p_j)^{m_1} + \sum_{s=0}^{k-1} \sum_{i=1}^{q} b_i(s) z(s-q_i)^{m_2}, \tag{7}$$

where $m_1, m_2 > 0$; $p_p \geqslant q_q$; $a_j(k), b_i(k) : \mathbb{Z}^+ \to \mathbb{R}^+$, $j = 1,2,...,p$, $i = 1,2,...,q$; $z(k) \leqslant C_1 \leqslant 1, k = -p_p,\dots,0$ and $0 = p_1 < p_2 < \dots < p_p$; $0 = q_1 < q_2 < \dots < q_q$. Let $m = \min\{m_1, m_2\}$, $\bar{m} = \max\{m_1, m_2\}$, $d(s) = \sum_{j=1}^{p} a_j(s) + \sum_{i=1}^{q} b_i(s)$ and $\{C_k\}_{k\geqslant 1}$ be a sequence of nonnegative real numbers such that $C_{k+1} \leqslant C_k$, $C_k \in (0,1)$, $k \in \mathbb{Z}^+$.

(a) If $m_1, m_2 \leqslant 1$, then

$$z(k) \leqslant C_1^{m^k} \prod_{s=0}^{k-1} [1 + d(s)], \tag{8}$$

or

$$z(k) \leqslant (1 + C_k^{m^k}) \prod_{s=0}^{k-1} [1 + d(s)], \tag{9}$$

where $C_k^{m^k}$ is increasing, $k \geqslant -p_p$.

(b) If $m_1 \leqslant 1 < m_2$, then

$$z(k) \leqslant C_1^{m_1^k} \prod_{s=0}^{k-1} [1 + d(s)]^{m_2^{k-s-1}}. \tag{10}$$

(c) If $m_1, m_2 > 1$, *then*

$$z(k) \leqslant C_1 \prod_{s=0}^{k-1} [1+d(s)]^{\bar{m}^{k-s-1}}. \tag{11}$$

Proof. *Case (a)* $m_1, m_2 \leqslant 1$: We can prove this theorem by induction on $k \in \mathbb{Z}^+$. Letting $k = 1$, the inequality (7) gives

$$\begin{aligned} z(1) &\leqslant C_1 + \sum_{j=1}^{p} a_j(0) z(-p_j)^{m_1} + \sum_{i=1}^{q} b_i(0) z(-q_i)^{m_2} \\ &\leqslant C_1 + \sum_{j=1}^{p} a_j(0) C_1^m + \sum_{i=1}^{q} b_i(0) C_1^m \\ &\leqslant (C_1^m)(1+d(0)). \end{aligned}$$

We assume that (8) holds for $k = 1, 2, 3, \ldots, k-1$. Using (7) for the step k, we have

$$\begin{aligned} z(k) &\leqslant C_k + \sum_{s=0}^{k-1} \sum_{j=1}^{p} a_j(s) z(s-p_j)^{m_1} + \sum_{s=0}^{k-1} \sum_{i=1}^{q} b_i(s) z(s-q_i)^{m_2} \\ &\leqslant C_1 + \sum_{s=0}^{k-2} \sum_{j=1}^{p} a_j(s) z(s-p_j)^{m_1} + \sum_{s=0}^{k-2} \sum_{i=1}^{q} b_i(s) z(s-q_i)^{m_2} \\ &\quad + \sum_{j=1}^{p} a_j(k-1) z(k-1-p_j)^{m_1} + \sum_{i=1}^{q} b_i(k-1) z(k-1-q_i)^{m_2}. \end{aligned}$$

By induction assumption, we obtain

$$\begin{aligned} z(k) &\leqslant C_1^{m^{k-1}} \prod_{s=0}^{k-2} [1+d(s)] + \sum_{j=1}^{p} a_j(k-1) \{C_1^{m^{k-1-p_j}} \prod_{s=0}^{k-2-p_j} [1+d(s)]\}^{m_1} \\ &\quad + \sum_{i=1}^{q} b_i(k-1) \{C_1^{m^{k-1-q_i}} \prod_{s=0}^{k-2-q_i} [1+d(s)]\}^{m_2}. \end{aligned}$$

For $m \leqslant \{m_1, m_2\} \leqslant 1$, $C_k \leqslant 1$, $k \in \mathbb{Z}^+$, we see that $C_1^{m_1 . m^{k-1-p_j}} \leqslant C_1^{m^{k-p_j}} \leqslant C_1^{m^k}$, $C_1^{m_2 . m^{k-1-q_i}} \leqslant C_1^{m^{k-q_i}} \leqslant C_1^{m^k}$, $j = 1, \ldots, p$, $i = 1, \ldots, q$,

$$\begin{aligned} \prod_{s=0}^{k-2-p_j} [1+d(s)]^{m_1} &\leqslant \prod_{s=0}^{k-2} [1+d(s)], \quad j = 1, \ldots, p, \\ \prod_{s=0}^{k-2-q_i} [1+d(s)]^{m_2} &\leqslant \prod_{s=0}^{k-2} [1+d(s)], \quad i = 1, \ldots, q. \end{aligned}$$

Thus,

$$
\begin{aligned}
z(k) &\leqslant (C_1^{m^{k-1}})\prod_{s=0}^{k-2}[1+d(s)]+\sum_{j=1}^{p}a_j(k-1)\{(C_1^{m^k})\prod_{s=0}^{k-2}[1+d(s)]\}\\
&\quad+\sum_{i=1}^{q}b_i(k-1)\{(C_1^{m^k})\prod_{s=0}^{k-2}[1+d(s)]\}\\
&\leqslant (C_1^{m^k})\prod_{s=0}^{k-1}[1+d(s)],
\end{aligned}
$$

which implies that (8) holds for the step k.

Another of *case (a)*: It is easy to verify (9) for $k=1$. We assume that (9) holds for steps $k=1,2,3,\dots,k-1$. Using (7) for the step k, we have

$$
\begin{aligned}
z(k) &\leqslant C_k+\sum_{s=0}^{k-1}\sum_{j=1}^{p}a_j(s)z(s-p_i)^{m_1}+\sum_{s=0}^{k-1}\sum_{i=1}^{q}b_i(s)z(s-q_j)^{m_2}\\
&\leqslant C_{k-1}+\sum_{s=0}^{k-2}\sum_{j=1}^{p}a_j(s)z(s-p_j)^{m_1}+\sum_{s=0}^{k-2}\sum_{i=1}^{q}b_i(s)z(s-q_i)^{m_2}\\
&\quad+\sum_{j=1}^{p}a_j(k-1)z(k-1-p_j)^{m_1}+\sum_{i=1}^{q}b_i(k-1)z(k-1-q_i)^{m_2}.
\end{aligned}
$$

By induction assumption, we obtain

$$
\begin{aligned}
z(k) &\leqslant (1+C_{k-1}^{m^{k-1}})\prod_{s=0}^{k-2}[1+d(s)]\\
&\quad+\sum_{j=1}^{p}a_j(k-1)\{(1+C_{k-1-p_j}^{m^{k-1-p_j}})\prod_{s=0}^{k-2-p_j}[1+d(s)]\}^{m_1}\\
&\quad+\sum_{i=1}^{q}b_i(k-1)\{(1+C_{k-1-q_i}^{m^{k-1-q_i}})\prod_{s=0}^{k-2-q_i}[1+d(s)]\}^{m_2}.
\end{aligned}
$$

For $m\leqslant\{m_1,m_2\}\leqslant 1$, $C_k\leqslant 1$, $k\in\mathbb{Z}^+$, it is easy to see that $(1+C_{k-1-p_j}^{m^{k-1-p_j}})^{m_1}\leqslant(1+C_{k-1-p_j}^{m^{k-1-p_j}})\leqslant(1+C_{k-1}^{m^{k-1}})$, $(1+C_{k-1-q_i}^{m^{k-1-q_i}})^{m_1}\leqslant(1+C_{k-1-q_i}^{m^{k-1-q_i}})\leqslant(1+C_{k-1}^{m^{k-1}})$, $j=1,\dots,p$, $i=1,\dots,q$,

$$
\begin{aligned}
\prod_{s=0}^{k-2-p_j}[1+d(s)]^{m_1} &\leqslant \prod_{s=0}^{k-2-p}[1+d(s)], \quad j=1,\dots,p,\\
\prod_{s=0}^{k-2-q_i}[1+d(s)]^{m_2} &\leqslant \prod_{s=0}^{k-2}[1+d(s)], \quad i=1,\dots,q.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
z(k) &\leqslant (1+C_{k-1}^{m^{k-1}})\prod_{s=0}^{k-2}[1+d(s)] + \sum_{j=1}^{p} a_j(k-1)\{(1+C_{k-1}^{m^{k-1}})\prod_{s=0}^{k-2}[1+d(s)]\} \\
&\quad + \sum_{i=1}^{q} b_i(k-1)\{(1+C_{k-1}^{m^{k-1}})\prod_{s=0}^{k-2}[1+d(s)]\} \\
&\leqslant (1+C_{k-1}^{m^{k-1}})\prod_{s=0}^{k-1}[1+d(s)] \\
&\leqslant (1+C_{k}^{m^{k}})\prod_{s=0}^{k-1}[1+d(s)],
\end{aligned}
$$

which implies that (9) holds for the step k.

Case (b) $m_1 \leqslant 1 < m_2$: It is easy to verify (10) for $k = 1$. Assume that (10) holds for steps $k = 1,2,3,\ldots,k-1$. Using (7) for the step k and by induction assumption, we have

$$
\begin{aligned}
z(k) &\leqslant C_1^{m_1^{k-1}}\prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-2}} \\
&\quad + \sum_{j=1}^{p} a_j(k-1)\{C_1^{m_1^{k-1-p_j}}\prod_{s=0}^{k-2-p_j}[1+d(s)]^{m_2^{k-s-2-p_j}}\}^{m_1} \\
&\quad + \sum_{i=1}^{q} b_i(k-1)\{C_1^{m_1^{k-1-q_i}}\prod_{s=0}^{k-2-q_i}[1+d(s)]^{m_2^{k-s-2-q_i}}\}^{m_2}.
\end{aligned}
$$

Similar to Case (a), we see that $C_1^{m_1.m_1^{k-1-p_j}} \leqslant C_1^{m_1^{k-p_j}} \leqslant C_1^{m_1^k}$, $C_1^{m_2.m_1^{k-1-q_i}} \leqslant C_1^{m_1^{k-q_i}} \leqslant C_1^{m_1^k}$, $j = 1,\ldots,p$, $i = 1,\ldots,q$,

$$
\begin{aligned}
\prod_{s=0}^{k-2-p_j}[1+d(s)]^{m_1.m_2^{k-s-2-p_j}} &\leqslant \prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-1}}, \quad j = 1,\ldots,p, \\
\prod_{s=0}^{k-2-q_i}[1+d(s)]^{m_2.m_2^{k-s-2-p_j}} &\leqslant \prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-1}}, \quad i = 1,\ldots,q.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
z(k) &\leqslant (C_1^{m_1^k})\prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-1}} + \sum_{j=1}^{p} a_j(k-1)\{(C_1^{m_1^k})\prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-1}}\} \\
&\quad + \sum_{i=1}^{q} b_i(k-1)\{(C_1^{m_1^k})\prod_{s=0}^{k-2}[1+d(s)]^{m_2^{k-s-1}}\} \\
&\leqslant (C_1^{m_1^k})\prod_{s=0}^{k-1}[1+d(s)]^{m_2^{k-s-1}},
\end{aligned}
$$

which implies (10) for the step k.

Case (c) $m_1, m_2 > 1$: We can prove by the same way of case (b). Therefore, the proof of this theorem is complete. □

COROLLARY 1. *Let* $z(k) : \mathbb{Z}^+ \to \mathbb{R}^+$. *Assume that*

$$z(k) \leqslant C_k + \sum_{s=0}^{k-1} \sum_{j=1}^{p} a_j(s) z(s - p_j)^m$$

where $m > 0$; $p \geqslant 1$; $a_j(k) : \mathbb{Z}^+ \to \mathbb{R}^+$, $j = 1, 2, ..., p$; $z(k) \leqslant C_1 \leqslant 1$, $k = -p_p, \ldots, 0$ *and* $\{C_k\}_{k \geqslant 1}$ *is a sequence of nonnegative real numbers such that* $C_{k+1} \leqslant C_k$, $C_k \in (0, 1)$, $k \in \mathbb{Z}^+$.

(a) If $m \leqslant 1$, *then*

$$z(k) \leqslant C_1^{m^k} \prod_{s=0}^{k-1} [1 + \sum_{j=1}^{p} a_j(s)],$$

or

$$z(k) \leqslant (1 + C_k^{m^k}) \prod_{s=0}^{k-1} [1 + \sum_{j=1}^{p} a_j(s)].$$

where $C_k^{m^k}$ *is increasing,* $k \geqslant -p_p$.

(b) If $m > 1$, *then*

$$z(k) \leqslant C_1 \prod_{s=0}^{k-1} [1 + \sum_{j=1}^{p} a_j(s)]^{m^{k-s-1}}.$$

3. Stabilizability results

In this section, we present sufficient conditions for the stabilizability and stability of system (1). We consider the system (1) where $B_i(k) = 0$ of the form

$$x(k+1) = \sum_{j=1}^{p} A_j(k) x(k - p_j) + f_{p,q}(k, x_k, u_k), \quad k \in \mathbb{Z}^+. \tag{12}$$

Associated with condition (6), we consider two conditions

$$\exists K > 0,\ w \in (0,1), \quad \|G_s^k\| \leqslant K w^{2k-s}, \quad \forall k > s \geqslant 0, \tag{13}$$

and

$$\exists K > 0,\ w \in (0,1), \quad \|G_s^k\| \leqslant K w^{2\sum_{i=s}^{k-1} m^i}, \quad \forall k > s \geqslant 0. \tag{14}$$

In the sequel we assume that $\exists a_j(k), b_i(k) : \mathbb{Z}^+ \longrightarrow \mathbb{R}^+$, $j = 1, 2, ..., p$, $i = 1, 2, ..., q$, such that

$$\|f(k, x_1, \ldots, x_p, u_1, \ldots, u_q)\| \leqslant \sum_{j=1}^{p} a_j(k) \|x_j\|^{m_1} + \sum_{i=1}^{q} b_i(k) \|u_i\|^{m_2}, \tag{15}$$

where $m_1, m_2 > 0;\ p, q \in \mathbb{Z}^+$. Let us set

$$\begin{aligned} l_{p_j}(k) &= w^{m_1 \sum_{i=0}^{k-p_j-1} m_2^i - \sum_{i=0}^{k-1} m_2^i + 1}, \\ l_{q_i}(k) &= w^{m_2 \sum_{i=0}^{k-q_i-1} m_2^i - \sum_{i=0}^{k-1} m_2^i + 1}, \\ \bar{l}_{p_j}(k) &= w^{m_1 \sum_{i=0}^{k-p_j-1} \bar{m}^i - \sum_{i=0}^{k-1} \bar{m}^i + 1}, \\ \bar{l}_{q_i}(k) &= w^{m_2 \sum_{i=0}^{k-q_i-1} \bar{m}^i - \sum_{i=0}^{k-1} \bar{m}^i + 1}. \end{aligned}$$

THEOREM 2. *Assume that the conditions* (13) *and* (15) *are satisfied. Suppose that there exist* $D(k),\ k \geqslant -q_q$, *are* $(n \times m)$ *matrices.*

(a) If $m_1, m_2 \leqslant 1$, *there exists* $w \in (0,1)$ *so that*

$$\lim_{k\to\infty}\Big[\sum_{j=1}^{p} a_j(k) w^{-k+km_1} + \sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2} w^{-k+km_2}\Big] = 0. \tag{16}$$

Then the system (12) *is weakly stabilizable.*

In another of case (a), assume that the following conditions hold.

(i) There exist $\delta \in (0,1)$ *and* $w \in (0,1)$ *such that*

$$C_k^{m^k} \leqslant C_{k+1}^{m^{k+1}}, \quad k \geqslant -p_p \quad where \quad C_k = K_1 w^k \delta \leqslant 1 \tag{17}$$

where K_1 *is defined by Lemma* 1.

(ii) Equation (16) *holds. Then the system* (12) *is weakly stabilizable.*

(b) If $m_1 \leqslant 1 < m_2$, *and there exist* $w \in (0,1)$ *and* $K > 0$ *so that*

$$\overline{\lim_{k\to\infty}} \sum_{j=1}^{p} a_j(k) l_{p_j}(k) + \sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2} l_{q_i}(k)] = 0 \tag{18}$$

and we assume the condition (14) *instead of* (13)*. Then the system* (12) *is weakly stabilizable.*

(c) If $m_1, m_2 > 1$, *and there exist* $w \in (0,1)$ *and* $K > 0$ *so that*

$$\lim_{k\to\infty}\Big[\sum_{j=1}^{p} a_j(k)\bar{l}_{p_j}(k) + \sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2}\bar{l}_{q_i}(k)\Big] = 0 \tag{19}$$

and we assume the condition (14) *instead of* (13)*. Then the system* (12) *is stabilizable.*

The feedback controller of (12) *is given by* $u(k) = D(k)x(k)$.

Proof. Given $w \in (0,1)$ and choose $\delta \in (0,1)$ so that $K_1 w^k \delta \leqslant 1$. The solution of system (12) is given by

$$x(k) = P_k x_0 + \sum_{s=0}^{k-1} G_{s+1}^k f(s, x(s-p_1), \ldots, x(s-p_p), u(s-q_1), \ldots, u(s-q_q)).$$

Then, we have

$$\|x(k)\| \leqslant K_1 w^{2k}\|x_0\| + \sum_{s=0}^{k-1} K w^{k-s-1}\Big(\sum_{j=1}^{p} a_j(k)\|x_{k-p_j}\|^{m_1} + \sum_{i=1}^{q} b_i(k)\|u_{k-q_i}\|^{m_2}\Big).$$

Setting $u(k) = D(k)x(k)$, $k \geqslant -q_q$, we obtain

$$\begin{aligned} w^{-k}\|x(k)\| \leqslant K_1 w^k \delta + \sum_{s=0}^{k-1} K w^{-s-1} \sum_{j=1}^{p} a_j(k)\|x_{k-p_j}\|^{m_1} \\ + \sum_{s=0}^{k-1} K w^{-s-1} \sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2}\|x(k-q_i)\|^{m_2}. \end{aligned}$$

for $\|x_0\| < \delta$. Let us set

$$\begin{aligned} C_1 &= K_1 w^{-p_p}\delta, \\ C_k &= K_1 w^k \delta, \\ z(k) &= w^{-k}\|x(k)\|, \\ \bar{a}_j(k) &= K w^{-k-1+km_1-m_1 p_j} a_j(k), \\ \bar{b}_i(k) &= K w^{-k-1+km_2-m_2 q_i} b_i(k)\|D(k-q_i)\|^{m_2}, \\ \bar{d}(k) &= \sum_{j=1}^{p} \bar{a}_j(k) + \sum_{i=1}^{q} \bar{b}_i(k). \end{aligned}$$

We have

$$\|z(k)\| \leqslant C_k + \sum_{s=0}^{k-1}\sum_{j=1}^{p} \bar{a}_j(s)\|z(s-p_j)\|^{m_1} + \sum_{s=0}^{k-1}\sum_{i=1}^{q} \bar{b}_i(s)\|z(s-q_i)\|^{m_2}. \tag{20}$$

Case (a) $m_1, m_2 \leqslant 1$: Applying Theorem 1 to the inequality (20), we obtain

$$\begin{aligned} \|x(k)\| \leqslant C_1^{m^k} \prod_{s=0}^{k-1}\Big(w + \sum_{j=1}^{p} K w^{-s+sm_1-m_1 p_j} a_j(s) \\ + \sum_{s=0}^{k-1} K w^{-s+sm_2-m_2 q_i} b_i(s)\|D(k-q_i)\|^{m_2}\Big). \end{aligned}$$

By assumption (16), there are $N \in \mathbb{N}$, $l \in (0, 1-w)$ such that

$$w + \sum_{j=1}^{p} K w^{-k+km_1-m_1 p_j} a_j(k) + \sum_{s=0}^{k-1} K w^{-k+km_2-m_2 q_i} b_i(k)\|D(k-q_i)\|^{m_2} \leqslant w + l := v < 1,$$

for all $k \geqslant N$ and $C_1^{m^k} = (K_1 w^{-p_p}\delta)^{m^k} \leqslant 1$. Therefore, we obtain

$$\|x(k)\| \leqslant M_N v^{k-N}, \quad k \geqslant N.$$

It implies that $\lim_{k\to\infty}\|x_k\|=0$, whenever $\|x_0\|<\delta$. Therefore, the system (12) is weakly stabilizable by the feedback control $u(k)=D(k)x(k)$.

In another of case (a), it is very clear that the system (12) is weakly stabilizable by using the same way of case (a).

Case (b) $m_1\leqslant 1<m_2$: We consider the condition (18) and by the same arguments that used in the proof of Lemma 2.1 [2], we can find some number $K_2>0$ such that

$$\|P_k\|\leqslant K_2 w^{2\sum_{i=0}^{k-1} m^i},\quad k\in\mathbb{Z}^+.$$

We obtain the following estimation the solution of system (12) of the form

$$\|x(k)\|\leqslant K_2 w^{2\sum_{i=0}^{k-1} m_2^i}\|x_0\|+\sum_{s=0}^{k-1} K w^{\sum_{i=s}^{k-1} m_2^i}\sum_{j=1}^{p} a_j(k)\|x(k-p_j)\|^{m_1}$$
$$+\sum_{s=0}^{k-1} K w^{\sum_{i=s}^{k-1} m_2^i}\sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2}\|x(k-q_i)\|^{m_2}.$$

Multiplying both sides of inequality by $w^{-\sum_{i=0}^{k-1} m^i}$, we obtain

$$w^{-\sum_{i=0}^{k-1} m_2^i}\|x(k)\|\leqslant K_2 w^{\sum_{i=0}^{k-1} m_2^i}\delta+\sum_{s=0}^{k-1} K w^{-\sum_{i=0}^{s-1} m_2^i}\sum_{j=1}^{p} a_j(k)\|x(k-p_j)\|^{m_1}$$
$$+\sum_{s=0}^{k-1} K w^{\sum_{i=s}^{k-1} m_2^i}\sum_{i=1}^{q} b_i(k)\|D(k-q_i)\|^{m_2}\|x(k-q_i)\|^{m_2},$$

for $\|x_0\|<\delta$. Let us set

$$\begin{aligned}
C_1 &= K_2 w\delta,\\
C_k &= K_2 w^{\sum_{i=0}^{k-1} m_2^i}\delta,\\
z(k) &= w^{-\sum_{i=0}^{k-1} m^i}\|x(k)\|,\\
\bar a_j(k) &= K w^{m_1\sum_{i=0}^{k-p_j-1} m_2^i-\sum_{i=0}^{k-1} m_2^i} a_j(k),\\
\bar b_i(k) &= K w^{m_2\sum_{i=0}^{k-q_i-1} m_2^i-\sum_{i=0}^{k-1} m_2^i} b_i(k)\|D(k-q_i)\|^{m_2}\quad ,\\
\bar d(k) &= \sum_{j=1}^{p}\bar a_j(k)+\sum_{i=1}^{q}\bar b_i(k).
\end{aligned}$$

Applying Theorem 3.1 to above inequality, we obtain

$$\|x(k)\|\leqslant C_1^{m_1^k}\prod_{s=0}^{k-1}(w+\sum_{j=1}^{p} K w^{m_1\sum_{i=0}^{s-p_j-1} m_2^i-\sum_{i=0}^{s-1} m_2^i+1} a_j(s)$$
$$+\sum_{i=0}^{q} K w^{m_2\sum_{i=0}^{s-q_i-1} m_2^i-\sum_{i=0}^{s-1} m_2^i+1} b_i(k)\|D(k-q_i)\|^{m_2})^{m_2^{k-s-1}}.$$

By assumption (18), there are $N_1 \in \mathbb{N}$, $l \in (0,1-w)$ such that

$$w+\sum_{j=1}^{p} Kl_{p_j}(k)a_j(k)+\sum_{i=0}^{q} Kl_{q_i}(k)b_i(k)\|D(k-q_i)\|^{m_2} \leqslant w+l := v < 1,$$

for all $k \geqslant N_1$ and $C_1^{m_1^k} = (K_2 w\delta)^{m_1^k} \leqslant 1$. Therefore, we obtain

$$\|x(k)\| \leqslant M_{N_1} v^{k-N_1}, \quad k \geqslant N_1.$$

It implies that $\lim_{k\to\infty}\|x_k\| = 0$, whenever $\|x_0\| < \delta$. Therefore, the system (12) is weakly stabilizable by the feedback control $u(k) = D(k)x(k)$.

Case (c) $m_1, m_2 > 1$: The proof can show by the same way in case (b). Therefore, the proof of this theorem is complete. □

COROLLARY 2. *Assume the condition* (13) *holds and suppose that*

$$\|g(k,x_1,\ldots,x_p)\| \leqslant \sum_{j=1}^{p} a_j(k)\|x_j\|^m,$$

where $m > 0, a_j(k) : \mathbb{Z}^+ \to \mathbb{R}^+$, $j = 1,2,...,p$.

(a) If $m \leqslant 1$, *and there exists* $w \in (0,1)$ *so that*

$$\overline{\lim_{k\to\infty}} \sum_{j=1}^{p} a_j(k) w^{-k+km} = 0 \tag{21}$$

then the system (3) *is weakly asymptotically stable.*

In another of case (a), We assume that the following conditions hold.

(i) There exist $\delta \in (0,1)$ *and* $w \in (0,1)$ *such that*

$$C_k^{m^k} \leqslant C_{k+1}^{m^{k+1}}, \quad k \geqslant -p_p \quad where \quad C_k = K_1 w^k \delta \leqslant 1 \tag{22}$$

where K_1 *is defined by Lemma 1.*

(ii) Equation (21) *holds. Then the system* (3) *is weakly stabilizable.*

(b) If $m > 1$, *and there exist* $w \in (0,1)$ *and* $K > 0$ *so that*

$$\overline{\lim_{k\to\infty}} \sum_{j=1}^{p} a_j(k) w^{m\sum_{i=0}^{k-p_j-1} m^i - \sum_{i=0}^{k-1} m^i} = 0 \tag{23}$$

and we assume the condition (14) *instead of* (13). *Then the system* (3) *is asymptotically stable.*

4. Numerical examples

EXAMPLE 1. Consider the nonlinear difference controls delay system in $\mathbb{R}^2$ of the form

$$x_1(k+1) = \frac{1}{2^{k+3}}x_1(k) + \frac{1}{2^k}u^{1/3}(k) + \frac{1}{2^k}x_1^{1/3}(k), \tag{24}$$

$$x_2(k+1) = \frac{1}{2^{k+3}}x_1(k) - x_2(k) + \frac{1}{2^{k+3}}x_2(k-3) + ku(k) + \frac{1}{2^k}x_2^{1/3}(k-3), \tag{25}$$

where $x_1(k), x_2(k), u(k) \in \mathbb{R}$. This equations can be the form of system (1), where

$$A_1(k) = \begin{pmatrix} \frac{1}{2^{k+3}} & 0 \\ \frac{1}{2^{k+3}} & -1 \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2^{k+3}} \end{pmatrix}, \quad B_1(k) = [0 \quad k]^T,$$

$\|f(k, x(k), x(k-3), u(k))\| \leqslant 2^{-k}\|u(k)\|^{1/3} + 2^{-k}\|x_1(k)\|^{1/3} + 2^{-k}\|x_2(k-3)\|^{1/3}$.

We have $m_1 = 1/3$, $m_2 = 1/3$, $p = 2$, $p_2 = 3$, $q = 1$, $a_1(k) = 2^{-k}$, $a_2(k) = 2^{-k}$, $b_1(k) = 2^{-k}$. For the feedback control $u(k) = D(k)x(k)$ with $D(k) = [0 \quad 1/k]$, we obtain

$$C_1(k) = A_1(k) + B_1(k)D(k) = \begin{pmatrix} \frac{1}{2^{k+3}} & 0 \\ \frac{1}{2^{k+3}} & 0 \end{pmatrix}, \quad C_2(k) = A_2(k).$$

It is easy to verify that the transition matrix G_s^k of the equations (24)–(25) satisfies (13), where $K = 1$, $w = 1/2$ and the condition (16) can do it. Therefore, (24) and (25) are weakly stabilizable. $\square$

EXAMPLE 2. Consider the nonlinear difference controls delay system in $\mathbb{R}^2$ of the form

$$x_1(k+1) = \frac{1}{5^{k+4}}x_1(k) + \frac{1}{5^{k+4}}x_2(k) + \frac{1}{5^k}u^{1/4}(k) + \frac{1}{5^k}x_1^{1/4}(k), \tag{26}$$

$$x_2(k+1) = -kx_1(k) - x_2(k) + \frac{1}{5^{k+4}}x_2(k-3) + k^2u(k) + \frac{1}{5^k}x_2^{1/4}(k-3), \tag{27}$$

where $x_1(k), x_2(k), u(k) \in \mathbb{R}$. This system is the form of system (1), where

$$A_1(k) = \begin{pmatrix} \frac{1}{5^{k+4}} & \frac{1}{5^{k+4}} \\ -k & -1 \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{5^{k+4}} \end{pmatrix}, \quad B_1(k) = [0 \quad k^2]^T,$$

$\|f(k, x(k), x(k-3), u(k))\| \leqslant 5^{-k}\|u(k)\|^{1/4} + 5^{-k}\|x_1(k)\|^{1/4} + 5^{-k}\|x_2(k-3)\|^{1/4}$. We have $m_1 = 1/4$, $m_2 = 1/4$, $p = 2$, $p_2 = 3$, $q = 1$, $a_1(k) = 5^{-k}$, $a_2(k) = 5^{-k}$, $b_1(k) = 5^{-k}$. For the feedback control $u(k) = D(k)x(k)$ with $D(k) = [1/k \quad 1/k^2]$, we get

$$C_1(k) = A_1(k) + B_1(k)D(k) = \begin{pmatrix} \frac{1}{5^{k+4}} & \frac{1}{5^{k+4}} \\ 0 & 0 \end{pmatrix}, \quad C_2(k) = A_2(k).$$

It is easy to verify that the transition matrix G_s^k of the equations (26)–(27) satisfies (13), where $K = 1$, $w = 1/5$. $\|G_s^k\| \leqslant \frac{1}{5^{2k-s}}$ for $k > s \geqslant 0$. Next, we will show that (16) and (17) hold. We choose $w = 1/5$, $K_1 = 5$, $\delta = (1/5)^7$ and we can see that

$$C_k^{m^k} \leqslant C_{k+1}^{m^{k+1}}, \quad \forall k \geqslant -p_2 \quad \text{where} \quad C_k = 5\left(\frac{1}{5}\right)^{k+7}$$

and

$$\begin{aligned} &\lim_{k\to\infty} a_1(k)w^{-k+\frac{k}{4}} + a_2(k)w^{-k+\frac{k}{4}} + b_1(k)\left(\frac{1}{k} + \frac{1}{k^2}\right)w^{-k+\frac{k}{4}} \\ &= \lim_{k\to\infty} 5^{-k}5^{k-\frac{k}{4}} + 5^{-k}5^{k-\frac{k}{4}} + 5^{-k}\left(\frac{1}{k} + \frac{1}{k^2}\right)5^{k-\frac{k}{4}} \\ &= 0. \end{aligned}$$

Therefore, the equations (26) and (27) are weakly stabilizable. □

EXAMPLE 3. Consider the nonlinear difference controls delay system in $\mathbb{R}^2$ of the form

$$x_1(k+1) = \frac{1}{4^{2k+4}}x_1(k) + \frac{1}{4^{2k+4}}x_2(k) + 2^{-2^{k-1}-k}u^2(k) + 2^{-2^{k-1}-2^{k-2}-k}x_1^{1/4}(k), \quad (28)$$

$$x_2(k+1) = -k^2x_1(k) - kx_2(k) + \frac{1}{4^{2k+4}}x_2(k-3) + k^3u(k), \quad (29)$$

where $x_1(k), x_2(k), u(k) \in \mathbb{R}$. This system is the form of system (1), where

$$A_1(k) = \begin{pmatrix} \frac{1}{4^{2k+4}} & \frac{1}{4^{2k+4}} \\ -k^2 & -k \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4^{2k+4}} \end{pmatrix}, \quad B_1(k) = [0 \quad k^3]^T,$$

$\|f(k,x(k),x(k-3),u(k))\| \leqslant 2^{-2^{k-1}-k}\|u(k)\|^2 + 2^{-2^{k-1}-2^{k-2}-k}\|x_1(k)\|^{1/4}$. We have $m_1 = 1/4$, $m_2 = 2$, $p = 1$, $q = 1$, $a_1(k) = 2^{-2^{k-1}-2^{k-2}-k}$, $b_1(k) = 2^{-2^{k-1}-k}$. For the feedback control $u(k) = D(k)x(k)$ with $D(k) = [1/k \quad 1/k^2]$, we get

$$C_1(k) = A_1(k) + B_1(k)D(k) = \begin{pmatrix} \frac{1}{4^{2k+4}} & \frac{1}{4^{2k+4}} \\ 0 & 0 \end{pmatrix}, \quad C_2(k) = A_2(k).$$

It is easy to verify that the transition matrix G_s^k of the system (28)–(29) satisfies (14) for $K = 1$, $w = 1/4$ and the condition (18) can do it. Therefore, the system (28)–(29) is weakly stabilizable. □

EXAMPLE 4. Consider the nonlinear difference controls delay system in $\mathbb{R}^2$ of the form

$$x_1(k+1) = \frac{1}{3^{3^k+4}}x_1(k) + \frac{1}{3^{3^k+4}}x_2(k) + 3^{-3^{k-1}-k}u^3(k) + 3^{-3^{k-1}-k}x_1^3(k), \quad (30)$$

$$x_2(k+1) = -kx_1(k) - x_2(k) + \frac{1}{5^{3^k+4}}x_2(k-3) + ku(k), \quad (31)$$

where $x_1(k), x_2(k), u(k) \in \mathbb{R}$. This system is the form of control system (1), where

$$A_1(k) = \begin{pmatrix} \frac{1}{3^{3^k+4}} & \frac{1}{3^{3^k+4}} \\ -k & -1 \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{5^{3^k+4}} \end{pmatrix}, \quad B_1(k) = [0 \quad k^2]^T,$$

$\|f(k,x(k),x(k-3),u(k))\| \leqslant 3^{-3^{k-1}-k}\|u(k)\|^3 + 3^{-3^{k-1}-k}\|x_1(k)\|^3$. We have $m_1 = m_2 = 3$, $p = 2$, $p_2 = 3$, $q = 1$, $a_1(k) = 3^{-3^{k-1}-k}$, $a_2(k) = 0$, $b_1(k) = 3^{-3^{k-1}-k}$. For the feedback control $u(k) = D(k)x(k)$ with $D(k) = [1/k \quad 1/k^2]$, we get

$$C_1(k) = A_1(k) + B_1(k)D(k) = \begin{pmatrix} \frac{1}{3^{3^k+4}} & \frac{1}{3^{3^k+4}} \\ 0 & 0 \end{pmatrix}, \quad C_2(k) = A_2(k)$$

We consider in the same way of Example 3 for this feedback control. Therefore, the system (30)–(31) is stabilizable. □

5. Conclusion

In this paper, we study nonlinear difference controls systems with multiple delays on control and states. We present a generalized discrete Gronwall's inequality. Based on a new Gronwall's inequality, sufficient conditions for stabilizability are obtained. Numerical examples illustrate the results are given.

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