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SOME FIXED POINT PROPERTY FOR MULTIVALUED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. We show some geometric conditions on a Banach space X concerning the generalized James constant, the generalized Jordan-von Neumann constant, the generalized Zbagănu constant and the coefficient $R(1,X)$, which imply the existence of fixed points for multivalued nonexpansive mappings. Our results extend and improve some recent results.

1. Introduction

In 1969, Nadler [1] established the multivalued version of Banach contraction principle. Since then, the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued non-expansive mappings. However, many questions remain open, for instance, the possibility of extending the well-known Kirk's theorem [2], that is, do Banach spaces with weak normal structure have the fixed point property (FPP) for multivalued nonexpansive mappings?

Since weak normal structure is implied by different geometric properties of Banach spaces, it is natural to study if those properties imply the FPP for multivalued mappings. S. Dhompongsa et al. [3, 4] introduced the (DL)-condition and property (D) which imply the FPP for multivalued nonexpansive mappings. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D). In 2007, Domínguez Benavides and Gavira [7] had established FFP for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opial modulus. A. Kaewkhao [5] has established FFP for multivalued nonexpansive mappings in terms of the James constant, the Jordan-von Neumann constant, weak orthogonality. In 2010, Domínguez Benavides and Gavira [8] had given a survey of this subject and presented the main known results and current research directions.

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2. Preliminaries

Before going to the result, let us recall some concepts and results which will be used in the following sections. Let X be a Banach space with the unit ball $B_X = \{x \in X : \|x\| \leqslant 1\}$ and the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The following two constants of a Banach space

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y \in X \ \ and \ \ \|x\|+\|y\|>0\right\},$$

$$J(X) = \sup\{\min\{\|x+y\|,\|x-y\|\} : x,y \in S_X\}$$

are called the von Neumann-Jordan [2] and James constants [9], respectively, and are widely studied by many authors ([1, 3, 7, 8, 12, 21, 24, 25, 26, 27, 28]). Recently, both constants are generalized in the following ways (see [7, 8]): for $a \geqslant 0$

$$C_{\mathrm{NJ}}(a,X) = \sup\left\{\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} : x,y,z \in X \ \ and \ \ \|x\|+\|y\|+\|z\|>0,\right.$$

$$\left. and \ \|y-z\| \leqslant a\|x\| \right\},$$

$$J(a,X) = \sup\{\min\{\|x+y\|,\|x-z\|\} : x,y,z \in S_X, \ and \ \|y-z\| \leqslant a\|x\|\}.$$

It is clear that $C_{\mathrm{NJ}}(0,X) = C_{\mathrm{NJ}}(X)$ and $J(0,X) = J(X)$.

Recently, Gao and Saejung in[10] define a new constant: for $a \geqslant 0$

$$C_Z(a,X) = \sup\left\{\frac{2\|x+y\|\|x-z\|}{2\|x\|^2+\|y\|^2+\|z\|^2} : x,y,z \in X \ \ and \ \ \|x\|+\|y\|+\|z\|>0,\right.$$

$$\left. and \ \|y-z\| \leqslant a\|x\| \right\}$$

which is inspired by Zbăganu paper [23]. It is clear that

$$C_Z(0,X) = C_Z(X) = \sup\left\{\frac{\|x+y\|\|x-z\|}{\|x\|^2+\|y\|^2} : x,y \in X, \|x\|+\|y\|>0\right\}.$$

The modulus of convexity of X (see[4]) is a function $\delta_X(\varepsilon) : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1-\frac{\|x+y\|}{2} : x,y \in S_X, \|x-y\| \geqslant \varepsilon\right\}.$$

The function $\delta_X(\varepsilon)$ strictly increasing on $[\varepsilon_0(X),2]$. Here $\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X, and the space is called uniformly nonsquare if $\varepsilon_0(X) < 2$.

The following coefficient is defined by Domínguez Benavides [20] as

$$R(1,X) = \sup\{\liminf_{n\to\infty}\|x+x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant 1$ and all weakly null sequence (x_n) in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty}(\limsup_{m\to\infty} \|x_n - x_m\|) \leqslant 1.$$

Obvious, $1 \leqslant R(1,X) \leqslant 2$. Some geometric conditions sufficient for normal structure in term of the coefficient have been studied in [6], [14].

Let C be a nonempty subset of a Banach space X. We shall denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : C \to CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty) \leqslant \|x-y\|, x,y \in C$$

where $H(.,.)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\{\sup_{x\in A}\inf_{y\in B} \|x-y\|, \sup_{y\in B}\inf_{x\in A} \|x-y\|\}, A,B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r(C,\{x_n\})$ and the asymptotic center $A(C,\{x_n\})$ of $\{x_n\}$ in C are defined by

$$r(C,\{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in C\}$$

and

$$A(C,\{x_n\}) = \{x \in C : \limsup_n \|x_n - x\| = r(C,\{x_n\})\},$$

respectively. It is known that $A(C,\{x_n\})$ is a nonempty weakly compact convex set whenever C is. The sequence $\{x_n\}$ is called regular with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences x_{n_i} of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C,\{x_n\}) = A(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf_{x\in C}\sup_{y\in D} \|x-y\|.$$

S. Dhompongsa et al. [4] introduced the property (D) if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset C of X, any sequence $\{x_n\} \subset C$ which is regular asymptotically uniform relative to C, and any sequence $\{y_n\} \subset A(C,\{x_n\})$ which is regular asymptotically uniform relative to X we have

$$r(C,\{y_n\}) \leqslant \lambda r(C,\{x_n\}).$$

The Domínguez-Lorenzo condition((DL)-condition,in short) introduced in [3] is defined as follows: if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset C of X and for every bounded sequence $\{x_n\}$ in C which is regular with respect to C,

$$r_C(A(C,\{x_n\})) \leqslant \lambda r(C,\{x_n\}).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results shows that property (D) is stronger than weak normal structure and also implies the existence of fixed points for multivalued nonexpansive mappings ([4]).

THEOREM 2.1. *Let X be a Banach space satisfying ((DL)-condition) property (D). Then X has weak normal structure.*

THEOREM 2.2. *Let C be a nonempty weakly compact convex subset of a Banach space X which satisfies ((DL)-condition) the property (D). Let $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

3. Main results

First we recall some basic facts about ultrapowers. Let $\mathscr{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathscr{F}$, denoted by $\lim_{\mathscr{F}} x_n = x$ if for each neighborhood U of x, $\{n \in \mathbb{N} : x_n \in U\} \in \mathscr{F}$. A filter U on $\mathbb{N}$ is called to be an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $A : A \subset \mathbb{N}, n_0 \in A$ for some fixed $n_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $l_\infty(X)$ denote the subspace of the product space $\amalg_{n\in\mathbb{N}} X$ equipped with the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Let $\mathscr{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathscr{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathscr{U}} \|x_n\| = 0\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $l_\infty(X)/N_{\mathscr{U}}$ equipped with the quotient norm. Write $(x_n)_{\mathscr{U}}$ to denote the elements of the ultrapower. Note that if $\mathscr{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically.

THEOREM 3.1. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in C regular with respect to C, then for all $a \geqslant 0$,*

$$r_C(A(C,\{x_n\})) \leqslant \frac{J(a,X)}{\min\left\{\left|1+\frac{(1-a)}{R(1,X)}\right|, 1+\frac{1}{R(1,X)}\right\}} r(C,\{x_n\}).$$

Proof. Denote $r = r(C,\{x_n\})$ and $A = A(C,\{x_n\})$. We can assume $r > 0$, by passing to a subsequence if necessary, we can also assume that $\{x_n\}$ is weakly convergent to a point $x \in C$ and $d = \lim_{n\neq m} \|x_n - x_m\|$ exists. Since $\{x_n\}$ is regular with respect to C, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Observe that the norm is weakly lower semicontinuous, we have

$$\liminf_n \|x_n - x\| \leqslant \liminf_n \liminf_m \|x_n - x_m\| = \lim_{n\neq m} \|x_n - x_m\| = d.$$

Let $\varepsilon > 0$, taking a subsequence if necessary, we can assume that $\|x_n - x\| < d + \varepsilon$ for all n. Let $z \in A$, then we have $\limsup_n \|x_n - z\| = r$ and $\|x - z\| \leqslant \liminf_n \|x_n - z\| \leqslant r$. Denote $R = R(1,X)$, then by definition we have

$$R \geqslant \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} + \frac{z-x}{r} \right\| = \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} - \frac{x-z}{r} \right\|.$$

Convexity of C implies that $\frac{R-1}{R+1}x+\frac{2}{R+1}z\in C$, since the norm is weak lower semicontinuity, we have

$$\begin{aligned}
&\liminf_n \left\|\frac{x_n-z}{r}+\frac{1}{R}\left(\frac{x_n-x}{d+\varepsilon}-\frac{x-z}{r}\right)\right\| \\
&= \liminf_n \left\|\left(\frac{1}{r}+\frac{1}{R(d+\varepsilon)}\right)(x_n-x)+\left(\frac{1}{r}-\frac{1}{Rr}\right)x-\left(\frac{1}{r}-\frac{1}{Rr}\right)z\right\| \\
&\geqslant \left\|\left(\frac{1}{r}-\frac{1}{Rr}\right)x+\frac{2}{Rr}z-\left(\frac{1}{r}+\frac{1}{Rr}\right)z\right\| \\
&= \left(\frac{1}{r}+\frac{1}{Rr}\right)\left\|\frac{R-1}{R+1}x+\frac{2}{R+1}z-z\right\| \\
&\geqslant \left(1+\frac{1}{R}\right)\left(\frac{r_C(A)}{r}\right).
\end{aligned}$$

$$\begin{aligned}
&\liminf_n \frac{1}{Rr}\left\|R(x_n-z)-(1-a)\left[\frac{r(x_n-x)}{d+\varepsilon}-(x-z)\right]\right\| \\
&\geqslant \frac{1}{Rr}\|\left(R-\frac{(1-a)r}{d+\varepsilon}\right)(x_n-x)+(R+1-a)(x-z)\| \\
&\geqslant \left|1+\frac{(1-a)}{R}\right|\left(\frac{r_C(A)}{r}\right)
\end{aligned}$$

For every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

(1) $\|x_N-z\|\leqslant r+\varepsilon$.

(2) $\|\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\|\leqslant R(r+\varepsilon)$.

(3) $\frac{1}{Rr}\|R(x_N-z)+\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\|\geqslant(1+\frac{1}{R})\frac{r_C(A)}{r}\left(\frac{r-\varepsilon}{r}\right)$.

(4) $\frac{1}{Rr}\left\|R(x_N-z)-(1-a)\left[\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right]\right\|\geqslant\left|1+\frac{(1-a)}{R}\right|\left(\frac{r_C(A)}{r}\right)\left(\frac{r-\varepsilon}{r}\right)$.

Now, put $\tilde{u}=(\frac{x_N-z}{r+\varepsilon})_{\mathscr{U}}$, $\tilde{v}=\frac{1}{R(r+\varepsilon)}(\frac{r(x_N-x)}{d+\varepsilon}-(x-z))_{\mathscr{U}}$, $\tilde{\omega}=\frac{1-a}{R(r+\varepsilon)}(\frac{r(x_N-x)}{d+\varepsilon}-(x-z))_{\mathscr{U}}$. Using the above estimates, we obtain $\tilde{u},\tilde{v},\tilde{\omega}\in B_X$, $\|\tilde{v}-\tilde{\omega}\|\leqslant a\|\tilde{u}\|$ and

$$\begin{aligned}
\|\tilde{u}+\tilde{v}\| &= \frac{1}{R(r+\varepsilon)}\left\|R(x_N-z)+\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right\| \\
&\geqslant \left(1+\frac{1}{R}\right)\frac{r_C(A)}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right), \\
\|\tilde{u}-\tilde{\omega}\| &= \frac{1}{R(r+\varepsilon)}\left\|R(x_n-z)-(1-a)\left[\frac{r(x_n-x)}{d+\varepsilon}-(x-z)\right]\right\| \\
&\geqslant \left|1+\frac{(1-a)}{R}\right|\left(\frac{r_C(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}$$

From the definition of $J(a,\tilde{X})$, then

$$\begin{aligned} J(a,\tilde{X}) &\geqslant \|\tilde{u}+\tilde{v}\| \wedge \|\tilde{u}-\tilde{\omega}\| \\ &\geqslant \left(1+\frac{1}{R}\right)\frac{r_C(A)}{r}\left(\frac{r-\varepsilon}{r+\varepsilon}\right) \wedge \left|1+\frac{(1-a)}{R}\right|\left(\frac{r_C(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right). \end{aligned}$$

Since the above inequality is true for every $\varepsilon > 0$ and $J(a,X) = J(a,\tilde{X})$, we obtain

$$r_C(A) \leqslant \frac{J(a,X)}{\min\left\{\left|1+\frac{(1-a)}{R}\right|, 1+\frac{1}{R}\right\}} r. \quad \square$$

COROLLARY 3.2. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $J(a,X) < \min\left\{\left|1+\frac{(1-a)}{R(1,X)}\right|, 1+\frac{1}{R(1,X)}\right\}$ and $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof. If $J(a,X) < \min\left\{\left|1+\frac{(1-a)}{R(1,X)}\right|, 1+\frac{1}{R(1,X)}\right\}$, then X satisfy the (DL)-condition by Theorem 3.1, so T has a fixed point by Theorem 2.2. $\square$

COROLLARY 3.3. *Let X be a Banach space such that*

$$J(a,X) < \min\left\{\left|1+\frac{(1-a)}{R(1,X)}\right|, 1+\frac{1}{R(1,X)}\right\},$$

then X has normal structure.

Proof. In fact, from the Theorem 2.1, it is easy to prove that X has weak normal structure. Since $1 \leqslant R(1,X) \leqslant 2$, we have $J(a,X) < \min\left\{\left|1+\frac{(1-a)}{R(1,X)}\right|, 1+\frac{1}{R(1,X)}\right\} < 2$ for some $a \geqslant 0$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weak normal structure coincide with normal structure. $\square$

REMARK 3.4. In particular, when $a = 0$, Theorem 3.1 and Corollary 3.2 includes the above result of B. Gavira [6, Theorem 3 and Corollary 2], Corollary 3.3 includes the result of E. M. Mazcuñán-Navarro [21, Corollary 24]

THEOREM 3.5. *Let C be a weakly compact convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in C regular with respect to C.*

$$r_C(A(C,\{x_n\})) \leqslant \frac{R(1,X)\sqrt{C_{\mathrm{NJ}}(a,X)[3+(1-a)^2]}}{\sqrt{(R(1,X)+1)^2+(R(1,X)+|1-a|)^2}} r(C,\{x_n\}).$$

Proof. Denote $R = R(1,X)$, repeating the arguments in the proof of Theorem 3.1, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \leqslant r + \varepsilon$.

(2) $\|\frac{r(x_N-x)}{d+\varepsilon} - (x-z)\| \leqslant R(r+\varepsilon)$.

(3) $\|R(x_N - z) + \frac{r(x_N-x)}{d+\varepsilon} - (x-z)\| \geqslant \left(R+1\right) r_C(A)\left(\frac{r-\varepsilon}{r}\right)$

(4) $\left\|R(x_N - z) - (1-a)\left[\frac{r(x_N-x)}{d+\varepsilon} - (x-z)\right]\right\| \geqslant \left|R+1-a\right| r_C(A)\left(\frac{r-\varepsilon}{r}\right)$.

Now, put $\tilde{u} = R(x_N - z)_{\mathscr{U}}$, $\tilde{v} = (\frac{r(x_N-x)}{d+\varepsilon} - (x-z))_{\mathscr{U}}$, $\tilde{\omega} = (1-a)(\frac{r(x_N-x)}{d+\varepsilon} - (x-z))_{\mathscr{U}}$. Using the above estimates, we obtain $\|\tilde{u}\| \leqslant R(r+\varepsilon)$, $\|\tilde{v}\| \leqslant R(r+\varepsilon)$, $\|\tilde{\omega}\| \leqslant |1-a|R(r+\varepsilon)$, $\|\tilde{v} - \tilde{\omega}\| \leqslant a\|\tilde{u}\|$ and

$$\begin{aligned}
\|\tilde{u}+\tilde{v}\| &= \left\|R(x_N - z) + \frac{r(x_N-x)}{d+\varepsilon} - (x-z)\right\| \\
&\geqslant \left(R+1\right) r_C(A)\left(\frac{r-\varepsilon}{r}\right), \\
\|\tilde{u}-\tilde{\omega}\| &= \left\|R(x_n - z) - (1-a)\left[\frac{r(x_n-x)}{d+\varepsilon} - (x-z)\right]\right\| \\
&\geqslant \left|R+1-a\right| r_C(A)\left(\frac{r-\varepsilon}{r}\right).
\end{aligned}$$

From the definition of $C_{\mathrm{NJ}}(a,\tilde{X})$, then

$$\begin{aligned}
C_{\mathrm{NJ}}(a,\tilde{X}) &\geqslant \left\{\frac{\|\tilde{u}+\tilde{v}\|^2 + \|\tilde{u}-\tilde{\omega}\|^2}{2\|\tilde{u}\|^2 + \|\tilde{v}\|^2 + \|\tilde{\omega}\|^2}\right\} \\
&\geqslant \frac{(R+1)^2 + (R+1-a)^2}{3R^2 + (1-a)^2R^2} \frac{r_C^2(A)}{r^2}\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^2.
\end{aligned}$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{\mathrm{NJ}}(a,X) = C_{\mathrm{NJ}}(a,\tilde{X})$, we obtain

$$r_C(A) \leqslant \frac{R\sqrt{C_{\mathrm{NJ}}(a,X)[3+(1-a)^2]}}{\sqrt{(R+1)^2 + (R+1-a)^2}}\, r. \quad \square$$

COROLLARY 3.6. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{\mathrm{NJ}}(a,X) < \frac{(R+1)^2+(R+1-a)^2}{[3+(1-a)^2]R^2}$ and $T : C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

Proof. If $C_{\mathrm{NJ}}(a,X) < \frac{(R+1)^2+(R+1-a)^2}{[3+(1-a)^2]R^2}$, then X satisfy the (DL)-condition by Theorem 3.5, so T has a fixed point by Theorem 2.2. $\square$

COROLLARY 3.7. *Let X be a Banach space such that $C_{\mathrm{NJ}}(a,X) < \frac{(R+1)^2+(R+1-a)^2}{[3+(1-a)^2]R^2}$, then X has normal structure.*

Proof. In fact, from the Theorem 2.1, it is easy to prove that X has weak normal structure. Since $1 \leqslant R(1,X) \leqslant 2$, we have $C_{\mathrm{NJ}}(a,X) < \frac{(R+1)^2+(R+1-a)^2}{[3+(1-a)^2]R^2} < 2$ for some $a \geqslant 0$. This implies that X is uniformly nonsquare, then X is reflexive, therefore weak normal structure coincide with normal structure.

Repeating the arguments in the proof of Theorem 3.5 and the definition of $C_{\mathrm{Z}}(a,X)$, we can easily get the following conclusion. □

COROLLARY 3.8. *Let C be a nonempty bounded closed convex subset of a Banach space X such that $C_{\mathrm{Z}}(a,X) < \frac{2(R+1)|R+1-a|}{[3+(1-a)^2]R^2}$ and $T: C \to KC(C)$ be a multivalued nonexpansive mapping, then T has a fixed point.*

COROLLARY 3.9. *Let X be a Banach space such that $C_{\mathrm{Z}}(a,X) < \frac{2(R+1)|R+1-a|}{[3+(1-a)^2]R^2}$, then X has normal structure.*

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