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A MULTIPLE OPIAL TYPE INEQUALITY FOR THE RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. The aim of this paper is to prove a multiple Opial type inequality for RL fractional derivatives which is proved for two factors and ordinary derivatives by Fink in [6]. Two methods are applied and a comparison of the obtained estimations is also given.

1. Introduction

In 1960. Z. Opial [8] proved the following inequality:
Let $f \in C^1[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$. Then

$$\int_0^h \left|f(x)f'(x)\right| dx \leqslant \frac{h}{4}\int_0^h \left[f'(x)\right]^2 dx, \tag{1.1}$$

where $h/4$ is the best possible.

One of many it's generalization over the last 50 years is the next inequality due to A. M. Fink [6]

$$\int_0^h \left|f^{(i)}(x)f^{(j)}(x)\right| dx \leqslant C(n,i,j,p)\, h^{2n-i-j+1-\frac{2}{p}} \left(\int_0^h \left|f^{(n)}(x)\right|^p dx\right)^{\frac{2}{p}}, \tag{1.2}$$

where $0 \leqslant i \leqslant j \leqslant n-1$, $f \in AC^n[0,h]$, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)} \in L_p[0,h]$ where $p \geqslant 1$. The constant $C(n,i,j,p)$ was explicitly computed and it is the best possible for $j = i+1$.

R. P. Agarwal and P. Y. H. Pang noticed in [9] that (1.2) does not hold for $j = i$ and $C(n,i,i,2)$. In the same paper they gave a weighted multiple version of (1.2) for ordinary derivatives using different method and without the best possible cases.

The main goal of this paper is to give a multiple version of (1.2) for the Riemann-Liouville fractional derivatives using Fink's method.

First we survey some facts about the Riemann-Liouville fractional derivatives needed in this paper. For more details see monographs [7, Chapter 2] and [10, Chapter 1].

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Let $x > 0$. By $C^m[0,x]$ we denote the space of all functions on $[0,x]$ which have continuous derivatives up to order m, and $AC[0,x]$ is the space of all absolutely continuous functions on $[0,x]$. By $AC^m[0,x]$ we denote the space of all functions $g \in C^{m-1}[0,x]$ with $g^{(m-1)} \in AC[0,x]$.

By $L_p[0,x]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[0,x]$, and by $L_\infty[0,x]$ the set of all functions measurable and essentially bounded on $[0,x]$. Clearly, $L_\infty[0,x] \subset L_p[0,x]$ for all $p \geqslant 1$.

Let $\nu > 0$ and $n = [\nu] + 1$ where $[\nu]$ is the integral part of ν. For $f \in L_1[0,x]$ the *Riemann-Liouville fractional integral of* f of order ν is defined by

$$J^\nu f(s) = \frac{1}{\Gamma(\nu)} \int_0^s (s-t)^{\nu-1} f(t)\,dt\,, \quad s \in [0,x],$$

where Γ is the gamma function, and for $f : [0,x] \to \mathbb{R}$ the *Riemann-Liouville fractional derivative of* f of order ν by

$$D^\nu f(s) = \frac{d^n J^{n-\nu} f}{ds^n}(s) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{ds^n} \int_0^s (s-t)^{n-\nu-1} f(t)\,dt\,.$$

In addition, we stipulate

$$D^0 f := f =: J^0 f, \quad J^{-\nu} f := D^\nu f \text{ if } \nu > 0.$$

If $\nu \in \mathbb{N}$ then $D^\nu f = \frac{d^\nu f}{ds^\nu}$ is the ordinary ν-order derivative.

The space $J^\nu(L_1[0,x])$ is defined as the set of all functions f on $[0,x]$ of the form $f = J^\nu \varphi$ for some $\varphi \in L_1[0,x]$, [10, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [10, p. 43], the latter characterization is equivalent to the condition

$$J^{n-\nu} f \in AC^n[0,x], \tag{1.3}$$

$$\frac{d^j J^{n-\nu} f}{ds^j}(0) = 0, \quad j = 0, 1, \ldots, n-1.$$

A function $f \in L_1[0,x]$ satisfying (1.3) is sad to have an *integrable fractional derivative* $D^\nu f$, [10, Chapter 1, Definition 2.4].

LEMMA 1. [2, Lemma 1.2] *Let* $\nu > 0$ *and* $n = [\nu] + 1$. *A function* $f \in L_1[0,x]$ *has an integrable fractional derivative* $D^\nu f$ *if and only if*

$$D^{\nu-k} f \in C[0,x], k = 1, \ldots, n, \quad \textit{and} \quad D^{\nu-1} f \in AC[0,x].$$

Furthermore, $f \in J^\nu(L_1[0,x])$ *if and only if* f *has an integrable fractional derivative* $D^\nu f$ *and satisfies the conditions*

$$D^{\nu-k} f(0) = 0 \quad \textit{for } k = 1, \ldots, n.$$

LEMMA 2. [10, Chapter 1, Theorem 2.5] *The law of indices*

$$J^u J^\nu f = J^{u+\nu} f$$

is valid in the following cases

(i) $v>0,\ u+v>0$, *and* $f\in L_1[0,x]$.

(ii) $v<0,\ u>0$, *and* $f\in J^{-v}(L_1[0,x])$.

(iii) $u<0,\ u+v<0$, *and* $f\in J^{-u-v}(L_1[0,x])$.

The following theorem is a simple consequence of Lemma 2 (ii) and gives the basic identity in deducing Opial type inequalities.

THEOREM 1. [2, Theorem 1.4] *Let* $\nu>\gamma\geqslant 0$. *Assume* $f\in L_1[0,x]$ *has an integrable fractional derivative* $D^\nu f\in L_\infty[0,x]$ *such that* $D^{\nu-k}f(0)=0$ *for* $k=1,\dots,[\nu]+1$. *Then*

$$D^\gamma f(s)=\frac{1}{\Gamma(\nu-\gamma)}\int_0^s (s-t)^{\nu-\gamma-1}D^\nu f(t)\,dt\,,\quad s\in[0,x]\,. \tag{1.4}$$

In Section 2 we give two versions of our multiple Opial type inequality involving the Riemann-Liouville fractional derivatives. The proof of the first version is based on the Fink's idea from [6] and the second on the method presented in [9]. Comparison of the obtained estimations is also given. In Section 3 we present another approach to identity (1.4).

2. Opial type inequalities

Our main result is the following theorem which is a multiple generalization of the main theorem from [6].

THEOREM 2. *Let* $\nu>\mu_1\geqslant\mu_i+1,\ \mu_i\geqslant 0,\ i=2,\dots,m,\ m\in\mathbb{N},\ m\geqslant 2$. *Suppose that* $f:[0,x]\to\mathbb{R}$ *is such that identity* (1.4) *holds for all pairs* $\{\nu,\mu_i\}$, $i=1,\dots,m$. *If* $p,q>1$ *are such that* $1/p+1/q=1$ *and* $D^\nu f\in L_p[0,x]$, *then*

$$\int_0^x\prod_{i=1}^m|D^{\mu_i}f(\tau)|\,d\tau\leqslant T_1(x)\left(\int_0^x|D^\nu f(t)|^p\,dt\right)^{\frac{m}{p}} \tag{2.1}$$

where $T_1(x)$ *is given by*

$$T_1(x)=\frac{x^{\sum\limits_{i=1}^m(\nu-\mu_i)+1-\frac{m}{p}}}{m^{\frac{1}{p}}\,q^{\frac{m}{q}}\,(\nu-\mu_1)\left(\nu-\mu_1+\frac{1}{q}\right)^{\frac{m-1}{q}}\prod\limits_{i=1}^m\Gamma(\nu-\mu_i)\left[\sum\limits_{i=1}^m(\nu-\mu_i)+1-\frac{m}{p}\right]^{\frac{1}{q}}}.$$

Inequality (2.1) *is sharp for* $\mu_1=\mu_2+1=\cdots=\mu_m+1$, *and equality in this case is achieved for a function* f *such that* $D^\nu f(t)=C(x-t)^{\frac{q}{p}(\nu-\mu_1)}$.

Proof. Write $\alpha_i = \nu - \mu_i - 1$, $i = 1, \dots, m$. Using identity (1.4), the triangle inequality and the Fubini's theorem, we have

$$\begin{aligned}
&\int_0^x \prod_{i=1}^m |D^{\mu_i} f(\tau)|\,d\tau \\
&\leqslant \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)} \int_0^x \left(\prod_{i=1}^m \int_0^x |D^\nu f(t_i)|(\tau - t_i)_+^{\alpha_i}\,dt_i \right) d\tau \\
&= \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)} \int_{[0,x]^m} \prod_{i=1}^m |D^\nu f(t_i)| \left(\int_0^x \prod_{i=1}^m (\tau - t_i)_+^{\alpha_i}\,d\tau \right) dt_1 \cdots dt_m \\
&= \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)} \int_{[0,x]^m} \prod_{i=1}^m |D^\nu f(t_i)| \left(\int_{\max\{t_1,\dots,t_m\}}^x \prod_{i=1}^m (\tau - t_i)^{\alpha_i}\,d\tau \right) dt_1 \cdots dt_m \\
&= \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)} \int_{\Delta_m} \prod_{i=1}^m |D^\nu f(t_i)| \left(\int_{t_m}^x \sum_{\pi\in S_m} \prod_{i=1}^m (\tau - t_i)^{\alpha_{\pi(i)}}\,d\tau \right) dt_m \cdots dt_1,
\end{aligned} \tag{2.2}$$

where $(\tau - t_i)_+ = \frac{\tau - t_i + |\tau - t_i|}{2}$, $\Delta_m = \{(t_1,\dots,t_m) : 0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant x\}$ and S_m is the group of all permutations of the set $\{1,2,\dots,m\}$. The last equality in (2.2) follows by dividing the cube $[0,x]^m$ into parts where $0 \leqslant t_{\pi(1)} \leqslant t_{\pi(2)} \leqslant \cdots \leqslant t_{\pi(m)} \leqslant x$ for some $\pi \in S_m$ and symmetry of the involved expressions. Suppose that $\pi \in S_m$ is given and suppose that $j \in \{1,\dots,m\}$ is such that $\pi(j) = 1$. We have

$$\begin{aligned}
&\prod_{i=1}^m (\tau - t_i)^{\alpha_{\pi(i)}} \\
&= \prod_{i\neq j} (\tau - t_i)^{\alpha_{\pi(i)} - \alpha_1 - 1} (\tau - t_j)^{\alpha_1} \prod_{i\neq j} (\tau - t_i)^{\alpha_1+1} \\
&\leqslant (\tau - t_1)^{\sum_{i=2}^m \alpha_i - (m-1)(\alpha_1+1)} (\tau - t_j)^{\alpha_1} \prod_{i\neq j} (\tau - t_i)^{\alpha_1+1}.
\end{aligned} \tag{2.3}$$

Using (2.3) we obtain estimation

$$\begin{aligned}
&\int_{t_m}^x \sum_{\pi\in S_m} \prod_{i=1}^m (\tau - t_i)^{\alpha_{\pi(i)}}\,d\tau \\
&\leqslant \frac{(m-1)!}{\alpha_1+1} \int_{t_m}^x (\tau - t_1)^{\sum_{i=2}^m \alpha_i - (m-1)(\alpha_1+1)} \frac{d}{d\tau} \left(\prod_{i=1}^m (\tau - t_i)^{\alpha_1+1} \right) d\tau \\
&\leqslant \frac{(m-1)!}{\alpha_1+1} (x - t_1)^{\sum_{i=2}^m \alpha_i - (m-2)(\alpha_1+1)} \prod_{i=2}^m (x - t_i)^{\alpha_1+1},
\end{aligned} \tag{2.4}$$

where the last estimation in (2.4) easily follows by using integration by parts.

Set

$$A = \frac{(m-1)!}{(\alpha_1+1)\prod_{i=1}^{m}\Gamma(\alpha_i+1)}.$$

It follows

$$\begin{aligned}
&\int_0^x \prod_{i=1}^{m} |D^{\mu_i} f(\tau)|\, d\tau \\
&\leqslant \frac{1}{\prod_{i=1}^{m}\Gamma(\alpha_i+1)} \int_{\Delta_m} \prod_{i=1}^{m} |D^{\nu} f(t_i)| \left(\int_{t_m}^{x} \sum_{\pi\in S_m} \prod_{i=1}^{m} (\tau - t_i)^{\alpha_{\pi(i)}}\, d\tau \right) dt_m \cdots dt_1 \\
&\leqslant A \int_{\Delta_m} \prod_{i=1}^{m} |D^{\nu} f(t_i)| (x-t_1)^{\sum_{i=2}^{m}\alpha_i - (m-2)(\alpha_1+1)} \prod_{i=2}^{m} (x-t_i)^{\alpha_1+1} dt_m \cdots dt_1 \\
&= A \int_0^x (x-t_1)^{\sum_{i=2}^{m}\alpha_i - (m-2)(\alpha_1+1)} |D^{\nu} f(t_1)| dt_1 \int_{t_1}^{x} (x-t_2)^{\alpha_1+1} |D^{\nu} f(t_2)| dt_2 \\
&\qquad \cdots \int_{t_{m-1}}^{x} (x-t_m)^{\alpha_1+1} |D^{\nu} f(t_m)| dt_m . \qquad (2.5)
\end{aligned}$$

Applying Hölder's inequality on the last integral in (2.5) we get

$$\begin{aligned}
&\int_{t_{m-1}}^{x} (x-t_m)^{\alpha_1+1} |D^{\nu} f(t_m)|\, dt_m \\
&\leqslant \left(\int_{t_{m-1}}^{x} (x-t_m)^{q(\alpha_1+1)}\, dt_m \right)^{\frac{1}{q}} \left(\int_{t_{m-1}}^{x} |D^{\nu} f(t_m)|^p\, dt_m \right)^{\frac{1}{p}} \\
&= \frac{(x-t_{m-1})^{\alpha_1+1+\frac{1}{q}}}{[q(\alpha_1+1)+1]^{\frac{1}{q}}} \left(\int_{t_{m-1}}^{x} |D^{\nu} f(t_m)|^p\, dt_m \right)^{\frac{1}{p}} . \qquad (2.6)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{t_{m-2}}^{x} (x-t_{m-1})^{2(\alpha_1+1)+\frac{1}{q}} |D^{\nu} f(t_{m-1})| \left(\int_{t_{m-1}}^{x} |D^{\nu} f(t_m)|^p\, dt_m \right)^{\frac{1}{p}} dt_{m-1} \\
&\leqslant \left(\int_{t_{m-2}}^{x} (x-t_{m-1})^{2q(\alpha_1+1)+1}\, dt_{m-1} \right)^{\frac{1}{q}} \\
&\qquad \times \left(\int_{t_{m-2}}^{x} |D^{\nu} f(t_{m-1})|^p \int_{t_{m-1}}^{x} |D^{\nu} f(t_m)|^p dt_m\, dt_{m-1} \right)^{\frac{1}{p}} \\
&= \frac{(x-t_{m-2})^{2(\alpha_1+1)+\frac{2}{q}}}{[2q(\alpha_1+1)+2]^{\frac{1}{q}}} \frac{1}{2^{\frac{1}{p}}} \left(\int_{t_{m-2}}^{x} |D^{\nu} f(t_m)|^p dt_m \right)^{\frac{2}{p}} . \qquad (2.7)
\end{aligned}$$

Next step gives us

$$\begin{aligned}
&\int_{t_{m-3}}^{x}(x-t_{m-2})^{3(\alpha_1+1)+\frac{2}{q}}|D^{\nu}f(t_{m-2})|\left(\int_{t_{m-2}}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^{\frac{2}{p}}dt_{m-2}\\
&\leqslant\left(\int_{t_{m-3}}^{x}(x-t_{m-2})^{3q(\alpha_1+1)+2}dt_{m-2}\right)^{\frac{1}{q}}\\
&\qquad\times\left[\int_{t_{m-3}}^{x}|D^{\nu}f(t_{m-2})|^p\left(\int_{t_{m-2}}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^2 dt_{m-2}\right]^{\frac{1}{p}}\\
&=\frac{(x-t_{m-3})^{3(\alpha_1+1)+\frac{3}{q}}}{[3q(\alpha_1+1)+3]^{\frac{1}{q}}}\frac{1}{3^{\frac{1}{p}}}\left(\int_{t_{m-3}}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^{\frac{3}{p}}.
\end{aligned}\tag{2.8}$$

Finally, we have

$$\begin{aligned}
&\int_0^x\prod_{i=1}^{m}|D^{\mu_i}f(\tau)|\,d\tau\\
&\leqslant\frac{A}{(m-2)![q(\alpha_1+1)+1]^{\frac{m-2}{q}}}\int_0^x(x-t_1)^{\sum\limits_{i=2}^{m}\alpha_i-(m-2)(\alpha_1+1)}|D^{\nu}f(t_1)|\,dt_1\\
&\qquad\times\frac{(x-t_1)^{(m-1)(\alpha_1+1)+\frac{m-1}{q}}}{[(m-1)q(\alpha_1+1)+m-1]^{\frac{1}{q}}(m-1)^{\frac{1}{p}}}\left(\int_{t_1}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^{\frac{m-1}{p}}dt_1\\
&=\frac{A}{(m-1)![q(\alpha_1+1)+1]^{\frac{m-1}{q}}}\int_0^x(x-t_1)^{\sum\limits_{i=1}^{m}\alpha_i+1+\frac{m-1}{q}}|D^{\nu}f(t_1)|\,dt_1\\
&\qquad\times\left(\int_{t_1}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^{\frac{m-1}{p}}dt_1\\
&\leqslant\frac{1}{\prod\limits_{i=1}^{m}\Gamma(\alpha_i+1)\,(\alpha_1+1)\,[q(\alpha_1+1)+1]^{\frac{m-1}{q}}}\left(\int_0^x(x-t_1)^{q\sum\limits_{i=1}^{m}\alpha_i+q+m-1}dt_1\right)^{\frac{1}{q}}\\
&\qquad\times\left[\int_0^x|D^{\nu}f(t_1)|^p\left(\int_{t_1}^{x}|D^{\nu}f(t_m)|^p dt_m\right)^{m-1}dt_1\right]^{\frac{1}{p}}\\
&=\frac{1}{\prod\limits_{i=1}^{m}\Gamma(\alpha_i+1)\,(\alpha_1+1)\,[q(\alpha_1+1)+1]^{\frac{m-1}{q}}}\cdot\frac{x^{\sum\limits_{i=1}^{m}\alpha_i+1+\frac{m}{q}}}{[q(\sum\limits_{i=1}^{m}\alpha_i+1)+m]^{\frac{1}{q}}}\\
&\qquad\times\frac{1}{m^{\frac{1}{p}}}\left(\int_0^x|D^{\nu}f(t_m)|^p dt_m\right)^{\frac{m}{p}}.
\end{aligned}$$

This proves inequality (2.1).

Consider the case $\mu_1 = \mu_2 + 1 = \cdots = \mu_m + 1$ and a function f such that $D^\nu f(t) = C(x-t)^{\frac{q}{p}(\nu-\mu_1)}$. Straightforward calculation shows that in this case inequality (2.1) is sharp. Namely, in this case the both sides of inequality (2.1) are equal to

$$\frac{C^m x^{qm(\nu-\mu_1)+m}}{m\left[q(\nu-\mu_1)+1\right]^m\left[\Gamma(\nu-\mu_1)\right]^m(\nu-\mu_1)^m}. \tag{2.9}$$

This completes the proof of the theorem. □

REMARK 1. Let $m=2$, $\nu = n \in \mathbb{N}$, $\mu_1 = j \in \mathbb{N}$, $\mu_2 = i \in \mathbb{N}$, $i < j \leqslant n-1$ and $x = h$. Then inequality (2.1) becomes Fink's inequality (1.2) on $[0,h]$ with $T_1(h) = C(n,i,j,p)\,h^{2n-i-j+1-\frac{2}{p}}$.

Next we consider a special case of [3, Theorem 2.1], which gives us the same Opial type inequality as the previous one, but with a different constant. We give a short proof for the reader's convenience.

THEOREM 3. *Let $p,\ q > 1$ are such that $1/p + 1/q = 1$. Let $\nu > \mu_i + \frac{1}{p}$, $\mu_i \geqslant 0$, $i = 1,2,\ldots,m$, $m \in \mathbb{N}$, $m \geqslant 2$. Suppose that $f:[0,x] \to \mathbb{R}$ is such that identity (1.4) holds for all pairs $\{\nu,\mu_i\}$, $i=1,\ldots,m$. If $D^\nu f \in L_p[0,x]$, then*

$$\int_0^x \prod_{i=1}^m |D^{\mu_i} f(\tau)|\,d\tau \leqslant T_2(x)\left(\int_0^x |D^\nu f(t)|^p\,dt\right)^{\frac{m}{p}}$$

where $T_2(x)$ is given by

$$T_2(x) = \frac{x^{\sum_{i=1}^m(\nu-\mu_i)+1-\frac{m}{p}}}{q^{\frac{m}{q}}\left[\sum_{i=1}^m(\nu-\mu_i)+1-\frac{m}{p}\right]\prod_{i=1}^m\Gamma(\nu-\mu_i)\left(\nu-\mu_i-\frac{1}{p}\right)^{\frac{1}{q}}}.$$

Proof. Write $\alpha_i = \nu - \mu_i - 1$, $i = 1,\ldots,m$. Using identity (1.4), the triangle inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_0^x \prod_{i=1}^m |D^{\mu_i} f(\tau)|\,d\tau \\
&\leqslant \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)}\int_0^x \prod_{i=1}^m\left(\int_0^\tau (\tau-t)^{\alpha_i}|D^\nu f(t)|\,dt\right)d\tau \\
&\leqslant \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i+1)}\int_0^x \prod_{i=1}^m\left[\left(\int_0^\tau (\tau-t)^{q\alpha_i}\,dt\right)^{\frac{1}{q}}\left(\int_0^\tau |D^\nu f(t)|^p\,dt\right)^{\frac{1}{p}}\right]d\tau
\end{aligned}$$

$$= \frac{1}{\prod_{i=1}^{m} \Gamma(\alpha_i+1)(q\alpha_i+1)^{\frac{1}{q}}} \int_0^x \tau^{\sum_{i=1}^{m}\alpha_i+\frac{m}{q}} \left(\int_0^\tau |D^\nu f(t)|^p\,dt\right)^{\frac{m}{p}} d\tau$$

$$\leqslant \frac{1}{\prod_{i=1}^{m} \Gamma(\alpha_i+1)(q\alpha_i+1)^{\frac{1}{q}}} \left(\int_0^x |D^\nu f(t)|^p\,dt\right)^{\frac{m}{p}} \frac{x^{\sum_{i=1}^{m}\alpha_i+1+\frac{m}{q}}}{\sum_{i=1}^{m}\alpha_i+1+\frac{m}{q}}. \qquad \square$$

REMARK 2. Although the constant T_1 gives the best possible estimation in the case $\mu_1 = \mu_i + 1$, $i = 2, \ldots, m$ (see Theorem 2), it seems the constant T_2 from Theorem 3 gives the more uniform estimation which is partially justified by the discussion below.

Notice

$$\frac{T_1(x)}{T_2(x)} = \frac{\left[\sum_{i=1}^{m}(\nu-\mu_i)+1-m+\frac{m}{q}\right]^{\frac{1}{p}} \prod_{i=1}^{m}\left(\nu-\mu_i-1+\frac{1}{q}\right)^{\frac{1}{q}}}{m^{\frac{1}{p}}(\nu-\mu_1)\left(\nu-\mu_1+\frac{1}{q}\right)^{\frac{m-1}{q}}}.$$

Set $\nu - \mu_1 = d$ and $\mu_1 - \mu_i = \delta_i \geqslant 1$ for $i = 2, \ldots, m$. Then $T_2(x) < T_1(x)$ is equivalent to

$$\frac{1}{\left(md+\sum_{i=2}^{m}\delta_i+1-m+\frac{m}{q}\right)^{1-\frac{1}{q}} \prod_{i=2}^{m}\left(d+\delta_i-1+\frac{1}{q}\right)^{\frac{1}{q}}} < \frac{\left(d-1+\frac{1}{q}\right)^{\frac{1}{q}}}{m^{1-\frac{1}{q}} d \left(d+\frac{1}{q}\right)^{\frac{m-1}{q}}}. \tag{2.10}$$

If δ_i are big enough, then the left side of (2.10) tends to zero, while the right side depends only of d. Therefore, in this case $T_2(x) < T_1(x)$.

Let $\delta_i = 1$, that is $\mu_1 = \mu_i + 1$, $i = 2, \ldots, m$ (see the discussion of sharpness in Theorem 2). Then the reverse inequality in (2.10) is equivalent to

$$\frac{1}{\left(md+\frac{m}{q}\right)^{1-\frac{1}{q}}\left(d+\frac{1}{q}\right)^{\frac{m-1}{q}}} > \frac{\left(d-1+\frac{1}{q}\right)^{\frac{1}{q}}}{m^{1-\frac{1}{q}} d \left(d+\frac{1}{q}\right)^{\frac{m-1}{q}}}.$$

that is

$$\frac{qd+1}{qd-q+1} > \left(1+\frac{1}{qd}\right)^q,$$

which is equivalent to inequality

$$\left(\frac{qd+1-q}{qd+1}\right)^{\frac{1}{q}} < \frac{qd}{1+qd}$$

and this is a simple consequence of the Bernoulli's inequality. This is in accordance with Theorem 2 and implies that T_2 is not the best possible estimation in this case.

Numerical calculations indicate that there is a very narrow area around the best possible case $\mu_1 = \mu_i + 1$, $i = 2, \ldots, m$, where T_1 gives better estimation than T_2.

3. On composition identity (law of indices) for the Riemann-Liouville fractional derivatives

Here we give another approach to composition identity given in Theorem 1.

THEOREM 4. *Let* $\nu > \gamma \geqslant 0$ *and let* $f \in AC^n[0,x]$ *be such that* $D^\nu f \in L_1[0,x]$ *and* $D^\gamma f \in L_1[0,x]$.

(i) *If* $\nu - \gamma \notin \mathbb{N}$ *and* f *is such that* $D^{\nu-k} f(0) = 0$ *for* $k = 1,\dots,[\nu]+1$ *and* $D^{\gamma-k} f(0) = 0$ *for* $k = 1,\dots,[\gamma]+1$, *then*

$$D^\gamma f(s) = \frac{1}{\Gamma(\nu-\gamma)} \int_0^s (s-t)^{\nu-\gamma-1} D^\nu f(t)\, dt, \quad s \in [0,x]. \tag{3.1}$$

(ii) *If* $\nu - \gamma = l \in \mathbb{N}$ *and* f *is such that* $D^{\nu-k} f(0) = 0$ *for* $k = 1,\dots,l$, *then* (3.1) *holds.*

Proof. Let $n = [\nu]+1$, $m = [\gamma]+1$.

(i) Define auxiliary function $h : [0,\infty) \to \mathbb{R}$ with

$$h(t) = \begin{cases} f(t), & t \in [0,x] \\ \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(t-x)^k, & t \geqslant x \end{cases}. \tag{3.2}$$

Obviously $h \in AC^n(0,\infty)$ and $D^{\nu-k} h(0) = 0$, $k = 1,\dots,n$ and $D^{\gamma-k} h(0) = 0$, $k = 1,\dots,m$. Also h has polynomial growth at ∞, so the Laplace transform of h exists. Notice that both sides of (3.1) are integrable functions. The identity (3.1) will follow if we prove that

$$\begin{aligned} &\frac{1}{\Gamma(m-\gamma)} \frac{d^m}{ds^m} \int_0^s (s-t)^{m-\gamma-1} h(t)\, dt \\ &= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_0^s (s-t)^{\nu-\gamma-1} \frac{d^n}{dt^n} \int_0^t (t-y)^{n-\nu-1} h(y)\, dy\, dt \end{aligned} \tag{3.3}$$

holds for $s \geqslant 0$. Using standard properties of the Laplace transform, for the right side of the equality (3.3) we have

$$\begin{aligned} &\mathscr{L}\left(\frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_0^s (s-t)^{\nu-\gamma-1} \frac{d^n}{dt^n} \int_0^t (t-y)^{n-\nu-1} h(y)\, dy\, dt \right)(p) \\ &= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \mathscr{L}\left(s^{\nu-\gamma-1}\right)(p)\, \mathscr{L}\left(\frac{d^n}{dt^n} \int_0^t (t-y)^{n-\nu-1} h(y)\, dy \right)(p) \end{aligned}$$

$$= \frac{1}{p^{\nu-\gamma}\Gamma(n-\nu)}\left[p^n \mathscr{L}\left(\int_0^t (t-y)^{n-\nu-1} h(y)\,dy\right)(p)\right.$$
$$\left. - \sum_{k=0}^{n-1} p^k \frac{d^{n-k-1}}{dt^{n-k-1}}\left(\int_0^t (t-y)^{n-\nu-1} h(y)\,dy\right)(0)\right]$$
$$= \frac{p^{n-\nu+\gamma}}{\Gamma(n-\nu)} \mathscr{L}\left(t^{n-\nu-1}\right)(p)\mathscr{L}(h)(p) - \frac{1}{p^{\nu-\gamma}}\sum_{k=1}^{n} p^{k-1}D^{\nu-k}h(0) \tag{3.4}$$
$$= p^{\gamma}\mathscr{L}(h)(p). \tag{3.5}$$

For the left side of (3.3) we have

$$\mathscr{L}\left(\frac{1}{\Gamma(m-\gamma)}\frac{d^m}{ds^m}\int_0^s (s-t)^{m-\gamma-1} h(t)\,dt\right)(p)$$
$$= \frac{1}{\Gamma(m-\gamma)}\left[p^m \mathscr{L}\left(\int_0^s (s-t)^{m-\gamma-1} h(t)\,dt\right)(p)\right.$$
$$\left. - \sum_{k=0}^{m-1} p^k \frac{d^{m-k-1}}{ds^{m-k-1}}\left(\int_0^s (s-t)^{m-\gamma-1} h(t)\,dt\right)(0)\right]$$
$$= \frac{1}{\Gamma(m-\gamma)} p^m \mathscr{L}\left(s^{m-\gamma-1}\right)(p)\mathscr{L}(h)(p) - \sum_{k=1}^{m} p^{k-1}D^{\gamma-k}h(0) \tag{3.6}$$
$$= p^{\gamma}\mathscr{L}(h)(p). \tag{3.7}$$

Using (3.5) and (3.7) it follows that both sides of (3.3) have the same Laplace transform so we conclude that equality holds in (3.3) for $s \geqslant 0$ (see [11]). This completes the proof of the (i).

(ii) Notice that from $\nu = \gamma + l$, $l \in \mathbb{N}$, it follows $n = m + l$. Using (3.6) and (3.4) it is enough to prove that

$$\sum_{k=1}^{m} p^{k-1}D^{\gamma-k}h(0) = \frac{1}{p^{\nu-\gamma}}\sum_{k=1}^{n} p^{k-1}D^{\nu-k}h(0),$$

which is equivalent to

$$\sum_{k=1}^{m} p^{k-1}D^{\gamma-k}h(0) = \sum_{k=1-l}^{m} p^{k-1}D^{\gamma-k}h(0),$$

which obviously holds by assumption $D^{\nu-k}f(0) = 0$ for $k = 1,\ldots,l$. □

There is no an easy way to check the boundary conditions given in Theorem 4. To give simpler conditions we use the identity

$$D^{\nu}f(t) = \sum_{i=0}^{n-1} \frac{f^{(i)}(0)\,t^{-\nu+i}}{\Gamma(-\nu+i+1)} + \frac{1}{\Gamma(n-\nu)}\int_0^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau)\,d\tau, \tag{3.8}$$

which holds for $f \in AC^n[0,x]$ (see for example [7]).

PROPOSITION 1. *Let* $\nu > 0$, $n = [\nu] + 1$ *and* $f \in AC^n[0,x]$.

(*i*) *If* $f(0) = f'(0) = \cdots = f^{(n-2)}(0) = 0$, *then* $D^{\nu-1}f(0) = D^{\nu-2}f(0) = \cdots = D^{\nu-n}f(0) = 0$.

(*ii*) *If* $\nu \notin \mathbb{N}$ *and* $D^{\nu-1}f$ *is bounded in a neighborhood of* $t = 0$, *then* $f(0) = f'(0) = \cdots = f^{(n-2)}(0) = 0$.

Proof. (i) Using (3.8) for $\nu \equiv \nu - k$ we have

$$D^{\nu-k}f(t) = \sum_{i=0}^{n-k-1} \frac{f^{(i)}(0)t^{-\nu+i+k}}{\Gamma(-\nu+i+k+1)} + \frac{1}{\Gamma(n-\nu)}\int_0^t (t-\tau)^{n-\nu-1} f^{(n-k)}(\tau)\,d\tau, \quad (3.9)$$

where $k = 1,\ldots,n-1$, and the implication obviously follows. Notice that $f^{(n-k)} \in C^{n-k}[0,x]$ for $k = 1,\ldots,n-1$ and $D^{\nu-n}f(0) = J^{n-\nu}f(0) = 0$ since $f \in C[0,x]$.

(ii) Using (3.9) for $k = 1$ we have

$$D^{\nu-1}f(t) = \sum_{i=0}^{n-2} \frac{f^{(i)}(0)t^{-\nu+i+1}}{\Gamma(-\nu+i+2)} + \frac{1}{\Gamma(n-\nu)}\int_0^t (t-\tau)^{n-\nu-1} f^{(n-1)}(\tau)\,d\tau. \quad (3.10)$$

For $0 < \nu < 1$ there is nothing to prove. Suppose $\nu > 1$. Multiplying (3.10) with $t^{\nu-1}$ and taking $\lim_{t\to 0}$ of the both sides of (3.10) it follows $f(0) = 0$. For $1 < \nu < 2$ the proof is complete. For $\nu > 2$ we proceed by induction analogously. □

In the following corollary we summarize conditions for identity (3.1).

COROLLARY 1. *Let* $\nu > \gamma \geqslant 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$. *Identity* (3.1) *is valid if one of the following conditions holds:*

(*i*) $f \in J^{\nu}(L_1[0,x])$.

(*ii*) $J^{n-\nu}f \in AC^n[0,x]$ *and* $D^{\nu-k}f(0) = 0$ *for* $k = 1,\ldots n$.

(*iii*) $D^{\nu-k}f \in C[0,x]$ *for* $k = 1,\ldots,n$, $D^{\nu-1}f \in AC[0,x]$ *and* $D^{\nu-k}f(0) = 0$ *for* $k = 1,\ldots n$.

(*iv*) $f \in AC^n[0,x]$, $D^{\nu}f \in L_1[0,x]$, $D^{\gamma}f \in L_1[0,x]$, $\nu - \gamma \notin \mathbb{N}$, $D^{\nu-k}f(0) = 0$ *for* $k = 1,\ldots,n$ *and* $D^{\gamma-k}f(0) = 0$ *for* $k = 1,\ldots,m$.

(*v*) $f \in AC^n[0,x]$, $D^{\nu}f \in L_1[0,x]$, $D^{\gamma}f \in L_1[0,x]$, $\nu - \gamma = l \in \mathbb{N}$, $D^{\nu-k}f(0) = 0$ *for* $k = 1,\ldots,l$.

(*vi*) $f \in AC^n[0,x]$, $D^{\nu}f \in L_1[0,x]$, $D^{\gamma}f \in L_1[0,x]$ *and* $f(0) = f'(0) = \cdots = f^{(n-2)}(0) = 0$.

(*vii*) $f \in AC^n[0,x]$, $D^{\nu}f \in L_1[0,x]$, $D^{\gamma}f \in L_1[0,x]$, $\nu \notin \mathbb{N}$ *and* $D^{\nu-1}f$ *is bounded in a neighborhood of* $t = 0$.

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