Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 151–159 doi:10.7153/jmi-07-14

A UNITARY APPROACH TO SOME CLASSICAL INEQUALITIES

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we will give a unitary approach to some classical inequalities. We will show that these results could be proved in the same manner.

1. Introduction

It has been said that the relation which truly governs mathematics is that of inequality, equality being a special case. In this context, the study of inequalities is a natural interest of the mathematicians. We find more references on this subject, from famous books like [6] or [8] to new books like [15] or [5]. Over the years, some results have remained popular because of the authors. In 2003, Bullen even made a dictionary of inequalities ([3]), where we can find a large number of known results.

Among these inequalities, there are some very special ones, such as the *AM-GM inequality*, *Cauchy's inequality*, *Hölder's inequality* or *Minkowski's inequality*. They have a fundamental role in many branches of mathematics and are also called *classical inequalities*. In this article we want to present another point of view on these inequalities. We will see that all these inequalities have three common elements: *homogeneity*, *convexity* and *subadditivity*.

2. Preliminary results

In this paragraph, we recall some useful definitions and results. Let $f:(0,\infty)^n \to \mathbb{R}$ be a function.

DEFINITION 2.1. A function f is called *positively homogeneous* if

$$f(tx_1,tx_2,\ldots,tx_n) = tf(x_1,x_2,\ldots,x_n),$$

for all $t\in(0,\infty)$ and $x_1,x_2,\ldots,x_n\in(0,\infty)$.

EXAMPLE. The function

$$f:(0,\infty)^n \to \mathbb{R}, \qquad f(x_1,x_2,\ldots,x_n) = \sqrt[n]{x_1x_2\ldots x_n}$$

Mathematics subject classification (2010): 26D20, 26A51, 26B25.
Keywords and phrases: classical inequalities, convexity, superadditivity.

is a positively homogeneous function.

DEFINITION 2.2. A function f is called a *subadditive function*, respectively a *superadditive function* if

$$f(x_1+y_1,x_2+y_2,...,x_n+y_n) \leqslant f(x_1,x_2,...,x_n)+f(y_1,y_2,...,y_n),$$

respectively

$$f(x_1+y_1,x_2+y_2,...,x_n+y_n) \geqslant f(x_1,x_2,...,x_n)+f(y_1,y_2,...,y_n),$$

for all $x_1,x_2,...,x_n,y_1,y_2,...,y_n \in (0,\infty)$.

EXAMPLE. The function

$$f:(0,\infty)^n \to \mathbb{R}, \qquad f(x_1,x_2,...,x_n) = |x_1|+|x_2|+...+|x_n|$$

is a subadditive function.

Using induction, it easy to prove the following results.

LEMMA 2.3. *Let $f:(0,\infty)^n \to \mathbb{R}$ be a subadditive function. Then*

$$f(x_1+x_2+...+x_k) \leqslant f(x_1)+f(x_2)+...+f(x_k),$$

for all $x_1,x_2,...,x_k \in (0,\infty)^n$.

LEMMA 2.4. *Let $f:(0,\infty)^n \to \mathbb{R}$ be a superadditive function. Then*

$$f(x_1+x_2+...+x_k) \geqslant f(x_1)+f(x_2)+...+f(x_k),$$

for all $x_1,x_2,...,x_k \in (0,\infty)^n$.

3. A very powerful result

The main result of this paper is presented in this paragraph. We will find that convexity is equivalent to subadditivity under the condition of positive homogeneity. It must be specified that this results is based on the Theorem 1.4.6. from [14].

First, we introduce some notations. For a function $f:(0,\infty)^n \to \mathbb{R}$, we define n other functions as follows:

$$g_1,g_2,...,g_n:(0,\infty)^{n-1} \to \mathbb{R},$$

$$g_1(y_1,y_2,...,y_{n-1}) = f(1,y_1,y_2,...,y_{n-1}),$$
$$g_2(y_1,y_2,...,y_{n-1}) = f(y_1,1,y_2,...,y_{n-1}),$$
$$\ldots$$
$$g_n(y_1,y_2,...,y_{n-1}) = f(y_1,y_2,...,y_{n-1},1).$$

In this context, we can prove the theorem below.

THEOREM 3.1. *Let $f:(0,\infty)^n \to \mathbb{R}$ be a positively homogeneous functions. Then f is subadditive if and only if there exists $k \in \{1,2,\dots,n\}$ such that the function g_k is a convex function.*

Proof. First, let f be a subadditive function. Let $a_1,a_2,\dots,a_{n-1} \in (0,\infty)$, $b_1,b_2,\dots,b_{n-1} \in (0,\infty)$ and $t \in (0,1)$. We will prove that g_1 is a convex function. Then

$$\begin{aligned}
&g_1\left(ta_1+(1-t)b_1, ta_2+(1-t)b_2, \dots, ta_{n-1}+(1-t)b_{n-1}\right)\\
&= f\left(1, ta_1+(1-t)b_1, ta_2+(1-t)b_2, \dots, ta_{n-1}+(1-t)b_{n-1}\right)\\
&= f\left(t+(1-t), ta_1+(1-t)b_1, ta_2+(1-t)b_2, \dots, ta_{n-1}+(1-t)b_{n-1}\right)\\
&\leqslant f\left(t, ta_1, ta_2, \dots, ta_{n-1}\right) + f\left((1-t), (1-t)b_1, (1-t)b_2, \dots, (1-t)b_{n-1}\right)\\
&= tf\left(1, a_1, a_2, \dots, a_{n-1}\right) + (1-t)f\left(1, b_1, b_2, \dots, b_{n-1}\right)\\
&= tg_1\left(a_1, a_2, \dots, a_{n-1}\right) + (1-t)g_1\left(b_1, b_2, \dots, b_{n-1}\right),
\end{aligned}$$

so g_1 is a convex function.

Conversely, without loss of generality, suppose that g_1 is a convex function. For any $x_1,x_2,\dots,x_n \in (0,\infty)$ and $y_1,y_2,\dots,y_n \in (0,\infty)$, we have

$$\begin{aligned}
&f\left(x_1+y_1, x_2+y_2, \dots, x_n+y_n\right)\\
&= (x_1+y_1) f\left(1, \frac{x_2+y_2}{x_1+y_1}, \dots, \frac{x_n+y_n}{x_1+y_1}\right)\\
&= (x_1+y_1) g_1\left(\frac{x_2+y_2}{x_1+y_1}, \dots, \frac{x_n+y_n}{x_1+y_1}\right)\\
&= (x_1+y_1) g_1\left(\frac{x_1}{x_1+y_1}\cdot\frac{x_2}{x_1} + \frac{y_1}{x_1+y_1}\cdot\frac{y_2}{y_1}, \dots, \frac{x_1}{x_1+y_1}\cdot\frac{x_n}{x_1} + \frac{y_1}{x_1+y_1}\cdot\frac{y_n}{y_1}\right)\\
&\leqslant (x_1+y_1)\left(\frac{x_1}{x_1+y_1}\cdot g_1\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) + \frac{y_1}{x_1+y_1}\cdot g_1\left(\frac{y_2}{y_1}, \dots, \frac{y_n}{y_1}\right)\right)\\
&= x_1 g_1\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) + y_1 g_1\left(\frac{y_2}{y_1}, \dots, \frac{y_n}{y_1}\right)\\
&= x_1 f\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) + y_1 f\left(1, \frac{y_2}{y_1}, \dots, \frac{y_n}{y_1}\right)\\
&= f\left(x_1, x_2, \dots, x_n\right) + f\left(y_1, y_2, \dots, y_n\right),
\end{aligned}$$

so f is a subadditive function and the proof is finished. □

A similar result exists for superadditive functions.

THEOREM 3.2. *Let $f:(0,\infty)^n \to \mathbb{R}$ be a positively homogeneous functions. Then f is superadditive if and only if there exists $k \in \{1,2,\dots,n\}$ such that the function g_k is a concave function.*

Proof. We apply the previous theorem for the function $-f$. □

4. Proofs for classical inequalities

In the context of the results from the previous paragraphs, we can present a general method to prove inequalities. We find a positively homogeneous function, study a partial convexity/concavity, apply Theorem 3.1./3.2. and obtain the conclusion using subadditivity. If necessary, we extend the results with Lemma 2.3./2.4. We start with two easy examples.

EXAMPLE 4.1. *Radon's inequality.* (see e.g. [6], p. 61, ex. 65)
Let $x_1, x_2, \ldots, x_n \in (0,\infty)$ and $y_1, y_2, \ldots, y_n \in (0,\infty)$. For any $p > 0$, we have

$$\sum_{i=1}^{n} \frac{x_i^{p+1}}{y_i^p} \geqslant \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{p+1}}{\left(\sum\limits_{i=1}^{n} y_i\right)^{p}}.$$

Proof. Consider the function

$$f : (0,\infty)^2 \to \mathbb{R}, \qquad f(x,y) = \frac{x^{p+1}}{y^p}.$$

It is a positively homogeneous function. Then, the function $g : (0,\infty) \to \mathbb{R}$, $g(x) = f(x,1) = x^{p+1}$ is a convex function, because $g''(x) = p(p+1)x^{p-1} > 0$ for all $x > 0$. Now, applying Theorem 3.1. and Lemma 2.3, we obtain

$$\sum_{i=1}^{n} f(x_i, y_i) \geqslant f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right),$$

which is equivalent to the conclusion. $\square$

EXAMPLE 4.2. *Milne's inequality.* (see e.g. [6], p. 61, ex. 67)
For any $x_1, x_2, \ldots, x_n \in (0,\infty)$ and $y_1, y_2, \ldots, y_n \in (0,\infty)$, we have

$$\sum_{i=1}^{n} (x_i + y_i) \sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i} \leqslant \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i.$$

Proof. Consider the function

$$f : (0,\infty)^2 \to \mathbb{R}, \qquad f(x,y) = \frac{xy}{x+y}.$$

This is a positively homogeneous function. The function $g : (0,\infty) \to \mathbb{R}$, $g(x) = f(x,1) = \frac{x}{x+1}$ has $g''(x) = -\frac{2}{(x+1)^3} < 0$, hence g is a concave function. Applying Theorem 3.2. and Lemma 2.4. we have,

$$\sum_{i=1}^{n} f(x_i, y_i) \leqslant f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right),$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i} \leqslant \frac{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} (x_i + y_i)}$$

and the conclusion. □

The next examples contain proofs for the most famous inequalities from mathematics.

EXAMPLE 4.3. *Cauchy's inequality – strong form.*
Let $a_1, a_2, ..., a_n \in \mathbb{R}$ and $b_1, b_2, ..., b_n \in \mathbb{R}$. Then

$$(|a_1 b_1| + |a_2 b_2| + ... + |a_n b_n|)^2 \leqslant \left(a_1^2 + a_2^2 + ... + a_n^2\right)\left(b_1^2 + b_2^2 + ... + b_n^2\right).$$

Proof. Consider the positively homogeneous function

$$f : (0,\infty)^2 \to \mathbb{R}, \qquad f(x,y) = \sqrt{xy}.$$

Then the function $g : (0,\infty) \to \mathbb{R}$, $g(x) = f(x,1) = \sqrt{x}$ is concave. From Theorem 3.2. and Lemma 2.4. we obtain

$$\sqrt{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} \geqslant \sum_{i=1}^{n} \sqrt{x_i y_i},$$

which is equivalent to

$$\left(\sum_{i=1}^{n} \sqrt{x_i y_i}\right)^2 \leqslant \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i.$$

Replace x_i with a_i^2 and y_i with b_i^2 and the conclusion follows. □

EXAMPLE 4.4. *Hölder's inequality.* Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in (0,\infty)$ and $p, q > 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

Proof. We use the function

$$f : (0,\infty)^2 \to \mathbb{R}, \qquad f(x,y) = x^{\frac{1}{p}} y^{\frac{1}{q}}.$$

For any $t > 0$ we have

$$f(tx,ty) = (tx)^{\frac{1}{p}} (ty)^{\frac{1}{q}} = t^{\frac{1}{p}+\frac{1}{q}} x^{\frac{1}{p}} y^{\frac{1}{q}} = t f(x,y).$$

Let $g:(0,\infty)\to\mathbb{R},\ g(x)=f(x,1)=x^{\frac{1}{p}}$. We have

$$g''(x)=\frac{1}{p}\left(\frac{1}{p}-1\right)x^{\frac{1}{p}-2}=-\frac{1}{pq}x^{\frac{1}{p}-2},$$

so g'' is a negative function. Then g is concave and f superadditive. Applying Theorem 3.2. and Lemma 2.4. gives

$$\sum_{i=1}^{n}x_i^{\frac{1}{p}}y_i^{\frac{1}{q}}\leqslant\left(\sum_{i=1}^{n}x_i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}y_i\right)^{\frac{1}{q}}.$$

Replace x_i with a_i^p and y_i with b_i^q and we are done. □

EXAMPLE 4.5. *Minkowski's inequality.*
Let $a_1,a_2,..,a_n\in(0,\infty)$ and $b_1,b_2,..,b_n\in(0,\infty)$. Then

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}}\leqslant\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}},$$

for any real number $p>1$.

Proof. The function

$$g:(0,\infty)\to\mathbb{R},g(x)=(x^p+1)^{\frac{1}{p}}$$

is convex because $g''(x)=(p-1)x^{p-2}(x^p+1)^{\frac{1}{p}-2}>0$, for all $x\in(0,\infty)$. In this context, the function

$$f:(0,\infty)^2\to\mathbb{R},f(x,y)=(x^p+y^p)^{\frac{1}{p}}$$

is subadditive because $g(x)=f(x,1)$. Lemma 2.3. concludes the proof. □

EXAMPLE 4.6. *AM − GM inequality.* For any real numbers $x_1,x_2,...,x_n>0$, have

$$\frac{x_1+x_2+...+x_n}{n}\geqslant\sqrt[n]{x_1x_2...x_n}.$$

Proof. Consider the function

$$f:(0,\infty)^n\to\mathbb{R},f(x_1,x_2,...,x_n)=\frac{x_1+x_2+...+x_n}{n}-\sqrt[n]{x_1x_2...x_n}$$

and we have

$$f(tx_1,tx_2,...,tx_n)=\frac{tx_1+tx_2+...+tx_n}{n}-\sqrt[n]{tx_1tx_2...tx_n}=tf(x_1,x_2,...,x_n),$$

for any $t>0$. This shows us that f is positively homogeneous. Consider the function

$$g:(0,\infty)^{n-1}\to\mathbb{R},\qquad g(x_1,x_2,...,x_{n-1})=f(x_1,x_2,...,x_{n-1},1).$$

First, we prove the following short lemma.

LEMMA. *Let n be a natural number, $n \geqslant 2$. For any real numbers $a_1, a_2, ..., a_n$, the matrix $A_n = (a_{ij})_{i,j=\overline{1,n}}$ where*

$$a_{ij} = \begin{cases} na_i^2 & if\ i = j \\ -a_i a_j & if\ i \neq j \end{cases}$$

is positive-definte.

Proof. Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. We compute xAx^T and obtain

$$\begin{aligned} xAx^T &= n\sum_{i=1}^{n} a_i^2 x_i^2 - 2\sum_{1=i<j=n} a_i a_j x_i x_j \\ &= \sum_{i=1}^{n} a_i^2 x_i^2 + \sum_{1=i<j=n} (a_i x_i - a_j x_j)^2 \end{aligned}$$

which is equivalent to the conclusion. □

Now, we can study the function g. We have

$$\frac{\partial^2 g}{\partial x_i^2} = -\frac{1}{n}\left(\frac{1}{n} - 1\right) x_1^{\frac{1}{n}} ... x_{i-1}^{\frac{1}{n}} x_i^{\frac{1}{n}-2} x_{i+1}^{\frac{1}{n}} ... x_{n-1}^{\frac{1}{n}}$$

for $i = \overline{1,n}$ and

$$\frac{\partial^2 g}{\partial x_i \partial x_j} = -\frac{1}{n^2} x_1^{\frac{1}{n}} ... x_{i-1}^{\frac{1}{n}} x_i^{\frac{1}{n}-1} x_{i+1}^{\frac{1}{n}} ... x_{j-1}^{\frac{1}{n}} x_j^{\frac{1}{n}-1} x_{j+1}^{\frac{1}{n}} .. x_{n-1}^{\frac{1}{n}}$$

for $1 \leqslant i < j \leqslant n$. After some algebraic manipulations, we can write the hessian of g under the form

$$H_g = \frac{1}{n^2} x_1^{\frac{1}{n}} x_2^{\frac{1}{n}} ... x_{n-1}^{\frac{1}{n}} \begin{pmatrix} \frac{n-1}{x_1^2} & -\frac{1}{x_1 x_2} & .... & -\frac{1}{x_1 x_{n-1}} \\ -\frac{1}{x_2 x_1} & \frac{n-1}{x_2^2} & .... & -\frac{1}{x_2 x_{n-1}} \\ \dots & \dots & \dots & \dots \\ -\frac{1}{x_{n-1} x_1} & -\frac{1}{x_{n-1} x_2} & .... & \frac{n-1}{x_{n-1}^2} \end{pmatrix}.$$

But $x_1, x_2, ..., x_{n-1} > 0$, so H_g is a positive-definte matrix according to the previous lemma. Now, we can affirm that the function g is convex. Then, f is subadditive. So, we can write

$$\begin{aligned} & f(x_1, x_2, ..., x_n) + f(x_2, x_3 ..., x_n, x_1) + ... + f(x_n, x_1, ..., x_{n-1}) \\ \geqslant\ & f(x_1 + x_2 + ... + x_n, x_2 + ... + x_n + x_1, ..., x_n + x_1 + ... + x_{n-1}) \\ =\ & f(s, s, .., s) = 0, \end{aligned}$$

where $s = x_1 + x_2 + \ldots + x_n$. But

$$f(x_1, x_2, \ldots, x_n) = f(x_2, x_3 \ldots, x_n, x_1) = \ldots = f(x_n, x_1, \ldots, x_{n-1}),$$

hence $nf(x_1, x_2, \ldots, x_n) \geqslant 0$. We obtain $f(x_1, x_2, \ldots, x_n) \geqslant 0$ which is equivalent to the AM-GM inequality. $\square$

Finally, we briefly introduce other examples of inequalities that can be proven through the same method.

- Using the function

$$f : (0,\infty)^n \to \mathbb{R}, \qquad f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n},$$

we obtain the inequality

$$\sqrt[n]{(x_1 + y_1)(x_2 + y_2)\ldots(x_n + y_n)} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n} + \sqrt[n]{y_1 y_2 \ldots y_n},$$

for $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in (0,\infty)$. (see e.g. [2], p. 176, ex. 4.4.6)

- With the function

$$f : (0,\infty)^2 \to \mathbb{R}, \qquad f(x,y) = \frac{x^2 + y^2}{x + y},$$

we can prove the following inequality:

$$\sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i + y_i} \geqslant \frac{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i},$$

for any $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in (0,\infty)^n$.

- For any $p \in [1,\ 2]$, the function

$$f : (0,\infty)^n \to \mathbb{R}, \qquad f(x_1, x_2, \ldots, x_n) = \frac{x_1^p + x_2^p + \ldots + x_n^p}{x_1^{p-1} + x_2^{p-1} + \ldots + x_n^{p-1}}$$

is useful to prove the inequality

$$\frac{\sum_{i=1}^{n} (x_i + y_i)^p}{\sum_{i=1}^{n} (x_i + y_i)^{p-1}} \leqslant \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}} + \frac{\sum_{i=1}^{n} y_i^p}{\sum_{i=1}^{n} y_i^{p-1}},$$

which holds for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in (0,\infty)^n$. (see e.g. [1], p. 25, theorem 9.)

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(Received March 25, 2012)

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