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SHARP BOUNDS FOR TOADER MEAN IN TERMS OF CONTRAHARMONIC MEAN WITH APPLICATIONS

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Abstract. We find the greatest value λ and the least value μ in $(1/2,1)$ such that the double inequality $C(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a)<T(a,b)<C(\mu a+(1-\mu)b,\mu b+(1-\mu)a)$ holds for all $a,b>0$ with $a\neq b$, and give new bounds for the perimeter of an ellipse. Here, $T(a,b)=\frac{2}{\pi}\int\limits_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta$, and $C(a,b)=(a^2+b^2)/(a+b)$ denote the Toader, and contraharmonic means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a\neq b$, the Toader mean $T(a,b)$ was introduced by Toader [11] as follows:

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta \\ &= \begin{cases} 2a\mathscr{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a>b, \\ 2b\mathscr{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a<b, \end{cases} \end{aligned} \tag{1.1}$$

where $\mathscr{E}(r)=\int\limits_0^{\pi/2}(1-r^2\sin^2 t)^{1/2}dt$, $r\in[0,1)$ is the complete elliptic integrals of the second kind. In particular, the perimeter $L(a,b)$ of an ellipse with the semiaxes a and b can be written as $L(a,b)=2\pi T(a,b)$.

In the recent past, investigation of the inequalities between Toader and other means has attracted the attention of a considerable number of mathematicians [1–6, 8–13].

Let $M_p(a,b)=[(a^p+b^p)/2]^{1/p}$, $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$, $S(a,b)=(a-b)/[2\arctan((a-b)/(a+b))]$, and

$$C(a,b)=\frac{a^2+b^2}{a+b} \tag{1.2}$$

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be the p-th power, harmonic, geometric, arithmetic, Seiffert, and contraharmonic means of two distinct positive numbers a and b, respectively. Then it is well-known that

$$\begin{aligned}\min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < A(a,b)\\ = M_1(a,b) < S(a,b) < C(a,b) < \max\{a,b\}\end{aligned}$$

for all $a,b>0$ with $a\neq b$.

Vuorinen [12] conjectured that

$$M_{3/2}(a,b) < T(a,b)$$

for all $a,b>0$ with $a\neq b$. This conjecture was proved by Barnard, Pearce and Richards in [4].

In [2], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b)$$

for all $a,b>0$ with $a\neq b$.

Very recently, Chu et al. [10] proved that

$$T(a,b) < S(a,b) \tag{1.3}$$

for all $a,b>0$ with $a\neq b$.

For fixed $a,b>0$ with $a\neq b$ and $x\in[1/2,1]$, let

$$g(x) = C(xa+(1-x)b, xb+(1-x)a).$$

Then it is not difficult to verify that $g(x)$ is continuous and strictly increasing in $[1/2,1]$. Note that $g(1/2)=A(a,b)<T(a,b)$ and $g(1)=C(a,b)>T(a,b)$. Therefore, it is natural to ask what are the greatest value λ and the least value μ in $(1/2,1)$ such that the double inequality $C(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a) < T(a,b) < C(\mu a+(1-\mu)b, \mu b+(1-\mu)a)$ holds for all $a,b>0$ with $a\neq b$. The main purpose of this paper is to answer these questions. Our main result is the following Theorem 1.1.

THEOREM 1.1. *If $\lambda,\mu\in(1/2,1)$, then the double inequality*

$$C(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a) < T(a,b) < C(\mu a+(1-\mu)b, \mu b+(1-\mu)a) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leqslant 3/4$ and $\mu\geqslant 1/2+\sqrt{4\pi-\pi^2}/(2\pi)$.

In order to establish our main result we need several formulas (see [3, Appendix E, pp. 474–475]).

Let $r\in[0,1)$, $\mathscr{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}dt$ be the complete elliptic integrals of the first kind. Then

$$\mathscr{K}(0)=\pi/2,\quad \mathscr{K}(1^-)=+\infty,\quad \mathscr{E}(0)=\pi/2,\quad \mathscr{E}(1^-)=1,$$

$$\frac{d\mathscr{K}(r)}{dr}=\frac{\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)}{r(1-r^2)},\quad \frac{d\mathscr{E}(r)}{dr}=\frac{\mathscr{E}(r)-\mathscr{K}(r)}{r},$$

$$\frac{d[\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)]}{dr}=r\mathscr{K}(r),\quad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)}{1+r}.$$

Moreover, for each $c\in[1/4,\infty)$ the function $f(r)\equiv(1-r^2)^c\mathscr{K}(r)$ is decreasing from $[0,1)$ onto $(0,\pi/2]$ (see [3, Theorem 3.21(7)]).

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\alpha=3/4$ and $\beta=1/2+\sqrt{4\pi-\pi^2}/(2\pi)$. Then from the monotonicity of the function $g(x)=C(xa+(1-x)b,xb+(1-x)a)$ in $[1/2,1]$ we know that to prove inequality (1.4) we only need to prove that inequalities

$$T(a,b)>C(\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a) \tag{2.1}$$

and

$$T(a,b)<C(\beta a+(1-\beta)b,\beta b+(1-\beta)a) \tag{2.2}$$

hold for all $a,b>0$ with $a\neq b$.

Without loss of generality, we assume that $a>b$. Let $t=b/a\in(0,1)$, $r=(1-t)/(1+t)\in(0,1)$ and $p\in[1/2,1]$. Then from (1.1) and (1.2) one has

$$\begin{aligned}
&T(a,b)-C(pa+(1-p)b,pb+(1-p)a)\\
=&\frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-(b/a)^2}\right)-a\frac{[p+(1-p)(b/a)]^2+[p(b/a)+1-p]^2}{1+b/a}\\
=&\frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-t^2}\right)-a\frac{[p+(1-p)t]^2+[pt+1-p]^2}{1+t}\\
=&\frac{2a}{\pi}\frac{2\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)}{1+r}-a\frac{[1-(1-2p)r]^2+[1+(1-2p)r]^2}{2(1+r)}\\
=&\frac{a}{1+r}\left\{\frac{2}{\pi}\left[2\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)\right]-(1-2p)^2r^2-1\right\}.
\end{aligned} \tag{2.3}$$

Let

$$f(r)=\frac{2}{\pi}\left[2\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)\right]-(1-2p)^2r^2-1, \tag{2.4}$$

$f_1(r)=rf'(r)$, and $f_2(r)=f_1{}'(r)/r$. Then simple computations lead to

$$f(0)=0, \tag{2.5}$$

$$f(1^-)=\frac{4}{\pi}-1-(1-2p)^2, \tag{2.6}$$

$$f_1(r)=\frac{2}{\pi}\left[\mathscr{E}(r)-(1-r^2)\mathscr{K}(r)\right]-2(1-2p)^2r^2,$$

$$f_1(0)=0, \tag{2.7}$$

$$f_1(1^-) = \frac{2}{\pi} - 2(1-2p)^2, \tag{2.8}$$

$$f_2(r) = \frac{2}{\pi}\mathscr{K}(r) - 4(1-2p)^2, \tag{2.9}$$

$$f_2(0) = 1 - 4(1-2p)^2, \tag{2.10}$$

$$f_2(1^-) = +\infty. \tag{2.11}$$

We divide the proof into two cases.

Case 1. $p = \alpha = 3/4$. Then equation (2.10) reduces to

$$f_2(0) = 0. \tag{2.12}$$

From (2.12), (2.9), (2.7) and (2.5) we clearly see that $f(r) > 0$ for $r \in (0,1)$. Therefore, inequality (2.1) follows from (2.3) and (2.4) together with $f(r) > 0$.

Case 2. $p = \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$. Then equations (2.6), (2.8) and (2.10) lead to

$$f(1^-) = 0, \tag{2.13}$$

$$f_1(1^-) = (2\pi - 6)/\pi > 0, \tag{2.14}$$

$$f_2(0) = (5\pi - 16)/\pi < 0. \tag{2.15}$$

From (2.11) and (2.15) together with the monotonicity of $f_2(r)$ we clearly see that there exists $r_0 \in (0,1)$ such that $f_2(r) < 0$ for $r \in (0, r_0)$ and $f_2(r) > 0$ for $r \in (r_0, 1)$. Hence $f_1(r)$ is strictly decreasing in $(0, r_0)$ and strictly increasing in $(r_0, 1)$.

It follows from (2.7) and (2.14) together with the piecewise monotonicity of $f_1(r)$ that there exists $r_1 \in (0,1)$ such that $f_1(r) < 0$ for $r \in (0, r_1)$ and $f_1(r) > 0$ for $r \in (r_1, 1)$. Hence $f(r)$ is strictly decreasing in $(0, r_1)$ and strictly increasing in $(r_1, 1)$.

Therefore, inequality (2.2) follows from (2.3)–(2.5) and (2.13) together with the piecewise monotonicity of $f(r)$.

Next, we prove that the parameter $\alpha = 3/4$ is the best possible parameter in $(1/2, 1)$ such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $p > \alpha = 3/4$, then equation (2.10) leads to $f_2(0) < 0$. From the continuity of $f(r)$, $f_1(r)$ and $f_2(r)$ we know that there exists $\delta_1 = \delta_1(p) > 0$ such that

$$f(r) < 0 \tag{2.16}$$

for $r \in (0, \delta_1)$.

It follows from (2.3), (2.4) and (2.16) that $T(a,b) < C(pa + (1-p)b, pb + (1-p)a)$ for $b/a \in ((1-\delta_1)/(1+\delta_1), 1)$.

Finally, we prove that the parameter $\beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$ is the best possible parameter in $(1/2, 1)$ such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $1/2 < p < \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$, then equation (2.6) leads to $f(1^-) > 0$. Hence, there exists $\delta_2 = \delta_2(p) \in (0,1)$ such that

$$f(r) > 0 \tag{2.17}$$

for $r \in (1-\delta_2, 1)$.

Therefore, $T(a,b) > C(pa+(1-p)b, pb+(1-p)a)$ for $b/a \in (0, \delta_2/(2-\delta_2))$ follows from equations (2.3) and (2.4) together with inequality (2.17).

REMARK 2.1. Let $\beta = 1/2+\sqrt{4\pi-\pi^2}/(2\pi))$ and $a,b>0$ with $a \neq b$. Then from inequality $S(a,b) > C(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$ [7] and Theorem 1.1 we get

$$T(a,b) < C(\beta a+(1-\beta)b, \beta b+(1-\beta)a) < S(a,b),$$

which is a refinement of inequality (1.3).

The following Corollary 2.2 can be derived directly from Theorem 1.1.

COROLLARY 2.2. *The double inequality*

$$2\pi C(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a) < L(a,b) < 2\pi C(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$$

holds for $\alpha = 3/4, \beta = 1/2+\sqrt{4\pi-\pi^2}/(2\pi)$ *and* $a,b>0$ *with* $a \neq b$.

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