SHARP BOUNDS FOR TOADER MEAN IN TERMS OF
CONTRAHARMONIC MEAN WITH APPLICATIONS

YU-MING CHU, MIAO-KUN WANG AND XIAO-YAN MA

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Abstract. We find the greatest value \( \lambda \) and the least value \( \mu \) in \((1/2,1)\) such that the double inequality
\[
C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < T(a,b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)
\]
holds for all \(a,b > 0\) with \(a \neq b\), and give new bounds for the perimeter of an ellipse. Here,
\[
T(a,b) = \frac{2}{\pi} \int_{0}^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta,
\]
and \(C(a,b) = (a^2 + b^2)/(a + b)\) denote the Toader and contraharmonic means of two positive numbers \(a\) and \(b\), respectively.

1. Introduction

For \(a,b > 0\) with \(a \neq b\), the Toader mean \(T(a,b)\) was introduced by Toader [11] as follows:
\[
T(a,b) = \frac{2}{\pi} \int_{0}^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta
= \begin{cases} 
2a \mathcal{E}(\sqrt{1 - (b/a)^2})/\pi, & a > b, \\
2b \mathcal{E}(\sqrt{1 - (a/b)^2})/\pi, & a < b,
\end{cases}
\]
where \(\mathcal{E}(r) = \int_{0}^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} \, dt\), \(r \in [0,1)\) is the complete elliptic integrals of the second kind. In particular, the perimeter \(L(a,b)\) of an ellipse with the semiaxes \(a\) and \(b\) can be written as \(L(a,b) = 2\pi T(a,b)\).

In the recent past, investigation of the inequalities between Toader and other means has attracted the attention of a considerable number of mathematicians [1–6, 8–13].

Let \(M_p(a,b) = [(a^p + b^p)/2]^{1/p}\), \(H(a,b) = 2ab/(a+b)\), \(G(a,b) = \sqrt{ab}\), \(A(a,b) = (a+b)/2\), \(S(a,b) = (a - b)/[2 \arctan((a-b)/(a+b))]\), and
\[
C(a,b) = \frac{a^2 + b^2}{a+b}
\]

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be the $p$-th power, harmonic, geometric, arithmetic, Seiffert, and contraharmonic means of two distinct positive numbers $a$ and $b$, respectively. Then it is well-known that

$$
\min \{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < A(a, b) = M_1(a, b) < S(a, b) < C(a, b) < \max \{a, b\}
$$

for all $a, b > 0$ with $a \neq b$.

Vuorinen [12] conjectured that

$$
M_{3/2}(a, b) < T(a, b)
$$

for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Barnard, Pearce and Richards in [4].

In [2], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$
T(a, b) < M_{\log 2/\log (\pi/2)}(a, b)
$$

for all $a, b > 0$ with $a \neq b$.

Very recently, Chu et al. [10] proved that

$$
T(a, b) < S(a, b)
$$

(1.3)

for all $a, b > 0$ with $a \neq b$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [1/2, 1]$, let

$$
g(x) = C(xa + (1 - x)b, xb + (1 - x)a).
$$

Then it is not difficult to verify that $g(x)$ is continuous and strictly increasing in $[1/2, 1]$. Note that $g(1/2) = A(a, b) < T(a, b)$ and $g(1) = C(a, b) > T(a, b)$. Therefore, it is natural to ask what are the greatest value $\lambda$ and the least value $\mu$ in $(1/2, 1)$ such that the double inequality

$$
C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < T(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)
$$

(1.4)

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq 3/4$ and $\mu \geq 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$.

In order to establish our main result we need several formulas (see [3, Appendix E, pp. 474–475]).

Let $r \in [0, 1)$, $\Theta(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} dt$ be the complete elliptic integrals of the first kind. Then

$$
\Theta(0) = \pi/2, \quad \Theta(1^-) = +\infty, \quad \Theta(0) = \pi/2, \quad \Theta(1^-) = 1,
$$

THEOREM 1.1. If $\lambda, \mu \in (1/2, 1)$, then the double inequality

$$
C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < T(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)
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$$
\Theta(0) = \pi/2, \quad \Theta(1^-) = +\infty, \quad \Theta(0) = \pi/2, \quad \Theta(1^-) = 1,
$$
Moreover, for each $c \in [1/4, \infty)$ the function $f(r) \equiv (1 - r^2)^c \mathcal{K}(r)$ is decreasing from $[0, 1]$ onto $(0, \pi/2]$ (see [3, Theorem 3.21(7)]).

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\alpha = 3/4$ and $\beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$. Then from the monotonicity of the function $g(x) = C(xa + (1-x)b, xb + (1-x)a)$ in $[1/2, 1]$ we know that to prove inequality (1.4) we only need to prove that inequalities

$$T(a, b) > C(\alpha a + (1 - \alpha) b, \alpha b + (1 - \alpha) a)$$

and

$$T(a, b) < C(\beta a + (1 - \beta) b, \beta b + (1 - \beta) a)$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = b/a \in (0, 1)$, $r = (1-t)/(1+t) \in (0, 1)$ and $p \in [1/2, 1]$. Then from (1.1) and (1.2) one has

$$T(a, b) - C(pa + (1 - p)b, pb + (1 - p)a)$$

$$= \frac{2a}{\pi} \mathcal{E} \left( \sqrt{1 - (b/a)^2} \right) - a \frac{[p+(1-p)(b/a)]^2+[p(b/a)+1-p]^2}{1+b/a}$$

$$= \frac{2a}{\pi} \mathcal{E} \left( \sqrt{1 - r^2} \right) - a \frac{[p+(1-p)t]^2+[pt+1-p]^2}{1+t}$$

$$= \frac{2a}{\pi} \mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)$$

$$= \frac{2a}{\pi} \left[ \frac{2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)}{1 + r} - a \frac{[1 - (1 - 2p)r] + [1 + (1 - 2p)r]^2}{2(1 + r)} \right].$$

Let

$$f(r) = \frac{2}{\pi} \left[ 2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r) \right] - (1 - 2p)^2 r^2 - 1,$$

$f_1(r) = rf'(r)$, and $f_2(r) = f_1'(r)/r$. Then simple computations lead to

$$f(0) = 0,$$

$$f(1) = \frac{4}{\pi} - 1 - (1 - 2p)^2,$$

$$f_1(r) = \frac{2}{\pi} \left[ \mathcal{E}(r) - (1 - r^2) \mathcal{K}(r) \right] - 2(1 - 2p)^2 r^2,$$

$$f_1(0) = 0,$$
\[ f_1(1^-) = \frac{2}{\pi} - 2(1 - 2p)^2, \quad (2.8) \]
\[ f_2(r) = \frac{2}{\pi} \mathcal{K}(r) - 4(1 - 2p)^2, \quad (2.9) \]
\[ f_2(0) = 1 - 4(1 - 2p)^2, \quad (2.10) \]
\[ f_2(1^-) = +\infty. \quad (2.11) \]

We divide the proof into two cases.

**Case 1.** \( p = \alpha = 3/4 \). Then equation (2.10) reduces to
\[ f_2(0) = 0. \quad (2.12) \]

From (2.12), (2.9), (2.7) and (2.5) we clearly see that \( f(r) > 0 \) for \( r \in (0, 1) \). Therefore, inequality (2.1) follows from (2.3) and (2.4) together with \( f(r) > 0 \).

**Case 2.** \( p = \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi) \). Then equations (2.6), (2.8) and (2.10) lead to
\[ f(1^-) = 0, \quad (2.13) \]
\[ f_1(1^-) = (2\pi - 6)/\pi > 0, \quad (2.14) \]
\[ f_2(0) = (5\pi - 16)/\pi < 0. \quad (2.15) \]

From (2.11) and (2.15) together with the monotonicity of \( f_2(r) \) we clearly see that there exists \( r_0 \in (0, 1) \) such that \( f_2(r) < 0 \) for \( r \in (0, r_0) \) and \( f_2(r) > 0 \) for \( r \in (r_0, 1) \). Hence \( f_1(r) \) is strictly decreasing in \((0, r_0)\) and strictly increasing in \((r_0, 1)\).

It follows from (2.7) and (2.14) together with the piecewise monotonicity of \( f_1(r) \) that there exists \( r_1 \in (0, 1) \) such that \( f_1(r) < 0 \) for \( r \in (0, r_1) \) and \( f_1(r) > 0 \) for \( r \in (r_1, 1) \). Hence \( f(r) \) is strictly decreasing in \((0, r_1)\) and strictly increasing in \((r_1, 1)\).

Therefore, inequality (2.2) follows from (2.3)–(2.5) and (2.13) together with the piecewise monotonicity of \( f(r) \).

Next, we prove that the parameter \( \alpha = 3/4 \) is the best possible parameter in \((1/2, 1)\) such that inequality (2.1) holds for all \( a, b > 0 \) with \( a \neq b \). In fact, if \( p > \alpha = 3/4 \), then equation (2.10) leads to \( f_2(0) < 0 \). From the continuity of \( f(r), f_1(r) \) and \( f_2(r) \) we know that there exists \( \delta_1 = \delta_1(p) > 0 \) such that
\[ f(r) < 0 \quad (2.16) \]
for \( r \in (0, \delta_1) \).

It follows from (2.3), (2.4) and (2.16) that \( T(a, b) < C(pa + (1 - p)b, pb + (1 - p)a) \) for \( b/a \in \left((1 - \delta_1)/(1 + \delta_1), 1\right) \).

Finally, we prove that the parameter \( \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi) \) is the best possible parameter in \((1/2, 1)\) such that inequality (2.2) holds for all \( a, b > 0 \) with \( a \neq b \). In fact, if \( 1/2 < p < \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi) \), then equation (2.6) leads to \( f(1^-) > 0 \). Hence, there exists \( \delta_2 = \delta_2(p) \in (0, 1) \) such that
\[ f(r) > 0 \quad (2.17) \]
for \( r \in (1 - \delta_2, 1) \).

Therefore, \( T(a, b) > C(pa + (1 - p)b, pb + (1 - p)a) \) for \( b/a \in (0, \delta_2/(2 - \delta_2)) \) follows from equations (2.3) and (2.4) together with inequality (2.17).

**Remark 2.1.** Let \( \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi) \) and \( a, b > 0 \) with \( a \neq b \). Then from inequality \( S(a, b) > C(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \) [7] and Theorem 1.1 we get

\[
T(a, b) < C(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) < S(a, b),
\]

which is a refinement of inequality (1.3).

The following Corollary 2.2 can be derived directly from Theorem 1.1.

**Corollary 2.2.** The double inequality

\[
2\pi C(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < L(a, b) < 2\pi C(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)
\]

holds for \( \alpha = 3/4, \beta = 1/2 + \sqrt{4\pi - \pi^2}/(2\pi) \) and \( a, b > 0 \) with \( a \neq b \).

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**References**


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Yu-Ming Chu  
*College of Mathematics and Computation Science*  
*Hunan City University*  
Yiyang 413000, China  
*e-mail*: chuyuming@hutc.zj.cn

Miao-Kun Wang  
*Department of Mathematics*  
*Huzhou Teachers College*  
Huzhou 313000, China

Xiao-Yan Ma  
*Department of Mathematics*  
*Zhejiang Sci-Tech University*  
Hangzhou 310018, China