Journal of
Mathematical
Inequalities
Volume 7, Number 2 (2013), 167–174
doi:10.7153/jmi-07-16

ANOTHER PROOF OF SPIRA'S INEQUALITY AND ITS APPLICATION TO THE RIEMANN HYPOTHESIS

SADEGH NAZARDONYAVI AND SEMYON YAKUBOVICH

(*Communicated by B. Uhrin*)

Abstract. By using new inequalities involving powers of rational functions, we give another proof of an important Spira's relation for the Riemann zeta-function $|\zeta(1-s)| \leqslant |\zeta(s)|$ in the strip $0 < \Re s < 1/2$, $|\Im s| \geqslant 12$. Moreover, we establish a sufficient condition of the validity of the Riemann hypothesis in terms of the derivative of $|\zeta(s)|^2$ with respect to $\Re s$ and conjecture its necessity.

1. Introduction and main result

As it is known, the Riemann zeta-function is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad (\Re s > 1). \tag{1}$$

Moreover, it admits an analytic continuation over the whole complex plane, having as its only singularity a simple pole with residue 1 at $s = 1$ ([5], pp. 1-3). The Riemann hypothesis (RH), stated by Riemann in 1859, concerns the complex zeros of the Riemann zeta function. The RH states that the non-real zeros of the Riemann zeta function $\zeta(s)$ all lie on the critical line $\sigma = 1/2$ ([6]).

Let $\zeta(1-s) = g(s)\zeta(s)$. It is easily seen from the functional equation for the Riemann zeta-function ([10], p. 16), that $g(s)$ can be written in two equivalent forms

(1) $\quad g(s) = 2(2\pi)^{-s} \cos\dfrac{\pi s}{2}\Gamma(s),$

(2) $\quad g(s) = \pi^{\frac{1}{2}-s}\Gamma\left(\dfrac{s}{2}\right) \Big/ \Gamma\left(\dfrac{1-s}{2}\right).$

In 1965, Spira [9] proved that $|g(s)| < 1, s = \sigma + it$, for $0 < \sigma < 1/2$, $|t| \geqslant 10$ using its representation in the form (1). To do this he also employed Stirling's formula for $\log\Gamma(s)$. For the strong inequality $|\zeta(1-s)| < |\zeta(s)|$ in the same strip, he proved

Mathematics subject classification (2010): 11M26, 11M99, 26Dxx, 41A17, 33B15.
Keywords and phrases: Riemann zeta-function, size of the Riemann zeta-function, critical strip.
The work of the first author is supported by the Calouste Gulbenkian Foundation, under Ph.D. grant number CB/C02/2009/32. Research partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under the project PEst-C/MAT/UI0144/2011.

that it is equivalent to the Riemann hypothesis. In 1966, Dixon and Schoenfeld [3] established that $|g(s)| < 1$ holds for a wider domain $0 < \sigma < 1/2$, $|t| \geqslant 6.8$.

Our goal here is to exhibit another proof of Spira's inequality, taking representation (2) and avoiding the use of Stirling's formula. Our method relies on infinite product representations of the Euler gamma-function together with two auxiliary lemmas involving powers of rational functions. Indeed, we state the main result of this note by the following

THEOREM 1. *For* $0 < \sigma < \frac{1}{2}$, $|t| \geqslant 12$

$$|\zeta(1-s)| \leqslant |\zeta(s)| \tag{2}$$

where the equality takes place only if $\zeta(s) = 0$.

Finally, we will give a sufficient condition for the validity of the Riemann hypothesis in terms of the partial derivative with respect to σ of $|\zeta(s)|^2$, conjecturing its necessity as well.

2. Auxiliary lemmas

In order to prove Theorem 1, we will need some auxiliary elementary inequalities involving rational and logarithmic functions. Precisely, we have (see [7], §2)

$$\frac{1}{x+1} < \log\left(1+\frac{1}{x}\right) < \frac{1}{x}, \qquad (x < -1, \text{ or } x > 0), \tag{3}$$

$$\frac{1}{x+\frac{1}{2}} < \log\left(1+\frac{1}{x}\right) < \frac{1}{x}, \qquad x > 0, \tag{4}$$

$$\frac{2x}{2+x} < \log(1+x) < \frac{x(2+x)}{2(1+x)}, \qquad (x > 0), \tag{5}$$

$$\frac{x(2+x)}{2(1+x)} < \log(1+x) < \frac{2x}{2+x}, \qquad (-1 < x < 0). \tag{6}$$

Next we give some possibly new inequalities, whose proofs are based on elementary calculus and will be omitted.

LEMMA 1. *For any* $t \geqslant 1$

$$\left(1+\frac{1}{tx+t-1}\right)^t \leqslant 1+\frac{1}{x}, \qquad (x \leqslant -1,\ x > 0), \tag{7}$$

$$\left(1+\frac{x}{t}\right)^t \leqslant 1+\frac{2tx}{(1-t)x+2t}, \qquad (0 \leqslant x \leqslant 2). \tag{8}$$

Finally, for $0 \leqslant a \leqslant 1$

$$\left(1+\frac{1}{x}\right)^a \geqslant 1+\frac{a}{x+1-a}, \qquad (x \leqslant -1,\ x > 0), \tag{9}$$

where the equality holds only if $a=0,\ 1$ *or* $x=-1$*, and*

$$\left(1+\frac{1}{x}\right)^{a} \geqslant 1+\frac{a}{x+\frac{1-a}{2}}, \qquad (x>0), \tag{10}$$

$$\left(1+\frac{1}{x}\right)^{a} \leqslant 1+\frac{a}{x+\frac{1-a}{2}}, \qquad (x\leqslant -1), \tag{11}$$

where it becomes equality only if $a=0,\ 1$.

Now we will prove a key lemma, which is used in the next section to prove the main result.

LEMMA 2. *Let* $0<\sigma<1/2$, $t\in\mathbb{R}$ *and* $x\geqslant(1+\sqrt{3})/4$. *Then*

$$\frac{(2x+1-\sigma)^2+t^2}{(2x+\sigma)^2+t^2}<\left\{\left(\frac{2x+1}{2x}\right)^2\left(1-\frac{(1+4x)((-1+\sigma)\sigma+t^2)}{(1+2x)^2((-1+\sigma)\sigma+t^2+4x^2)}\right)\right\}^{1-2\sigma}. \tag{12}$$

If $t\geqslant 1/2$*, it has*

$$\frac{(1-\sigma)^2+t^2}{\sigma^2+t^2}<\left(1+\frac{1}{(-1+\sigma)\sigma+t^2}\right)^{1-2\sigma}. \tag{13}$$

Finally, for $t\geqslant 12$*, the following inequality holds*

$$\left(\frac{(1-\sigma)^2+t^2}{\sigma^2+t^2}\right)\prod_{n=1}^{3}\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2}<\left(\frac{1}{4}\prod_{n=1}^{3}\left(\frac{2n+1}{2n}\right)^2\right)^{1-2\sigma}. \tag{14}$$

Proof. Let $1-2\sigma=1/y$. Then (12) is equivalent to

$$\left(1+\frac{4(1+4x)}{y((-1/y+1+4x)^2+4t^2)}\right)^{y}<1+\frac{4(1+4x)y^2}{1+(-1+4t^2+16x^2)y^2}. \tag{15}$$

It is not difficult to verify

$$0<\frac{4(1+4x)}{(-1/y+1+4x)^2+4t^2}\leqslant 2, \qquad \left(x\geqslant\frac{1+\sqrt{3}}{4},\ t\in\mathbb{R}\right). \tag{16}$$

But relation (15) is just inequality (8), where

$$x:=\frac{4(1+4x)}{(-1/y+1+4x)^2+4t^2}, \qquad t:=y.$$

So we proved (12). In the same manner we establish (13). To prove (14) it is enough to verify the following inequality

$$\left(1+\frac{1}{(-1+\sigma)\sigma+t^2}\right)\prod_{n=1}^{3}\left(1-\frac{(1+4n)((-1+\sigma)\sigma+t^2)}{(1+2n)^2((-1+\sigma)\sigma+t^2+4n^2)}\right)<\frac{1}{4}.$$

Indeed, its left-hand side is increasing by σ and decreasing by t in the strip $]0,1/2[\times]1/2,\infty[$. Therefore, we may put $\sigma=1/2$ and $t=12$ and see by straightforward computation that it is less than $1/4$. □

3. Proof of the main result

Now we are ready to prove Theorem 1.

Proof. In fact, using representation (2) for $g(s)$ we show that for $0<\sigma<\frac{1}{2}$ and $t\geqslant 12,\quad |g(\sigma+it)|<1$. To do this, we appeal to the infinite product for the sine function ([1], p. 197)

$$\sin\pi z=\pi z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2}),\qquad z\in\mathbb{C},$$

and letting $z=\frac{1}{2}$, we arrive at the known Wallis's formula

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}.$$

Moreover, the Gauss infinite product formula for the gamma function ([2], p. 61)

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}},$$

yields

$$\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}=\frac{1-s}{s}\prod_{n=1}^{\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}-s}\left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}}\right).$$

Hence

$$\begin{aligned}
g(s)&=\left(\frac{1-s}{s}\right)2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n)^2}{(2n-1)(2n+1)}\right)^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}-s}\left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}}\right)\\
&=\left(\frac{1-s}{s}\right)2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n)^2n}{(2n-1)(2n+1)(n+1)}\right)^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}}\\
&=\left(\frac{1-s}{s}\right)2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n)n}{(2n-1)(n+1)}\right)^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{\frac{1}{2}-s}\left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}}\right)\\
&=\left(\frac{1-s}{s}\right)2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n)n}{(2n-1)(n+1)}\right)^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{\frac{1}{2}-s}\left(\frac{2n+1-s}{2n+s}\right)\\
&=\left(\frac{1-s}{s}\right)2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n+1)n}{(2n-1)(n+1)}\right)^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{1-2s}\left(\frac{2n+1-s}{2n+s}\right).
\end{aligned}$$

Let

$$f(s)=2^{\frac{1}{2}-s}\prod_{n=1}^{\infty}\left(\frac{(2n+1)n}{(2n-1)(n+1)}\right)^{\frac{1}{2}-s},$$

and

$$h(s) = h_1(s)h_2(s),$$

where

$$h_1(s) = \frac{1-s}{s}, \qquad h_2(s) = \prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{1-2s}\frac{2n+1-s}{2n+s}.$$

For any N we have $\prod_{n=1}^{N}\{((2n+1)n)/((2n-1)(n+1))\} = (2N+1)/(N+1) < 2$ and so

$$\prod_{n=1}^{\infty}\frac{(2n+1)n}{(2n-1)(n+1)} = 2.$$

Hence

$$|f(s)| = 2^{1-2\sigma}.$$

Therefore, it is sufficient to show that for $0 < \sigma < \frac{1}{2}$ and $t \geqslant 12$, $|h(s)| < 2^{2\sigma-1}$.

Indeed, $|h_1(s)|$ is a decreasing function with respect to σ and t for $0 < \sigma < 1/2$ and $t > 0$. Meanwhile,

$$|h_2(s)| = \prod_{n=1}^{\infty}\left(\frac{2n}{2n+1}\right)^{1-2\sigma}\left|\frac{2n+1-s}{2n+s}\right| \tag{17}$$

is increasing with respect to σ in the strip $(\sigma,t) \in]0,1/2[\times[1/2,\infty[$ and decreasing with respect to t in the strip $(\sigma,t) \in]0,1/2[\times\mathbb{R}^{+}$.

Denoting by

$$h_{2,n}(\sigma,t) = \left(\frac{2n}{2n+1}\right)^{1-2\sigma}\left|\frac{2n+1-(\sigma+it)}{2n+(\sigma+it)}\right|$$

the general term of the product and assuming for now

$$h_{2,n}(\sigma,t) < 1, \qquad (0 < \sigma < \frac{1}{2},\ t \geqslant 0), \tag{18}$$

we easily come out with the inequality

$$\prod_{n=1}^{N+1} h_{2,n}(\sigma,t) < \prod_{n=1}^{N} h_{2,n}(\sigma,t), \qquad (0 < \sigma < \frac{1}{2},\ t \geqslant 0).$$

To verify (18) we need to show that

$$(1+\frac{1}{2n})^{1-2\sigma} > \sqrt{\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2}}, \qquad t \geqslant 0. \tag{19}$$

In fact,

$$\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2} = 1 + \frac{(1-2\sigma)(4n+1)}{(2n+\sigma)^2+t^2}. \tag{20}$$

Hence inequality (19) yields

$$(1+\frac{1}{2n})^{1-2\sigma} > \frac{2n+1-\sigma}{2n+\sigma} \geqslant \sqrt{\frac{(2n+1-\sigma)^2+t^2}{(2n+\sigma)^2+t^2}}. \tag{21}$$

However

$$\frac{2n+1-\sigma}{2n+\sigma} = 1+\frac{1-2\sigma}{2n+\sigma}.$$

So the first inequality in (21) follows immediately from (9), letting $x = 2n$ and $a = 1-2\sigma$. Thus we have inequality (18).

Further, we show that $\{h_{2,n}(\sigma,t)\}_{n=1}^{\infty}$ is an increasing sequence for any $(\sigma,t) \in]0,1/2[\times\mathbb{R}$. To do this we consider the function $H_2(y) = h_{2,y}(\sigma,t)$ and differentiate it with respect to y. Hence by straightforward calculations we derive

$$H_2'(y) = \frac{\frac{1-2\sigma}{y(2y+1)}\left(\frac{2y}{2y+1}\right)^{1-2\sigma}}{((2y+\sigma)^2+t^2)^2\sqrt{\frac{(2y+1-\sigma)^2+t^2}{(2y+\sigma)^2+t^2}}} \times\Big\{(2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma)+(1+6y(1+2y)-2(1-\sigma)\sigma)t^2+t^4\Big\}.$$

Since

$$\begin{aligned}&(2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma)+(1+6y(1+2y)-2(1-\sigma)\sigma)t^2+t^4\\ \geqslant\ &(2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma)>0,\end{aligned}$$

we get that the derivative is positive, and therefore $H_2(y)$ is increasing for $y>0$. Now fixing $t \geqslant 1/2$, we justify that $h_{2,n}(\sigma,t)$ is increasing by σ. Precisely,

$$\frac{\partial}{\partial\sigma}h_{2,n}(\sigma,t) = \frac{\left(\frac{2n}{2n+1}\right)^{1-2\sigma}}{\left|\frac{2n+1-(\sigma+it)}{2n+(\sigma+it)}\right|}\Big\{-(1+4n)(4n^2+2n+\sigma-\sigma^2+t^2) +2((2n+1-\sigma)^2+t^2)((2n+\sigma)^2+t^2)\log\left(1+\frac{1}{2n}\right)\Big\}$$

and we achieve the goal, showing that the latter multiplier is positive. But this is true due to inequality (4), because it is greater than

$$\begin{aligned}&\frac{-(1-2\sigma)^2(2n+1-\sigma)(2n+\sigma)+(8n(1+2n)+3-8(1-\sigma)\sigma)t^2+4t^4}{1+4n}\\ \geqslant\ &\frac{1+(1-\sigma)\sigma(8n(1+2n)-3+4(1-\sigma)\sigma)}{1+4n}>0, \qquad (0<\sigma<1/2,\ t\geqslant 1/2).\end{aligned}$$

Returning to (17) we conclude that $|h_2(\sigma,t)|$ is increasing with respect to σ for $0 < \sigma < \frac{1}{2}$ and $t \geqslant 1/2$, and by (20) it is decreasing with respect to t for $0 < \sigma < \frac{1}{2}$ and $t > 0$.

Further, since

$$|h_N(s)| = \left|\frac{1-s}{s}\right| \prod_{n=1}^{N} \left(\frac{2n}{2n+1}\right)^{1-2\sigma} \left|\frac{2n+1-s}{2n+s}\right| \tag{22}$$

is decreasing by N, we have

$$|h(s)| \leqslant |h_N(s)|.$$

As $|h_N(s)|$ is decreasing by t, it is enough to show that

$$|h_N(s)| < 2^{2\sigma-1}, \qquad \text{for} \qquad (t = 12,\ N = 3)$$

and this has been established in (14). Moreover, since $\zeta(s)$ is reflexive with respect to the real axis, i.e., $\zeta(\overline{s}) = \overline{\zeta(s)}$, inequality (2) holds also for $t \leqslant -12$. Theorem 1 is proved. □

REMARK 1. A computer simulation shows that the main result is still valid for $t \in]6.5, 12[$. However, a direct proof by this approach is more complicated, because to achieve the goal we should increase a number N of terms in the product (22).

4. An application to the Riemann hypothesis

Similar to [9], we announce the following proposition.

PROPOSITION 1. *The Riemann hypothesis is true if and only if*

$$|\zeta(1-s)| < |\zeta(s)|, \qquad \textit{for} \quad (0 < \sigma < \frac{1}{2},\ |t| > 6.5).$$

As it is known [4], zeros of the derivative $\zeta'(s)$ of Riemann's zeta-function are connected with the behavior of zeros of $\zeta(s)$ itself. Precisely, Speiser's theorem [8] states that the Riemann hypothesis (RH) is equivalent to $\zeta'(s)$ having no zeros on the left of the critical line.

PROPOSITION 2. *If*

$$\frac{\partial}{\partial\sigma}|\zeta(s)|^2 < 0, \qquad \textit{for} \quad (0 < \sigma < \frac{1}{2},\ |t| > 6.5), \tag{A}$$

then the Riemann hypothesis is true.

Proof. In fact, if the Riemann hypothesis were not true, then by Speiser's theorem [8], there exists a number $s \in]0, 1/2[\times \mathbb{R}$, such that $\zeta'(s) = 0$. Hence $\frac{\partial}{\partial\sigma}|\zeta(s)|^2 = 0$. □

Finally, we conjecture the necessity of the condition (A) for the validity of the Riemann hypothesis.

REFERENCES

[1] L. V. AHLFORS, *Complex Analysis, An Introduction to the Theory of Analytic Functions of One Complex Variable*, McGraw-Hill, Inc. 3rd edition, 1979.
[2] R. E. ATTAR, *Special Functions and Orthogonal Polynomials*, Lulu Press, 2006.
[3] R. D. DIXON AND LOWELL SCHOENFELD, *The size of the Riemann zeta-function at places symmetric with respect to the Point* $1/2$, Duke Math. J. **33** (1966), 291–292.
[4] E. DUEÑEZ et. al., *Roots of the derivative of the Riemann zeta function and of characteristic polynomials*, Nonlinearity **23** (2010), 2599–2621.
[5] A. IVIĆ, *The Riemann Zeta-Function*, Dover Publications Inc., 2003.
[6] J. C. LAGARIAS, *An elementary problem equivalent to the Riemann hypothesis*, Amer. Math. Monthly **109**, 6 (2002), 534–543.
[7] D. S. MITRINOVIĆ et. al., *Elementary Inequalities*, P. Noordhoff Ltd., Groningen, 1964.
[8] A. SPEISER, *Geometrisches zur Riemannschen Zetafunktion*, Math. Ann. **110** (1934), 514–521 (German).
[9] R. SPIRA, *An inequality for the Riemann zeta function*, Duke Math. J. **32** (1965), 247–250.
[10] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-Function*, 2nd edition, Clarendon Press Oxford University Press, Oxford, 1986.

(Received July 28, 2012)

Sadegh Nazardonyavi
Department of Mathematics, Faculty of Sciences
University of Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal
e-mail: sdnazdi@yahoo.com

Semyon Yakubovich
Department of Mathematics, Faculty of Sciences
University of Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal
e-mail: syakubov@fc.up.pt