Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 183–194 doi:10.7153/jmi-07-18

THE BEST CONSTANT IN A GEOMETRIC INEQUALITY RELATING MEDIANS, INRADIUS AND CIRCUMRADIUS IN A TRIANGLE

SHI-CHANG SHI AND YU-DONG WU

(*Communicated by L. Yang*)

Abstract. In this paper, the authors give a refinement of the inequality associated with the medians, inradius and circumradius in a triangle by making use of certain analytical techniques for systems of nonlinear algebraic equations.

1. Introduction and main results

For a given $\triangle ABC$, let a, b and c denote the side-lengths facing the angles A, B and C, respectively. Also let m_a, m_b and m_c denote the corresponding medians, h_a, h_b and h_c the altitudes, $s=\frac{1}{2}(a+b+c)$ the semi-perimeter, R the circumradius and r the inradius of $\triangle ABC$. In addition, we let

$$m_1=\frac{1}{2}\sqrt{(b+c)^2-a^2},$$

$$m_2=\frac{1}{2}\sqrt{2a^2+\frac{1}{4}(b+c)^2},$$

$$r_0=\frac{a\sqrt{s(s-a)}}{2s},$$

and

$$R_0=\frac{(b+c)^2}{8\sqrt{s(s-a)}}.$$

In 1986, Janous [3] posed the following conjecture involving the geometrical inequality

$$\frac{5}{s}<\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}. \tag{1.1}$$

Later, in 1988, Gmeiner and Janous [2] proved the inequality (1.1) by using calculus. Later, inequality (1.1) was sharpened by An [1], Shi [8, 9], Yang [13] and Srivastava et al. [7], etc. It is easy to prove the reverse of inequality (1.1)

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\leqslant\frac{1}{r} \tag{1.2}$$

Mathematics subject classification (2010): 51M16, 52A40.
Keywords and phrases: Geometric inequality,triangle, medians, inradius, circumradius.

with the well-known inequalities $m_a \geqslant h_a$, etc.

In 1996, Liu considered a refinement of inequality (1.2), and he [4] posed the following interesting and beautiful geometric inequality conjecture with regard to the medians, inradius and circumradius.

CONJECTURE 1.1. In $\triangle ABC$, prove or disprove

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leqslant \frac{2}{3}\left(\frac{1}{R}+\frac{1}{r}\right). \tag{1.3}$$

Recently, Liu [5] proved inequality (1.3). The main goal of this paper is to refine inequality (1.3) as follows.

THEOREM 1.1. *In $\triangle ABC$, the best constant k for the following inequality*

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leqslant \frac{1}{r}-k\left(\frac{1}{r}-\frac{2}{R}\right) \tag{1.4}$$

is the real root in the interval $\left(\frac{1}{3},\frac{2}{5}\right)$ of equation

$$354294k^6-509571k^5+1927260k^4-2145600k^3+133376k^2+99328k+12288=0. \tag{1.5}$$

Furthermore, the constant k is approximately equal to 0.3440653.

2. Preliminary results

In order to prove Theorem 1.1, we need the following results.

LEMMA 2.1. *In $\triangle ABC$, if $a \geqslant b \geqslant c$, then*

$$(m_2+m_b)(m_2+m_c) \geqslant s\left(a+2\sqrt{(s-b)(s-c)}\right). \tag{2.1}$$

Proof. From the well-known inequalities $m_b \geqslant \sqrt{s(s-b)}$, $m_c \geqslant \sqrt{s(s-c)}$ and the obvious inequality $m_2 \geqslant \sqrt{\frac{1}{2}as}$, we get

$$\begin{aligned}
&(m_2+m_b)(m_2+m_c)-s\left(a+2\sqrt{(s-b)(s-c)}\right)\\
\geqslant&\left(\sqrt{\frac{1}{2}as}+\sqrt{s(s-b)}\right)\left(\sqrt{\frac{1}{2}as}+\sqrt{s(s-c)}\right)-s\left(a+2\sqrt{(s-b)(s-c)}\right)\\
=&-\frac{1}{2}as+\sqrt{\frac{1}{2}as}\left(\sqrt{s(s-b)}+\sqrt{s(s-c)}\right)-\sqrt{s^2(s-b)(s-c)}\\
=&s\left(\sqrt{\frac{1}{2}a}-\sqrt{s-b}\right)\left(\sqrt{s-c}-\sqrt{\frac{1}{2}a}\right)\\
=&\frac{s(b-c)^2}{2(\sqrt{a}+\sqrt{2(s-b)})(\sqrt{a}+\sqrt{2(s-c)})} \geqslant 0.
\end{aligned}$$

Hence, inequality (2.1) holds true.

LEMMA 2.2. *In $\triangle ABC$, if inequality (1.4) holds, then $k \leqslant \frac{4}{9}$.*

Proof. Let $b=c=1$ and $a=x$ $(0<x<2)$, then inequality (1.2) is equivalent to

$$\frac{2}{\sqrt{4-x^2}}+\frac{4}{\sqrt{2x^2+1}} \leqslant \frac{2(2+x)}{x\sqrt{4-x^2}}-k\left[\frac{2(2+x)}{x\sqrt{4-x^2}}-2\sqrt{4-x^2}\right]$$
$$\Longleftrightarrow k\cdot\frac{2(2+x)(1-x)^2}{x\sqrt{4-x^2}} \leqslant \frac{2(2+x)}{x\sqrt{4-x^2}}-\frac{2}{\sqrt{4-x^2}}-\frac{4}{\sqrt{2x^2+1}}$$
$$\Longleftrightarrow k\cdot\frac{2(2+x)(1-x)^2}{x\sqrt{4-x^2}} \leqslant \frac{4}{x\sqrt{4-x^2}}-\frac{4}{\sqrt{2x^2+1}}$$
$$\Longleftrightarrow k\cdot\frac{2(2+x)(1-x)^2}{x\sqrt{4-x^2}} \leqslant \frac{4(x^2-1)^2}{x\sqrt{(2x^2+1)(4-x^2)}(x\sqrt{4-x^2}+\sqrt{2x^2+1})}.$$

Thus,

$$(2+x)k \leqslant \frac{2(x+1)^2}{\sqrt{2x^2+1}(x\sqrt{4-x^2}+\sqrt{2x^2+1})}. \tag{2.2}$$

Taking $x=1$ in inequality (2.2), we obtain that $k \leqslant \frac{4}{9}$.

LEMMA 2.3. *In $\triangle ABC$, if $a \geqslant b \geqslant c$, then*

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}-\frac{1}{m_1}-\frac{2}{m_2} \leqslant \frac{9(b+c)^2(b-c)^2}{32\sqrt{\frac{1}{2}as(s-b)(s-c)}\cdot s^2[a+2\sqrt{(s-b)(s-c)}]}. \tag{2.3}$$

Proof. It is obvious that

$$\frac{1}{m_a}-\frac{1}{m_1}=\frac{m_1^2-m_a^2}{m_a m_1(m_a+m_1)}=-\frac{(b-c)^2}{4m_a m_1(m_a+m_1)}. \tag{2.4}$$

For $a \geqslant b \geqslant c$, we have that

$$m_1 \leqslant m_a \leqslant m_b \leqslant m_2 \leqslant m_c, \tag{2.5}$$

then by Cauchy's Inequality, we get

$$\begin{aligned} m_c+m_2 &\geqslant m_b+m_c \\ &\geqslant m_b+m_2 \\ &\geqslant \frac{1}{2}\sqrt{a^2+2c^2}+\frac{1}{2}\sqrt{2a^2+\frac{1}{4}(b+c)^2} \\ &\geqslant \frac{a+2c}{2\sqrt{3}}+\frac{2a+\frac{b+c}{2}}{2\sqrt{3}} \\ &= \frac{6a+b+5c}{4\sqrt{3}} \\ &\geqslant \frac{\sqrt{3}}{2}(b+c). \end{aligned} \tag{2.6}$$

And

$$m_b^2+m_c^2-2m_2^2=\frac{(b-c)^2}{8}\geqslant 0,$$

so

$$m_b^2+m_c^2\geqslant 2m_2^2. \tag{2.7}$$

Hence, by inequalities $a\geqslant b\geqslant c$ and (2.5)–(2.7), we obtain

$$\begin{aligned}
&\frac{1}{m_b}+\frac{1}{m_c}-\frac{2}{m_2}\\
=&\frac{m_2^2-m_b^2}{m_bm_2(m_b+m_2)}+\frac{m_2^2-m_c^2}{m_cm_2(m_c+m_2)}\\
=&\frac{(5b+7c)(b-c)}{16m_bm_2(m_b+m_2)}+\frac{(7b+5c)(c-b)}{16m_cm_2(m_c+m_2)}\\
=&-\frac{(b-c)^2}{16m_bm_2(m_b+m_2)}-\frac{(b-c)^2}{16m_cm_2(m_c+m_2)}+\frac{3(b+c)(b-c)}{8m_bm_2(m_b+m_2)}+\frac{3(b+c)(c-b)}{8m_cm_2(m_c+m_2)}\\
=&-\frac{(b-c)^2}{16m_bm_2(m_b+m_2)}-\frac{(b-c)^2}{16m_cm_2(m_c+m_2)}+\frac{3(b+c)(c-b)[(m_b^2-m_c^2)+m_2(m_b-m_c)]}{8m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
=&-\frac{(b-c)^2}{16m_bm_2(m_b+m_2)}-\frac{(b-c)^2}{16m_cm_2(m_c+m_2)}+\frac{3(b+c)(c-b)(m_b^2-m_c^2)}{8m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
&+\frac{3(b+c)(c-b)(m_b^2-m_c^2)}{8m_bm_c(m_b+m_2)(m_c+m_2)(m_b+m_c)}\\
=&-\frac{(b-c)^2}{16m_bm_2(m_b+m_2)}-\frac{(b-c)^2}{16m_cm_2(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
&+\frac{9(b+c)^2(b-c)^2}{32m_bm_c(m_b+m_2)(m_c+m_2)(m_b+m_c)}\\
=&-\frac{[m_b^2+m_c^2+m_2(m_b+m_c)](b-c)^2}{16m_bm_cm_2(m_b+m_2)(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
&+\frac{9(b+c)^2(b-c)^2}{32m_bm_c(m_b+m_2)(m_c+m_2)(m_b+m_c)}\\
\leqslant&-\frac{[2m_2^2+m_2(m_b+m_c)](b-c)^2}{16m_bm_cm_2(m_b+m_2)(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
&+\frac{9(b+c)^2(b-c)^2}{32m_bm_c(m_b+m_2)(m_c+m_2)\cdot\frac{\sqrt{3}}{2}(b+c)}\\
=&-\frac{[2m_2+(m_b+m_c)](b-c)^2}{16m_bm_c(m_b+m_2)(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
&+\frac{9(b+c)(b-c)^2}{16\sqrt{3}m_bm_c(m_b+m_2)(m_c+m_2)}\\
\leqslant&-\frac{\sqrt{3}(b+c)(b-c)^2}{16m_bm_c(m_b+m_2)(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3\sqrt{3}(b+c)(b-c)^2}{16m_bm_c(m_b+m_2)(m_c+m_2)}\\
=&\frac{\sqrt{3}(b+c)(b-c)^2}{8m_bm_c(m_b+m_2)(m_c+m_2)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}\\
\leqslant&\frac{(b-c)^2}{4m_am_1(m_a+m_1)}+\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}. \qquad (2.8)
\end{aligned}
$$

From inequalities (2.4) and (2.8), we obtain

$$\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}-\frac{1}{m_1}-\frac{2}{m_2}\leqslant\frac{9(b+c)^2(b-c)^2}{32m_bm_cm_2(m_b+m_2)(m_c+m_2)}. \qquad (2.9)$$

With inequalities $m_b\geqslant\sqrt{s(s-b)}$, $m_c\geqslant\sqrt{s(s-c)}$, $m_2\geqslant\sqrt{\frac{1}{2}as}$, inequality (2.9), together with Lemma 2.1, we immediately obtain inequality (2.3).

LEMMA 2.4. *In* $\triangle ABC$*,*

$$\frac{1}{r_0}-\frac{1}{r}=-\frac{\sqrt{s}(b-c)^2}{a\sqrt{(s-a)(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}, \qquad (2.10)$$

if $a\geqslant b\geqslant c$*, then*

$$\frac{1}{R_0}-\frac{1}{R}\leqslant\frac{2\sqrt{s(s-a)}(b^2+c^2-a^2)(b-c)^2}{bc(b+c)^2\sqrt{(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}. \qquad (2.11)$$

Proof. Identity (2.10) just follows from the well-known formula

$$r=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

Now we prove inequality (2.11). From the well-known formula

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

and

$$2\sqrt{(s-b)(s-c)}\leqslant(s-b)+(s-c)=a,$$

we obtain

$$\begin{aligned}
\frac{1}{R_0}-\frac{1}{R} &= 4\sqrt{s(s-a)}\left[\frac{2}{(b+c)^2}-\frac{\sqrt{(s-b)(s-c)}}{abc}\right]\\
&= 4\sqrt{s(s-a)}\left[\frac{a-2\sqrt{(s-b)(s-c)}}{2abc}-\frac{(b+c)^2-4bc}{2bc(b+c)^2}\right]\\
&= 2\sqrt{s(s-a)}\left[\frac{(b-c)^2}{abc[a+2\sqrt{(s-b)(s-c)}]}-\frac{(b-c)^2}{bc(b+c)^2}\right]\\
&= \frac{2\sqrt{s(s-a)}\{(b+c)^2-a[a+2\sqrt{(s-b)(s-c)}]\}}{abc(b+c)^2[a+2\sqrt{(s-b)(s-c)}]}\cdot(b-c)^2\\
&= \frac{2\sqrt{s(s-a)}\{[(b+c)^2-a^2]\sqrt{(s-b)(s-c)}-2a(s-b)(s-c)\}}{abc(b+c)^2[a+2\sqrt{(s-b)(s-c)}]\sqrt{(s-b)(s-c)}}\cdot(b-c)^2\\
&\leqslant \frac{2\sqrt{s(s-a)}\{[(b+c)^2-a^2]\cdot\frac{a}{2}-\frac{a}{2}\cdot[a^2-(b-c)^2])\}}{abc(b+c)^2[a+2\sqrt{(s-b)(s-c)}]\sqrt{(s-b)(s-c)}}\cdot(b-c)^2\\
&= \frac{2\sqrt{s(s-a)}\cdot(b^2+c^2-a^2)}{bc(b+c)^2[a+2\sqrt{(s-b)(s-c)}]\sqrt{(s-b)(s-c)}}\cdot(b-c)^2.
\end{aligned}$$

LEMMA 2.5. *In $\triangle ABC$, let*

$$f(a,b,c)=\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}-\frac{1}{r}+k\left(\frac{1}{r}-\frac{2}{R}\right).$$

Then

$$f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=\frac{1}{m_1}+\frac{2}{m_2}-\frac{1}{r_0}+k\left(\frac{1}{r_0}-\frac{2}{R_0}\right).$$

If $a\geqslant b\geqslant c$ and $0<k\leqslant\frac{4}{9}$, then

$$f(a,b,c)\leqslant f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right). \tag{2.12}$$

Proof. If $a\geqslant b\geqslant c$, then

$$bc-2a(s-a)=(a-b)(a-c)\geqslant 0,$$

thus

$$a^2\geqslant\left(\frac{b+c}{2}\right)^2\geqslant bc\geqslant 2a(s-a),$$

and for $0<k\leqslant\frac{4}{9}$, hence,

$$\begin{aligned}
&9(b+c)^2\sqrt{2a(s-a)}-16(2-3k)s^3\\
\leqslant&9(b+c)^2\cdot\frac{b+c}{2}-16\left(2-3\cdot\frac{4}{9}\right)\cdot\frac{1}{8}\left(\frac{b+c}{2}+b+c\right)^3=0
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&8a(s-a)(b^2+c^2-a^2)-bc(b+c)^2\\
\leqslant&8a(s-a)(b^2+c^2-a^2)-2a(s-a)(b+c)^2\\
=&2a(s-a)(3b^2+3c^2-4a^2-2bc)\\
=&2a(s-a)[3(b^2-a^2)+(c^2-a^2)+2c(c-b)]\leqslant 0.
\end{aligned}\tag{2.14}$$

From Lemmas 2.3–2.4, inequalities (2.13)–(2.14), we obtain that

$$\begin{aligned}
&f(a,b,c)-f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)\\
=&\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}-\frac{1}{m_1}-\frac{2}{m_2}+(1-k)\left(\frac{1}{r_0}-\frac{1}{r}\right)+2k\left(\frac{1}{R_0}-\frac{1}{R}\right)\\
\leqslant&\frac{9(b+c)^2(b-c)^2}{32\sqrt{\frac{1}{2}as(s-b)(s-c)}\cdot s^2[a+2\sqrt{(s-b)(s-c)}]}\\
&-\frac{(1-k)\sqrt{s}(b-c)^2}{a\sqrt{(s-a)(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}\\
&+\frac{4k\sqrt{s(s-a)}(b^2+c^2-a^2)(b-c)^2}{bc(b+c)^2\sqrt{(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}\\
=&\frac{\sqrt{s}[9(b+c)^2\sqrt{2a(s-a)}-16(2-3k)s^3](b-c)^2}{32as^3\sqrt{(s-a)(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}\\
&+\frac{k\sqrt{s}[8a(s-a)(b^2+c^2-a^2)-bc(b+c)^2](b-c)^2}{2abc(b+c)^2\sqrt{(s-a)(s-b)(s-c)}[a+2\sqrt{(s-b)(s-c)}]}\leqslant 0.
\end{aligned}\tag{2.15}$$

Therefore, inequality (2.12) holds.

LEMMA 2.6. (see [6, 11, 12]) *Let*

$$F(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n,$$

and

$$G(x)=b_0x^m+b_1x^{m-1}+\cdots+b_m.$$

If $a_0\neq 0$ *or* $b_0\neq 0$*, then the polynomials* $F(x)$ *and* $G(x)$ *have a common root if and only if*

$$R(F,G):=\left|\begin{array}{cccccc}
a_0 & a_1 & \cdots & a_n & & \\
 & a_0 & a_1 & \cdots & a_n & \\
 & & \ddots & \ddots & & \ddots \\
 & & & a_0 & a_1 & \cdots\ a_n \\
b_0 & b_1 & \cdots & b_m & & \\
 & b_0 & b_1 & \cdots & b_m & \\
 & & \ddots & \ddots & & \ddots \\
 & & & b_0 & b_1 & \cdots\ b_m
\end{array}\right|\begin{array}{l}\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}m\\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}n\end{array}=0$$

where $R(F,G)$ ($(m+n)\times(m+n)$ determinant) *is Sylvester's Resultant of* $F(x)$ *and* $G(x)$.

LEMMA 2.7. (see [10, 12]) *Given a polynomial* $f(x)$ *with real coefficients*

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n,$$

if the number of the sign changes of the revised sign list of its discriminant sequence

$$\{D_1(f),D_2(f),\cdots,D_n(f)\}$$

is v, *then the number of the pairs of distinct conjugate imaginary roots of* $f(x)$ *equals* v. *Furthermore, if the number of non-vanishing members of the revised sign list is* l, *then the number of the distinct real roots of* $f(x)$ *equals* $l-2v$.

3. The proof of Theorem 1.1

Proof. If $k\leqslant 0$, then by inequality (1.2), we can easily find that inequality (1.4) holds. Hence, we only need consider the case $k>0$, and by Lemma 2.2, we only need consider the case $0<k\leqslant\frac{4}{9}$.

Now we determine the best constant k such that $f(a,b,c)\leqslant 0$. Since the inequality (1.4) is symmetrical with respect to the side-lengths a, b and c, there is no harm in supposing $a\geqslant b\geqslant c$. Thus, by Lemma 2.5, we only need to determine the best constant k such that

$$f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)\leqslant 0.$$

or, equivalently, that

$$\frac{2}{\sqrt{(b+c)^2-a^2}}+\frac{4}{\sqrt{2a^2+\left(\frac{b+c}{2}\right)^2}}-\frac{2s}{a\sqrt{s(s-a)}}+k\left(\frac{2s}{a\sqrt{s(s-a)}}-\frac{16\sqrt{s(s-a)}}{(b+c)^2}\right)\leqslant 0. \tag{3.1}$$

Without loss of generality, we can assume that

$$a=x \quad\text{and}\quad \frac{b+c}{2}=1 \quad (1\leqslant x<2),$$

because the inequality (3.1) is homogeneous with respect to a and $\frac{b+c}{2}$. Thus, clearly, the inequality (3.1) is equivalent to the following inequality:

$$\frac{2}{\sqrt{4-x^2}}+\frac{4}{\sqrt{2x^2+1}}-\frac{2(x+2)}{x\sqrt{4-x^2}}+k\left[\frac{2(x+2)}{x\sqrt{4-x^2}}-2\sqrt{4-x^2}\right]\leqslant 0. \tag{3.2}$$

We consider the following two cases separately.

Case 1. When $x=1$, the inequality (3.2) holds true for any $k\in\mathbb{R}:=(-\infty,+\infty)$.

Case 2. When $1<x<2$, the inequality (3.2) is equivalent to the following inequality:

$$k\leqslant\frac{2(x+1)^2}{(x+2)\sqrt{2x^2+1}\left(\sqrt{2x^2+1}+x\sqrt{4-x^2}\right)}. \tag{3.3}$$

Define the function

$$g(x) := \frac{2(x+1)^2}{(x+2)\sqrt{2x^2+1}(\sqrt{2x^2+1}+x\sqrt{4-x^2})}, \quad x \in (1,2).$$

Calculating the derivative of $g(x)$, we get

$$g'(x) = \frac{2(x+1)[4x^6+12x^5+5x^4-21x^3-28x^2-8-(2x^3+6x^2+7x-3)\sqrt{2x^2+1}\sqrt{4-x^2}]}{(x+2)^2(2x^2+1)^{\frac{3}{2}}(\sqrt{2x^2+1}+x\sqrt{4-x^2})^2\sqrt{4-x^2}}.$$

By setting $g'(x) = 0$, we obtain

$$4x^6+12x^5+5x^4-21x^3-28x^2-8-(2x^3+6x^2+7x-3)\sqrt{2x^2+1}\sqrt{4-x^2} = 0. \quad (3.4)$$

It is easy to see that the roots of the equation (3.4) are also solutions of the following equation:

$$(4x^6+12x^5+5x^4-21x^3-28x^2-8)^2-(2x^3+6x^2+7x-3)^2(2x^2+1)(4-x^2) = 0,$$

that is

$$(x-1)(x+2)(x+1)^4\varphi(x) = 0, \quad (3.5)$$

where

$$\varphi(x) = 16x^6+16x^5-16x^4-80x^3+5x^2-35x-14.$$

It is obvious that the following equation:

$$(x-1)(x+2)(x+1)^4 = 0 \quad (3.6)$$

has no real root on the interval $(1,2)$.

The revised sign list of the discriminant sequence of $\varphi(x)$ is given by

$$[1,\ 1,\ -1,\ -1,\ 1,\ 1]. \quad (3.7)$$

So the number of the sign changes of the revised sign list of (3.7) is 2. Thus, by applying Lemma 2.7, we find that the equation:

$$\varphi(x) = 0 \quad (3.8)$$

has 2 distinct real roots. Moreover, it is not difficult to check that

$$\begin{aligned} \varphi(-1) &= 90 > 0, \\ \varphi(0) &= -14 < 0, \\ \varphi(1) &= -108 < 0 \end{aligned}$$

and

$$\varphi(2) = 576 > 0.$$

We can conclude that the equation (3.8) has 2 distinct real roots in the following intervals:

$$(-1,0) \quad \text{and} \quad (1,2).$$

So that the equation (3.4) has only one real root x_0 given by $x_0 = 1.67073609778\cdots$ in the interval $(1,\ 2)$, and

$$g(x)_{min} = g(x_0) \approx 0.3440653 \in \left(\frac{1}{3},\ \frac{2}{5}\right). \tag{3.9}$$

Now we prove that $g(x_0)$ is the root of the equation (1.5). For this purpose, we consider the following system of nonlinear algebraic equations:

$$\begin{cases} \varphi(x_0) = 0, \\ 2x_0^2 + 1 - u_0^2 = 0, \\ 4 - x_0^2 - v_0^2 = 0, \\ 2(x+1)^2 - (x+2)\, u_0\, (u_0 + x\, v_0)\, k = 0. \end{cases} \tag{3.10}$$

It is easy to see that $g(x_0)$ is also the solution of the nonlinear algebraic equation system (3.10). If we eliminate the v_0, u_0 and x_0 ordinal by resultant (by using Lemma 2.6), then we get

$$348285173760000 \cdot \phi_1^2(k) \cdot \phi_2^2(k) = 0. \tag{3.11}$$

where

$$\begin{aligned} \phi_1(k) = {} & 472392\,k^6 - 2182626\,k^5 + 4000527\,k^4 \\ & - 4119168\,k^3 + 2375744\,k^2 - 690176\,k + 76800 \end{aligned}$$

and

$$\begin{aligned} \phi_2(k) = {} & 354294\,k^6 - 509571\,k^5 + 1927260\,k^4 \\ & - 2145600\,k^3 + 133376\,k^2 + 99328\,k + 12288. \end{aligned}$$

The revised sign list of the discriminant sequence of $\phi_1(k)$ is given by

$$[1,\ 1,\ -1,\ -1,\ -1,\ 1]. \tag{3.12}$$

The revised sign list of the discriminant sequence of $\phi_2(k)$ is given by

$$[1,\ -1,\ -1,\ -1,\ -1,\ 1]. \tag{3.13}$$

So the number of the sign changes of the revised sign list of (3.12) and (3.13) are both 2, Thus, by applying Lemma 2.6, we find that each of the equations:

$$\phi_1(k) = 0 \tag{3.14}$$

and

$$\phi_2(k) = 0 \tag{3.15}$$

has 2 distinct real roots. In addition, it is easy to check that

$$\phi_1\left(\frac{1}{5}\right) = \frac{102716437}{15625} > 0; \quad \phi_2\left(\frac{1}{3}\right) = \frac{26393}{9} > 0,$$
$$\phi_1\left(\frac{1}{3}\right) = -\frac{2381}{3} < 0; \quad \phi_2\left(\frac{2}{5}\right) = -\frac{287312544}{15625} < 0,$$
$$\phi_1(2) = -356432 < 0; \quad \phi_2(1) = -128625 < 0$$

and

$$\phi_1(3) = 46208769 > 0; \quad \phi_2(2) = 20784352 > 0.$$

We can thus find that the equation (3.14) has 2 distinct real roots in the following intervals:

$$\left(\frac{1}{5}, \frac{1}{3}\right) \quad \text{and} \quad (2, 3).$$

And the equation (3.15) has 2 distinct real roots in the following intervals:

$$\left(\frac{1}{3}, \frac{2}{5}\right) \quad \text{and} \quad (1, 2).$$

Hence, by (3.9), we can conclude that $g(x_0)$ is the root of the equation (1.5). The proof of Theorem 1.1 is thus completed.

Acknowledgement. The authors would like to thank the anonymous referees for their very careful reading and making some valuable comments which have essentially improved the presentation of this paper.

REFERENCES

[1] Z.-P. AN, *Generalizations of several inequalties in triangle*, M athematics Communications **8** (1991), 21–24. (in Chinese)
[2] W.GMEINER AND W.JANOUS, *Solution(completed) to problem 1137**, Crux Math. **14** (1988), 79–83.
[3] W.JANOUS, *Problem 1137**, Crux Math. **12** (1986), 177.
[4] J. LIU, *100 pending inequality problems in triangle*, In: C. Shan (ed.), Geometric Inequalities in China., Jiangsu Educational Press, Nanjing, 175–187 (1996).(in Chinese)
[5] J. LIU, *Testifying a conjecture of an median inequality of triangle*, Journal of East China Jiaotong University **25**, 1 (2008), 105–108.(in Chinese)
[6] J. J. SYLVESTER, *A method of determining by mere inspection the derivatives from two equations of any degree*, Phil. Mag. **16** (1840), 132–135.
[7] H. M. SRIVASTAVA, V. LOKESHA AND Y.-D. WU, *A new refinement of the Janous – Gmeiner inequality for a triangle*, Comput. Math. Appl. **62** (2011), 2349–2353.
[8] S.-C. SHI, *Sharpening on Janous's inequality*, Mathematics Communications **8** (1993), 26–28. (in Chinese)
[9] S.-C. SHI, *Sharpening on Janous-Gmeiner's inequality*, In: J. Chen and Z.-H. Ye (ed.), Forward Position of Elementary Mathematics, Jiangsu Educational Press (1996), 85–89. (in Chinese)
[10] L. YANG, X.-R. HOU AND Z.-B. ZENG, *A complete discrimination system for polynomials*, Sci. China (Ser. E) **39** (1996), 628–646.
[11] L. YANG AND B.-C. XIA, *Automated Proving and Discoverering on Inequalities*, Science Press, Beijing (2008), 6. (in Chinese)

[12] L. YANG, J.-Z. ZHANG AND X.-R. HOU, *Nonlinear Algebraic Equation System and Automated Theorem Proving*, Shanghai Scientific and Technological Education Press, 23–25, 1996. (in Chinese)

[13] X.-Z. YANG, *Consummation of Janous-Gmeiner's inequality*, In: J. Chen and Z.-H. Ye (ed.), Forward Position of Elementary Mathematics, Jiangsu Educational Press (1996), 77–84. (in Chinese)

(Received February 24, 2012)

Shi-Chang Shi
Zhejiang Teaching and Research Institute
Department of Education
Hangzhou, Zhejiang 310012
People's Republic of China

Yu-Dong Wu
Department of Mathematics
Zhejiang Xinchang High School
Shaoxing, Zhejiang 312500
People's Republic of China
e-mail: yudong.wu@hotmail.com