THE BEST CONSTANT IN A GEOMETRIC INEQUALITY RELATING MEDIANS, INRADIUS AND CIRCUMRADIUS IN A TRIANGLE

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Abstract. In this paper, the authors give a refinement of the inequality associated with the medians, inradius and circumradius in a triangle by making use of certain analytical techniques for systems of nonlinear algebraic equations.

1. Introduction and main results

For a given \( \triangle ABC \), let \( a, b \) and \( c \) denote the side-lengths facing the angles \( A, B \) and \( C \), respectively. Also let \( m_a, m_b \) and \( m_c \) denote the corresponding medians, \( h_a, h_b \) and \( h_c \) the altitudes, \( s = \frac{1}{2}(a+b+c) \) the semi-perimeter, \( R \) the circumradius and \( r \) the inradius of \( \triangle ABC \). In addition, we let

\[
\begin{align*}
  m_1 &= \frac{1}{2} \sqrt{(b+c)^2 - a^2}, \\
  m_2 &= \frac{1}{2} \sqrt{2a^2 + \frac{1}{4}(b+c)^2}, \\
  r_0 &= \frac{a \sqrt{s(s-a)}}{2s}, \\
  R_0 &= \frac{(b+c)^2}{8 \sqrt{s(s-a)}}.
\end{align*}
\]

In 1986, Janous [3] posed the following conjecture involving the geometrical inequality

\[
\frac{5}{s} < \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}.
\]

Later, in 1988, Gmeiner and Janous [2] proved the inequality (1.1) by using calculus. Later, inequality (1.1) was sharpened by An [1], Shi [8, 9], Yang [13] and Srivastava et al. [7], etc. It is easy to prove the reverse of inequality (1.1)

\[
\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{1}{r}.
\]
with the well-known inequalities \( m_a \geq h_a \), etc.

In 1996, Liu considered a refinement of inequality (1.2), and he [4] posed the following interesting and beautiful geometric inequality conjecture with regard to the medians, inradius and circumradius.

**Conjecture 1.1.** In \( \triangle ABC \), prove or disprove

\[
\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{2}{3} \left( \frac{1}{R} + \frac{1}{r} \right).
\] (1.3)

Recently, Liu [5] proved inequality (1.3). The main goal of this paper is to refine inequality (1.3) as follows.

**Theorem 1.1.** In \( \triangle ABC \), the best constant \( k \) for the following inequality

\[
\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{1}{r} - k \left( \frac{1}{r} - \frac{2}{R} \right)
\] (1.4)

is the real root in the interval \( \left( \frac{1}{3}, \frac{2}{5} \right) \) of equation

\[
354294k^6 - 509571k^5 + 1927260k^4 - 2145600k^3 + 133376k^2 + 99328k + 12288 = 0.
\] (1.5)

Furthermore, the constant \( k \) is approximately equal to 0.3440653.

### 2. Preliminary results

In order to prove Theorem 1.1, we need the following results.

**Lemma 2.1.** In \( \triangle ABC \), if \( a \geq b \geq c \), then

\[
(m_2 + m_b)(m_2 + m_c) \geq s \left( a + 2\sqrt{(s-b)(s-c)} \right).
\] (2.1)

**Proof.** From the well-known inequalities \( m_b \geq \sqrt{s(s-b)} \), \( m_c \geq \sqrt{s(s-c)} \) and the obvious inequality \( m_2 \geq \sqrt{\frac{1}{2}as} \), we get

\[
(m_2 + m_b)(m_2 + m_c) - s \left( a + 2\sqrt{(s-b)(s-c)} \right)
\geq \left( \sqrt{\frac{1}{2}as} + \sqrt{s(s-b)} \right) \left( \sqrt{\frac{1}{2}as} + \sqrt{s(s-c)} \right) - s \left( a + 2\sqrt{(s-b)(s-c)} \right)
= -\frac{1}{2}as + \sqrt{\frac{1}{2}as} (\sqrt{s(s-b)} + \sqrt{s(s-c)}) - \sqrt{s^2(s-b)(s-c)}
= s \left( \sqrt{\frac{1}{2}a - \sqrt{s-b}} \right) \left( \sqrt{s-c} - \sqrt{\frac{1}{2}a} \right)
= \frac{s(b-c)^2}{2(\sqrt{a} + \sqrt{2(s-b)})(\sqrt{a} + \sqrt{2(s-c)})} \geq 0.
\]

Hence, inequality (2.1) holds true.
LEMMA 2.2. In $\triangle ABC$, if inequality (1.4) holds, then $k \leq \frac{4}{9}$.

Proof. Let $b = c = 1$ and $a = x$ ($0 < x < 2$), then inequality (1.2) is equivalent to

$$\frac{2}{\sqrt{4 - x^2}} + \frac{4}{\sqrt{2x^2 + 1}} \leq \frac{2(2 + x)}{x\sqrt{4 - x^2}} - k \left[ \frac{2(2 + x)}{x\sqrt{4 - x^2}} - 2\sqrt{4 - x^2} \right]$$

$$\iff k \cdot \frac{2(2 + x)(1 - x)}{x\sqrt{4 - x^2}} \leq \frac{2(2 + x)}{x\sqrt{4 - x^2}} - \frac{2}{\sqrt{4 - x^2}} - \frac{4}{\sqrt{2x^2 + 1}}$$

$$\iff k \cdot \frac{2(2 + x)(1 - x)}{x\sqrt{4 - x^2}} \leq \frac{4}{x\sqrt{4 - x^2}} - \frac{4}{\sqrt{2x^2 + 1}}$$

$$\iff k \cdot \frac{2(2 + x)(1 - x)}{x\sqrt{4 - x^2}} \leq \frac{4(x^2 - 1)}{x\sqrt{(2x^2 + 1)(4 - x^2)(x\sqrt{4 - x^2} + \sqrt{2x^2 + 1})}}.$$ 

Thus,

$$(2 + x)k \leq \frac{2(x + 1)^2}{\sqrt{2x^2 + 1}(x\sqrt{4 - x^2} + \sqrt{2x^2 + 1})}. \quad (2.2)$$

Taking $x = 1$ in inequality (2.2), we obtain that $k \leq \frac{4}{9}$.

LEMMA 2.3. In $\triangle ABC$, if $a \geq b \geq c$, then

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} - \frac{1}{m_1} - \frac{2}{m_2} \leq \frac{9(b + c)^2(b - c)^2}{32\sqrt{\frac{1}{2}as(s - b)(s - c) \cdot s^2[a + 2\sqrt{(s - b)(s - c)}]}}. \quad (2.3)$$

Proof. It is obvious that

$$\frac{1}{m_a} - \frac{1}{m_1} = \frac{m_1^2 - m_a^2}{m_1m_a(m_a + m_1)} = -\frac{(b - c)^2}{4m_1m_a(m_a + m_1)}. \quad (2.4)$$

For $a \geq b \geq c$, we have that

$$m_1 \leq m_a \leq m_b \leq m_2 \leq m_c, \quad (2.5)$$

then by Cauchy’s Inequality, we get

$$m_c + m_2 \geq m_b + m_c$$

$$\geq m_b + m_2$$

$$\geq \frac{1}{2} \sqrt{a^2 + 2c^2} + \frac{1}{2} \sqrt{2a^2 + \frac{1}{4}(b + c)^2}$$

$$\geq \frac{a + 2c}{2\sqrt{3}} + \frac{2a + \frac{b + c}{2}}{2\sqrt{3}}$$

$$= \frac{6a + b + 5c}{4\sqrt{3}}$$

$$\geq \frac{\sqrt{3}}{2}(b + c). \quad (2.6)$$
And
\[ m_b^2 + m_c^2 - 2m_2^2 = \frac{(b-c)^2}{8} \geq 0, \]
so
\[ m_b^2 + m_c^2 \geq 2m_2^2. \] (2.7)
Hence, by inequalities \( a \geq b \geq c \) and (2.5)–(2.7), we obtain
\[
\frac{1}{m_b} + \frac{1}{m_c} - \frac{2}{m_2} = \frac{m_2^2 - m_b^2}{m_b m_2 (m_b + m_2)} + \frac{m_2^2 - m_c^2}{m_c m_2 (m_c + m_2)}
\]
\[
= \frac{(5b+7c)(b-c)}{16m_b m_2 (m_b + m_2)} + \frac{(7b+5c)(c-b)}{16m_c m_2 (m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{3(b+c)(b-c)}{8m_b m_2 (m_b + m_2)} + \frac{3(b+c)(c-b)}{8m_c m_2 (m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{3(b+c)(c-b)[(m_b^2 - m_c^2) + m_2(m_b - m_c)]}{8m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{3(b+c)(c-b)m_2^2 - m_c^2}{8m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{9(b+c)^2(b-c)}{32m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{9(b+c)^2(b-c)2(m_b + m_2)(m_c + m_2)}{32m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{9(b+c)^2(b-c)2(m_b + m_2)(m_c + m_2)}{32m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{9(b+c)^2(b-c)2(m_b + m_2)(m_c + m_2)}{32m_b m_c (m_b + m_2)(m_c + m_2)}
\]
\[
= -\frac{(b-c)^2}{16m_b m_2 (m_b + m_2)} - \frac{(b-c)^2}{16m_c m_2 (m_c + m_2)} + \frac{9(b+c)^2(b-c)2(m_b + m_2)(m_c + m_2)}{32m_b m_c (m_b + m_2)(m_c + m_2)}
\]
From inequalities (2.4) and (2.8), we obtain

\[
\frac{1}{ma} + \frac{1}{mb} + \frac{1}{mc} - \frac{1}{m1} - \frac{2}{m2} \leq \frac{9(b + c)^2(b - c)^2}{32mbmc(m_b + m_2)(m_c + m_2)}. \tag{2.9}
\]

With inequalities \(m_b \geq \sqrt{s(s-b)}, \ m_c \geq \sqrt{s(s-c)}, \ m_2 \geq \sqrt{\frac{1}{2}as},\) inequality (2.9), together with Lemma 2.1, we immediately obtain inequality (2.3).

**Lemma 2.4.** In \(\triangle ABC,\)

\[
\frac{1}{r_0} - \frac{1}{r} = -\frac{\sqrt{s(b - c)^2}}{a\sqrt{(s-a)(s-b)(s-c)[a + 2\sqrt{(s-b)(s-c)}]}}, \tag{2.10}
\]

if \(a \geq b \geq c,\) then

\[
\frac{1}{R_0} - \frac{1}{R} \leq \frac{2\sqrt{s(s-a)(b^2 + c^2 - a^2)}(b - c)^2}{bc(b + c)^2 \sqrt{(s-b)(s-c)[a + 2\sqrt{(s-b)(s-c)}]}}. \tag{2.11}
\]

**Proof.** Identity (2.10) just follows from the well-known formula

\[
r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.
\]

Now we prove inequality (2.11). From the well-known formula

\[
R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}},
\]

and

\[
2\sqrt{(s-b)(s-c)} \leq (s-b) + (s-c) = a,
\]
we obtain
\[
\frac{1}{R_0} - \frac{1}{R} = 4\sqrt{s(s-a)} \left[ \frac{2}{(b+c)^2} - \frac{\sqrt{(s-b)(s-c)}}{abc} \right]
= 4\sqrt{s(s-a)} \left[ \frac{a - 2\sqrt{(s-b)(s-c)} - (b+c)^2 - 4bc}{2abc} \right]
= 2\sqrt{s(s-a)} \left[ \frac{(b-c)^2}{abc[a + 2\sqrt{(s-b)(s-c)}]} - \frac{(b-c)^2}{bc(b+c)^2} \right]
= \frac{2\sqrt{s(s-a)}((b+c)^2 - a[a + 2\sqrt{(s-b)(s-c)}])}{abc(b+c)^2[a + 2\sqrt{(s-b)(s-c)}](s-b)(s-c)]} \cdot (b-c)^2
\leq \frac{2\sqrt{s(s-a)}[[((b+c)^2 - a^2)\cdot \frac{9}{4} - \frac{a}{2} \cdot [a^2 - (b-c)^2]]}{abc(b+c)^2[a + 2\sqrt{(s-b)(s-c)}](s-b)(s-c)]} \cdot (b-c)^2
\leq \frac{2\sqrt{s(s-a)}(b^2 + c^2 - a^2)}{bc(b+c)^2[a + 2\sqrt{(s-b)(s-c)}](s-b)(s-c)]} \cdot (b-c)^2.
\]

**Lemma 2.5.** In \(\triangle ABC\), let
\[
f(a,b,c) = \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} - \frac{1}{r} + k \left( \frac{1}{r} - \frac{2}{R} \right).
\]
Then
\[
f \left( a, \frac{b+c}{2}, \frac{b+c}{2} \right) = \frac{1}{m_1} + \frac{2}{m_2} - \frac{1}{r_0} + k \left( \frac{1}{r_0} - \frac{2}{R_0} \right).
\]
If \(a \geq b \geq c\) and \(0 < k \leq \frac{4}{9}\), then
\[
f(a,b,c) \leq f \left( a, \frac{b+c}{2}, \frac{b+c}{2} \right). \tag{2.12}
\]

**Proof.** If \(a \geq b \geq c\), then
\[
bc - 2a(s-a) = (a-b)(a-c) \geq 0,
\]
thus
\[
a^2 \geq \left( \frac{b+c}{2} \right)^2 \geq bc \geq 2a(s-a),
\]
and for \(0 < k \leq \frac{4}{9}\), hence,
\[
9(b+c)^2 \sqrt{2a(s-a)} - 16(2 - 3k)s^3
\leq 9(b+c)^2 \cdot \frac{b+c}{2} - 16 \left( 2 - 3 \cdot \frac{4}{9} \right) \cdot \frac{1}{8} \left( \frac{b+c}{2} + b + c \right)^3 = 0 \tag{2.13}
\]
and

\[ 8a(s - a)(b^2 + c^2 - a^2) - bc(b + c)^2 \]
\[ \leq 8a(s - a)(b^2 + c^2 - a^2) - 2a(s - a)(b + c)^2 \]
\[ = 2a(s - a)(3b^2 + 3c^2 - 4a^2 - 2bc) \]
\[ = 2a(s - a)[3(b^2 - a^2) + (c^2 - a^2) + 2c(c - b)] \leq 0. \]  \tag{2.14}

From Lemmas 2.3–2.4, inequalities (2.13)–(2.14), we obtain that

\[
\begin{align*}
& f(a, b, c) - f\left(\frac{b + c}{2}, \frac{b + c}{2}\right) \\
& = \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} - \frac{1}{m_1} - \frac{2}{m_2} + (1 - k) \left(\frac{1}{r_0} - \frac{1}{r}\right) + 2k \left(\frac{1}{R_0} - \frac{1}{R}\right) \\
& \leq \frac{\frac{1}{2}as(s - b)(s - c) \cdot s^2[a + 2\sqrt{(s - b)(s - c)}] - \frac{1}{a\sqrt{(s - a)(s - b)(s - c)[a + 2\sqrt{(s - b)(s - c)}]} + 4k\sqrt{s(s - a)(b^2 + c^2 - a^2)(b - c)^2}}{bc(b + c)^2\sqrt{(s - b)(s - c)[a + 2\sqrt{(s - b)(s - c)}]} - \frac{\sqrt{s}[9(b + c)^2\sqrt{2a(s - a)} - 16(2 - 3k)s^3](b - c)^2}{32as^3\sqrt{(s - a)(s - b)(s - c)[a + 2\sqrt{(s - b)(s - c)}]} + \frac{k\sqrt{s}[8a(s - a)(b^2 + c^2 - a^2) - bc(b + c)^2](b - c)^2}{2abc(b + c)^2\sqrt{(s - a)(s - b)(s - c)[a + 2\sqrt{(s - b)(s - c)}]} \leq 0.}
\end{align*}
\]

Therefore, inequality (2.12) holds.

**Lemma 2.6.** (see [6, 11, 12]) Let

\[ F(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \]

and

\[ G(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m. \]

If \( a_0 \neq 0 \) or \( b_0 \neq 0 \), then the polynomials \( F(x) \) and \( G(x) \) have a common root if and only if

\[
R(F, G) := \begin{vmatrix}
  a_0 & a_1 & \cdots & a_n \\
  a_0 & a_1 & \cdots & a_n \\
  \vdots & \vdots & \ddots & \vdots \\
  a_0 & a_1 & \cdots & a_n \\
  b_0 & b_1 & \cdots & b_m \\
  b_0 & b_1 & \cdots & b_m \\
  \vdots & \vdots & \ddots & \vdots \\
  b_0 & b_1 & \cdots & b_m \\
\end{vmatrix} = 0
\]
where \( R(F,G)((m+n) \times (m+n) \) determinant) is Sylvester’s Resultant of \( F(x) \) and \( G(x) \).

**Lemma 2.7.** (see [10, 12]) Given a polynomial \( f(x) \) with real coefficients
\[
f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,
\]
if the number of the sign changes of the revised sign list of its discriminant sequence
\[
\{D_1(f), D_2(f), \ldots, D_n(f)\}
\]
is \( v \), then the number of the pairs of distinct conjugate imaginary roots of \( f(x) \) equals \( v \). Furthermore, if the number of non-vanishing members of the revised sign list is \( l \), then the number of the distinct real roots of \( f(x) \) equals \( l - 2v \).

### 3. The proof of Theorem 1.1

**Proof.** If \( k \leq 0 \), then by inequality (1.2), we can easily find that inequality (1.4) holds. Hence, we only need consider the case \( k > 0 \), and by Lemma 2.2, we only need consider the case \( 0 < k \leq \frac{4}{9} \).

Now we determine the best constant \( k \) such that \( f(a,b,c) \leq 0 \). Since the inequality (1.4) is symmetrical with respect to the side-lengths \( a, b \) and \( c \), there is no harm in supposing \( a \geq b \geq c \). Thus, by Lemma 2.5, we only need to determine the best constant \( k \) such that
\[
f\left( a, \frac{b+c}{2}, \frac{b+c}{2} \right) \leq 0.
\]
or, equivalently, that
\[
\frac{2}{\sqrt{(b+c)^2 - a^2}} + \frac{4}{\sqrt{2a^2 + (\frac{b+c}{2})^2}} - \frac{2s}{a \sqrt{s(s-a)}} + k\left( \frac{2s}{a \sqrt{s(s-a)}} \frac{16 \sqrt{s(s-a)}}{(b+c)^2} \right) \leq 0.
\]
Without loss of generality, we can assume that
\[
a = x \quad \text{and} \quad \frac{b+c}{2} = 1 \quad (1 \leq x < 2),
\]
because the inequality (3.1) is homogeneous with respect to \( a \) and \( \frac{b+c}{2} \). Thus, clearly, the inequality (3.1) is equivalent to the following inequality:
\[
\frac{2}{\sqrt{4 - x^2}} + \frac{4}{\sqrt{2x^2 + 1}} - \frac{2(x+2)}{x \sqrt{4 - x^2}} + k\left[ \frac{2(x+2)}{x \sqrt{4 - x^2}} - 2 \sqrt{4 - x^2} \right] \leq 0.
\]

We consider the following two cases separately.

**Case 1.** When \( x = 1 \), the inequality (3.2) holds true for any \( k \in \mathbb{R} := (-\infty, +\infty) \).

**Case 2.** When \( 1 < x < 2 \), the inequality (3.2) is equivalent to the following inequality:
\[
k \leq \frac{2(x+1)^2}{(x+2) \sqrt{2x^2 + 1}(\sqrt{2x^2 + 1} + x \sqrt{4 - x^2})}.
\]
Define the function

\[ g(x) := \frac{2(x+1)^2}{(x+2)\sqrt{2x^2+1} + (\sqrt{2x^2+1} + x\sqrt{4-x^2})}, \quad x \in (1, 2). \]

Calculating the derivative of \( g(x) \), we get

\[ g'(x) = \frac{2(x+1)[4x^6+12x^5+5x^4-21x^3-28x^2-8-(2x^3+6x^2+7x-3)\sqrt{2x^2+1}\sqrt{4-x^2}]}{(x+2)^2(2x^2+1)^{3/2} \left( \sqrt{2x^2+1} + x\sqrt{4-x^2} \right)^2 \sqrt{4-x^2}}. \]

By setting \( g'(x) = 0 \), we obtain

\[ 4x^6 + 12x^5 + 5x^4 - 21x^3 - 28x^2 - 8 - (2x^3 + 6x^2 + 7x - 3) \sqrt{2x^2 + 1} \sqrt{4 - x^2} = 0. \quad (3.4) \]

It is easy to see that the roots of the equation (3.4) are also solutions of the following equation:

\[ (4x^6 + 12x^5 + 5x^4 - 21x^3 - 28x^2 - 8)^2 - (2x^3 + 6x^2 + 7x - 3)^2(2x^2 + 1)(4 - x^2) = 0, \]

that is

\[ (x-1)(x+2)(x+1)^4 \phi(x) = 0, \quad (3.5) \]

where

\[ \phi(x) = 16x^6 + 16x^5 - 16x^4 - 80x^3 + 5x^2 - 35x - 14. \]

It is obvious that the following equation:

\[ (x-1)(x+2)(x+1)^4 = 0 \quad (3.6) \]

has no real root on the interval \((1, 2)\).

The revised sign list of the discriminant sequence of \( \phi(x) \) is given by

\[ [1, 1, -1, -1, 1, 1]. \quad (3.7) \]

So the number of the sign changes of the revised sign list of (3.7) is 2. Thus, by applying Lemma 2.7, we find that the equation:

\[ \phi(x) = 0 \quad (3.8) \]

has 2 distinct real roots. Moreover, it is not difficult to check that

\[ \phi(-1) = 90 > 0, \]
\[ \phi(0) = -14 < 0, \]
\[ \phi(1) = -108 < 0 \]

and

\[ \phi(2) = 576 > 0. \]
We can conclude that the equation (3.8) has 2 distinct real roots in the following intervals:

\((-1, 0)\) and \((1, 2)\).

So that the equation (3.4) has only one real root \(x_0\) given by \(x_0 = 1.67073609778 \cdots\) in the interval \((1, 2)\), and

\[ g(x)_{\min} = g(x_0) \approx 0.3440653 \in \left(\frac{1}{3}, \frac{2}{5}\right). \]  

(3.9)

Now we prove that \(g(x_0)\) is the root of the equation (1.5). For this purpose, we consider the following system of nonlinear algebraic equations:

\[
\begin{cases}
\phi(x_0) = 0, \\
2x_0^2 + 1 - u_0^2 = 0, \\
4 - x_0^2 - v_0^2 = 0, \\
2(x + 1)^2 - (x + 2) (u_0 + x v_0) k = 0.
\end{cases}
\]  

(3.10)

It is easy to see that \(g(x_0)\) is also the solution of the nonlinear algebraic equation system (3.10). If we eliminate the \(v_0, u_0\) and \(x_0\) ordinal by resultant (by using Lemma 2.6), then we get

\[ 348285173760000 \cdot \phi_1^2(k) \cdot \phi_2^2(k) = 0. \]  

(3.11)

where

\[ \phi_1(k) = 472392 k^6 - 2182626 k^5 + 4000527 k^4 \\
- 4119168 k^3 + 2375744 k^2 - 690176 k + 76800 \]

and

\[ \phi_2(k) = 354294 k^6 - 509571 k^5 + 1927260 k^4 \\
- 2145600 k^3 + 133376 k^2 + 99328 k + 12288. \]

The revised sign list of the discriminant sequence of \(\phi_1(k)\) is given by

\([1, 1, -1, -1, -1, 1].\)  

(3.12)

The revised sign list of the discriminant sequence of \(\phi_2(k)\) is given by

\([1, -1, -1, -1, -1, 1].\)  

(3.13)

So the number of the sign changes of the revised sign list of (3.12) and (3.13) are both 2, Thus, by applying Lemma 2.6, we find that each of the equations:

\[ \phi_1(k) = 0 \]  

(3.14)

and

\[ \phi_2(k) = 0 \]  

(3.15)
has 2 distinct real roots. In addition, it is easy to check that
\[
\phi_1 \left( \frac{1}{5} \right) = \frac{102716437}{15625} > 0; \quad \phi_2 \left( \frac{1}{5} \right) = \frac{26393}{9} > 0,
\]
\[
\phi_1 \left( \frac{1}{3} \right) = -\frac{2381}{3} < 0; \quad \phi_2 \left( \frac{2}{5} \right) = -\frac{287312544}{15625} < 0,
\]
\[
\phi_1 (2) = -356432 < 0; \quad \phi_2 (1) = -128625 < 0
\]
and
\[
\phi_1 (3) = 46208769 > 0; \quad \phi_2 (2) = 20784352 > 0.
\]
We can thus find that the equation (3.14) has 2 distinct real roots in the following intervals:
\[
\left( \frac{1}{5}, \frac{1}{3} \right) \quad \text{and} \quad (2, 3).
\]
And the equation (3.15) has 2 distinct real roots in the following intervals:
\[
\left( \frac{1}{3}, \frac{2}{5} \right) \quad \text{and} \quad (1, 2).
\]
Hence, by (3.9), we can conclude that \( g(x_0) \) is the root of the equation (1.5). The proof of Theorem 1.1 is thus completed.

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