Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 195–212 doi:10.7153/jmi-07-19

POPOVICIU TYPE CHARACTERIZATION OF POSITIVITY OF SUMS AND INTEGRALS FOR CONVEX FUNCTIONS OF HIGHER ORDER

ASIF R. KHAN, JOSIP PEČARIĆ AND SANJA VAROŠANEC

(*Communicated by S. Abramovich*)

Abstract. Some very general identities of Abel and Popoviciu type for sums $\sum p_i f(x_i)$, $\sum\sum p_{ij} f(x_i, y_j)$ and integral $\int\int P(x,y) f(x,y)dxdy$ are deduced. Using obtained identities, positivity of these expressions are characterized for convex functions of higher order. An application in terms of exponential convexity is given.

1. Introduction

Let f be a real-valued function defined on $I = [a,b] \subset \mathbb{R}$. The n-th order divided difference of f at distinct points $x_i, x_{i+1}, \ldots, x_{i+n}$ in I is defined recursively by:

$$[x_j; f] = f(x_j), \quad i \leqslant j \leqslant i+n$$

$$[x_i, \ldots, x_{i+n}; f] = \frac{[x_{i+1}, \ldots, x_{i+n}; f] - [x_i, \ldots, x_{i+n-1}; f]}{x_{i+n} - x_i}.$$

It is easy to see that

$$[x_i, \ldots, x_{i+n}; f] = \sum_{k=0}^{n} \frac{f(x_{i+k})}{\prod\limits_{j=i, j\neq i+k}^{i+n} (x_{i+k} - x_j)}.$$

In this paper $[x_i, \ldots, x_{i+n}; f]$ is denoted by $\Delta^n f(x_i)$.

We say that $f: I \to \mathbb{R}$ is a convex function of order m (or m-convex function) if for all choices of $(n+1)$ distinct points $x_i, \ldots, x_{i+n}$ inequality $\Delta^m f(x_i) \geqslant 0$ holds. The function f is said to be ∇-convex of order m if for all choices of $(n+1)$ distinct points $x_i, \ldots, x_{i+n}$ inequality $\nabla^m f(x_i) = (-1)^m \Delta^m f(x_i) \geqslant 0$ holds.

It is well-known that if $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geqslant 0$.

Mathematics subject classification (2010): 26A51, 39B62, 26D15, 26D20, 26D99.
Keywords and phrases: Convex function, ∇-convex function, exponential-convexity, mean.
This research work is funded by Higher Education Commission Pakistan. The research of the second and the third authors was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 117-1170889-0888 and 058-1170889-1050.

Let f be a real-valued function defined on $I\times J$, $I=[a,b]$, $J=[c,d]$. Then the (n,m) divided difference of the function f at distinct points $x_i,\dots,x_{i+n}\in I$, $y_j,\dots,y_{j+m}\in J$ is defined by

$$\Delta^n_m f(x_i,y_j)=[x_i,\dots,x_{i+n};[y_j,\dots,y_{j+m};f]].$$

A function $f:I\times J\to\mathbb{R}$ is said to be convex of order (n,m) or (n,m)-convex if inequality

$$\Delta^n_m f(x_i,y_j)\geqslant 0$$

holds for all distinct points $x_i,\dots,x_{i+n}\in I$, $y_j,\dots,y_{j+m}\in J$.

It is known that if the partial derivative $\frac{\partial^{n+m}f}{\partial x^n\partial y^m}$ exists, then f is (n,m)-convex iff $\frac{\partial^{n+m}f}{\partial x^n\partial y^m}\geqslant 0$. For some other results about convex functions of higher order see the book [8].

Let us describe the structure of the paper. After brief introduction, we consider identities for sum $\sum_{i=1}^N p_i f(x_i)$ which involve divided differences $\Delta^n f$ and $\nabla^n f$. These identities are basic tools for getting necessary and sufficient conditions that inequality $\sum_{i=1}^N p_i f(x_i)\geqslant 0$ holds for every n-convex function or ∇-convex function of higher order. In the third section we obtain an identity for sum $\sum_{i=1}^N\sum_{j=1}^M p_{ij}f(x_i,y_j)$ and investigate the inequality $\sum_{i=1}^N\sum_{j=1}^M p_{ij}f(x_i,y_j)\geqslant 0$ for (n,m)-convex function of two variables. The fourth section is devoted to the integral case. We consider an identity for double integral $\Lambda(f)=\int\int P(x,y)f(x,y)dxdy$ and related inequality. Finally, we consider a functional $\Lambda(f)$, apply it on the family of certain exponentially convex functions $\varphi^{(p)}$ and give some properties of it.

2. Discrete case for function of one variable

In papers [3] and [4] the following results for a real sequence (a_n) were proved:

$$\sum_{i=1}^n p_i a_i=\sum_{k=0}^{m-1}\frac{1}{k!}\Delta^k a_1\sum_{i=k+1}^n(i-1)^{(k)}p_i+\frac{1}{(m-1)!}\sum_{k=m+1}^n\Big(\sum_{i=k}^n(i-k+m-1)^{(m-1)}p_i\Big)\Delta^m a_{k-m}\tag{2.1}$$

and

$$\sum_{i=1}^n p_i a_i=\sum_{k=0}^{m-1}\frac{1}{k!}\nabla^k a_{n-k}\sum_{i=1}^{n-k}(n-i)^{(k)}p_i+\frac{1}{(m-1)!}\sum_{k=1}^{n-m}\Big(\sum_{i=1}^{k}(k-i+m-1)^{(m-1)}p_i\Big)\nabla^m a_k\tag{2.2}$$

where $a^{(k)}=a(a-1)\dots(a-k+1)$, $a^{(0)}=1$, $\Delta^k a_i=k!\Delta^k a(i)$ and $\nabla^k a_i=k!\nabla^k a(i)$, where a is a function $a:i\mapsto a_i$, and p_i, $(i=1,2,\dots,n)$ are real numbers.

Similar result for the real function was proved by Popoviciu [9] and it is a generalization of (2.1). Namely, he proved that if f is a real function defined on I, $x_1,\dots,x_n$ distinct numbers from I and p_i, $(i=1,2,\dots,n)$ are real numbers, then

$$\sum_{i=1}^{n} p_i f(x_i) = \sum_{k=0}^{m-1}\Big(\sum_{i=k+1}^{n} p_i (x_i-x_1)^{(k)}\Big)\Delta^k f(x_1) + \sum_{k=m+1}^{n}\Big(\sum_{i=k}^{n} p_i (x_i-x_{k-m+1})^{(m-1)}\Big)\Delta^m f(x_{k-m})(x_k-x_{k-m}) \tag{2.3}$$

where $(x_i-x_k)^{(n+1)} = (x_i-x_k)(x_i-x_{k+1})\dots(x_i-x_{k+n})$ for $n\geqslant 0$ and $(x_i-x_k)^{(0)}=1$.

Now, let us prove an identity which is a generalization of formula (2.2). In fact, it is a formula which is similar to Popoviciu's result (2.3), but involving the operator ∇.

LEMMA 2.1. *Let m,n be integers $m\leqslant n$ and p_i, $(i=1,2,\dots,n)$ are real numbers. Let a function f be defined on I and let x_i, $(i=1,2,\dots,n)$, be mutually different elements from I. Then the following identity holds:*

$$\sum_{i=1}^{n} p_i f(x_i) = \sum_{k=0}^{m-1}\Big(\sum_{j=1}^{n-k} p_j (x_n-x_j)^{\{k\}}\Big)\nabla^k f(x_{n-k}) + \sum_{k=1}^{n-m}\Big(\sum_{j=1}^{k} p_j (x_{k+m-1}-x_j)^{\{m-1\}}\Big)\nabla^m f(x_k)(x_{k+m}-x_k) \tag{2.4}$$

where $(x_n-x_j)^{\{k+1\}} = (x_n-x_j)(x_{n-1}-x_j)\dots(x_{n-k}-x_j)$ *for* $k\geqslant 0$ *and* $(x_n-x_j)^{\{0\}}=1$.

Proof. For $m=1$ we have

$$\sum_{i=1}^{n} p_i f(x_i) = \sum_{j=1}^{n} p_j f(x_n) + \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{k} p_j\Big)(f(x_k)-f(x_{k+1}))$$

what is true.

Suppose that (2.4) is valid. Then

$$\begin{aligned}
&\sum_{k=0}^{m}\left(\sum_{j=1}^{n-k} p_j (x_n-x_j)^{\{k\}}\right)\nabla^k f(x_{n-k})\\
&\quad+\sum_{k=1}^{n-m-1}\left(\sum_{j=1}^{k} p_j (x_{k+m}-x_j)^{\{m\}}\right)\nabla^{m+1} f(x_k)(x_{k+m+1}-x_k)\\
&=A+\sum_{j=1}^{n-m} p_j (x_n-x_j)^{\{m\}}\nabla^m f(x_{n-m})\\
&\quad+\sum_{k=1}^{n-m-1} B(-1)^{m+1}([x_{k+1},\dots,x_{k+m+1};f]-[x_k,\dots,x_{k+m};f])
\end{aligned}$$

$$
\begin{aligned}
&= A+\sum_{j=1}^{n-m} p_j(x_n-x_j)^{\{m\}}\nabla^m f(x_{n-m})\\
&\quad+\sum_{j=1}^{n-m-1} p_j(x_{n-1}-x_j)^{\{m\}}(-1)^{m+1}[x_{n-m},\dots,x_n;f]\\
&\quad+\sum_{k=1}^{n-m-2} B(-1)^{m+1}[x_{k+1},\dots,x_{k+m+1};f]-\sum_{k=2}^{n-m-1} B(-1)^{m+1}[x_k,\dots,x_{k+m};f]\\
&\quad-p_1(x_{m+1}-x_1)^{\{m\}}(-1)^{m+1}[x_1,\dots,x_{1+m};f]\\
&= A+\sum_{j=1}^{n-m} p_j(x_{n-1}-x_j)^{\{m-1\}}\nabla^m f(x_{n-m})(x_n-x_m)\\
&\quad+\sum_{k=2}^{n-m-1}(-1)^m[x_k,\dots,x_{k+m};f](\sum_{j=1}^{k} p_j(x_{k+m}-x_j)^{\{m\}}\\
&\quad-\sum_{j=1}^{k-1} p_j(x_{k+m-1}-x_j)^{\{m\}})+p_1(x_{m+1}-x_1)^{\{m\}}\nabla^m f(x_1)\\
&= A+\sum_{j=1}^{n-m} p_j(x_{n-1}-x_j)^{\{m-1\}}\nabla^m f(x_{n-m})(x_n-x_m)\\
&\quad+\sum_{k=2}^{n-m-1}(\sum_{j=1}^{k} p_j(x_{k+m-1}-x_j)^{\{m-1\}})\nabla^m f(x_k)\\
&\quad+p_1(x_m-x_1)^{\{m-1\}}\nabla^m f(x_1)(x_{m+1}-x_1)=\sum_{i=1}^{n} p_i f(x_i).
\end{aligned}
$$

where $A=\sum_{k=0}^{m-1}(\sum_{j=1}^{n-k} p_j(x_n-x_j)^{\{k\}})\nabla^k f(x_{n-k})$ and $B=\sum_{j=1}^{k} p_j(x_{k+m}-x_j)^{\{m\}}$. Thus, identity (2.4) is proved. □

From identity (2.4) we can obtain the following result about necessary and sufficient conditions that inequality $\sum_{i=1}^{n} p_i f(x_i)\geqslant 0$ holds for every ∇-convex function of order m.

THEOREM 2.2. *Let assumptions of Lemma 2.1 are valid and $x_1<x_2<\ldots<x_n$. Inequality*

$$\sum_{i=1}^{n} p_i f(x_i)\geqslant 0$$

holds for every ∇-convex function f of order m iff

$$\sum_{j=1}^{n-k} p_j(x_n-x_j)^{\{k\}}=0 \quad k=0,\dots,m-1, \tag{2.5}$$

$$\sum_{j=1}^{k} p_j(x_{k+m-1}-x_j)^{\{m-1\}}\geqslant 0 \quad k=1,\dots,n-m. \tag{2.6}$$

Proof. If inequalities (2.5) and (2.6) are satisfied, then the first sum in identity (2.4) is equal to 0, the second sum is nonnegative and the inequality $\sum_{i=1}^{n} p_i f(x_i) \geqslant 0$ holds.

If for all ∇-convex functions of order m inequality $\sum_{i=1}^{n} p_i f(x_i) \geqslant 0$ holds, then we consider the functions $f_1(x) = x^r$ and $f_2(x) = -x^r$, $r \leqslant m-1$. Functions f_1 and f_2 are ∇-convex functions of order m and for $r \leqslant m-1$ we have

$$\sum_{i=1}^{n} p_i x_i^r = 0.$$

From this equality we obtain (2.5). For every $k \in \{1, \dots, n-m\}$, $m > 1$, the function

$$f_k(x) = \begin{cases} (x_{k+1} - x) \dots (x_{k+m-1} - x), & x < x_{k+1} \\ 0, & x \geqslant x_{k+1} \end{cases}$$

is ∇-convex of order m and using these facts we obtain (2.6). □

The next theorem is a generalization of the result from [6, pp. 121–122].

THEOREM 2.3. *Let the assumptions of Lemma 2.1 be valid and* $x_1 < x_2 < \ldots < x_n$.

a) Inequality

$$\sum_{i=1}^{n} p_i f(x_i) \geqslant 0$$

holds for every convex function f *of order* $j, j+1, \ldots, m$, $(j = 0, 1, 2, \ldots, m)$ *iff*

$$\sum_{i=k+1}^{n} p_i (x_i - x_1)^{(k)} = 0, \qquad k = 0, ..., j-1, \tag{2.7}$$

$$\sum_{i=k+1}^{n} p_i (x_i - x_1)^{(k)} \geqslant 0, \qquad k = j, ..., m-1, \tag{2.8}$$

$$\sum_{i=k}^{n} p_i (x_i - x_{k-m+1})^{(m-1)} \geqslant 0, \qquad k = m+1, \ldots, n. \tag{2.9}$$

If $j = 0$ *(or* $j = m$*), condition (2.7) (or (2.8)) can be omitted.*

b) Inequality

$$\sum_{i=1}^{n} p_i f(x_i) \geqslant 0$$

holds for every ∇*-convex function f of order* $j, j+1, \ldots, m$, $(j = 0, 1, \ldots, m)$ *iff*

$$\sum_{i=1}^{n-k} p_i (x_n - x_i)^{\{k\}} = 0, \qquad k = 0, \ldots, j-1, \tag{2.10}$$

$$\sum_{i=1}^{n-k} p_i(x_n - x_i)^{\{k\}} \geqslant 0, \qquad k = j, \dots, m-1, \tag{2.11}$$

$$\sum_{i=1}^{k} p_i(x_{k+m-1} - x_i)^{\{m-1\}} \geqslant 0, \qquad k = 1, \dots, n-m. \tag{2.12}$$

For $j = 0$ (or $j = m$), condition (2.10) (or (2.11)) can be omitted.

The proof is similar to the proof of Theorem 2.2 and we omit it. The result for the special case $f(x_i) = a_i$ can be found in [7], see also [8, p. 257].

3. Discrete case for function of two variables

Let us now consider a function of two variables defined on $I \times J$. Firstly, we obtain an identity for $\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j)$ which involves divided differences and then, in the next theorem, we consider necessary and sufficient conditions that inequality

$$\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) \geqslant 0$$

holds for every convex function of order (n,m).

THEOREM 3.1. *Let $x_1, \dots, x_N$ be mutually distinct numbers from $I = [a,b]$ and $y_1, \dots, y_M$ be mutually distinct numbers from $J = [c,d]$ and let f be a real-valued functions on $I \times J$. Let p_{ij}, $(i = 1, \dots, N)$, $(j = 1, \dots, M)$, be real numbers.*

Then the following identity holds:

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) \qquad (3.1)\\
&= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)} \Delta_k^t f(x_1, y_1)\\
&\quad + \sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_1)^{(k)} \Delta_k^n f(x_{t-n}, y_1)(x_t - x_{t-n})\\
&\quad + \sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_{k-m+1})^{(m-1)} \Delta_m^t f(x_1, y_{k-m})(y_k - y_{k-m})\\
&\quad + \sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\Big(\sum_{s=t}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_{k-m+1})^{(m-1)}\Big)\\
&\quad \times \Delta_m^n f(x_{t-n}, y_{k-m})(x_t - x_{t-n})(y_k - y_{k-m}).
\end{aligned}$$

Proof. We have

$$\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) = \sum_{i=1}^{N}\left(\sum_{j=1}^{M} q_j G_i(y_j)\right),$$

where $p_{ij} = q_j$ and $G_i : y \mapsto f(x_i, y)$. Using (2.3) on the inner sum we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) \\
&= \sum_{i=1}^{N}\sum_{k=0}^{m-1}\left(\sum_{j=k+1}^{M} q_j (y_j - y_1)^{(k)}\right)\Delta^k G_i(y_1) \\
&\quad + \sum_{i=1}^{N}\sum_{k=m+1}^{M}\left(\sum_{j=k}^{M} q_j (y_j - y_{k-m+1})^{(m-1)}\right)\Delta^m G_i(y_{k-m})(y_k - y_{k-m}) \\
&= \sum_{k=0}^{m-1}\left(\sum_{i=1}^{N}\left(\sum_{j=k+1}^{M} q_j (y_j - y_1)^{(k)}\right)\Delta^k G_i(y_1)\right) \\
&\quad + \sum_{k=m+1}^{M}\left(\sum_{i=1}^{N}\left(\sum_{j=k}^{M} q_j (y_j - y_{k-m+1})^{(m-1)}(y_k - y_{k-m})\right)\Delta^m G_i(y_{k-m})\right) \\
&= \sum_{k=0}^{m-1}\left(\sum_{i=1}^{N} w_i F(x_i)\right) + \sum_{k=m+1}^{M}\left(\sum_{i=1}^{N} v_i H(x_i)\right)
\end{aligned}$$

where $w_i = \sum_{j=k+1}^{M} q_j (y_j - y_1)^{(k)} = \sum_{j=k+1}^{M} p_{ij}(y_j - y_1)^{(k)}$, $F(x_i) = \Delta^k G_i(y_1)$, $v_i = \sum_{j=k}^{M} q_j (y_j - y_{k-m+1})^{(m-1)}(y_k - y_{k-m})$ and $H(x_i) = \Delta^m G(y_{k-m})$.

Applying again (2.3) on the inner sums we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) \\
&= \sum_{k=0}^{m-1}\sum_{r=0}^{n-1}\left(\sum_{i=r+1}^{N} w_i (x_i - x_1)^{(r)}\right)\Delta^r F(x_1) \\
&\quad + \sum_{k=0}^{m-1}\sum_{r=n+1}^{N}\left(\sum_{i=r}^{N} w_i (x_i - x_{r-n+1})^{(n-1)}\right)\Delta^n F(x_{r-n})(x_r - x_{r-n}) \\
&\quad + \sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{i=t+1}^{N} v_i (x_i - x_1)^{(t)}\Delta^t H(x_1) \\
&\quad + \sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{i=t}^{N} v_i (x_i - x_{t-n+1})^{(n-1)}\Delta^n H(x_{t-n})(x_t - x_{t-n}) \\
&= \sum_{k=0}^{m-1}\sum_{r=0}^{n-1}\sum_{i=r+1}^{N}\left(\sum_{j=k+1}^{M} p_{ij}(y_j - y_1)^{(k)}\right)(x_i - x_1)^{(r)}\Delta_k^r f(x_1, y_1) \\
&\quad + \sum_{k=0}^{m-1}\sum_{r=n+1}^{N}\sum_{i=r}^{N}\left(\sum_{j=k+1}^{M} p_{ij}(y_j - y_1)^{(k)}\right)(x_i - x_{r-n+1})^{(n-1)}\Delta_k^n f(x_{r-n}, y_1)(x_r - x_{r-n}) \\
&\quad + \sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{i=t+1}^{N}\sum_{j=k}^{M} p_{ij}(y_j - y_{k-m+1})^{(m-1)}(y_k - y_{k-m})(x_i - x_1)^{(t)}\Delta_m^t f(x_1, y_{k-m})
\end{aligned}$$

$$+\sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{i=t}^{N}\sum_{j=k}^{M}p_{ij}(y_j-y_{k-m+1})^{(m-1)}(y_k-y_{k-m})(x_i-x_{t-n+1})^{(n-1)}$$
$$\times\Delta_m^n f(x_{t-n},y_{k-m})(x_t-x_{t-n}).$$

If we substitute in the first and in the second sums $r\to t$, and in all sums change $i\to s$, $j\to r$, we get the identity (3.1). □

THEOREM 3.2. *Let p_{ij}, $(i=1,\dots,N)$, $(j=1,\dots,M)$, be real numbers and f be a real function defined on $I\times J$. Let $x_1<x_2<\dots<x_N$, $x_i\in I$, $y_1<y_2<\dots y_M$, $y_j\in J$.*

Inequality

$$\sum_{i=1}^{N}\sum_{j=1}^{M}p_{ij}f(x_i,y_j)\geqslant 0$$

holds for every convex function f of order (n,m) iff

$$\sum_{s=t+1}^{N}\sum_{r=k+1}^{M}p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}=0 \quad \begin{array}{l}k=0,\dots,m-1\\ t=0,\dots,n-1\end{array}$$
$$\sum_{s=t}^{N}\sum_{r=k+1}^{M}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}=0 \quad \begin{array}{l}k=0,\dots,m-1\\ t=n+1,\dots,N\end{array}$$
$$\sum_{s=t+1}^{N}\sum_{r=k}^{M}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}=0 \quad \begin{array}{l}k=m+1,\dots,M\\ t=0,\dots,n-1\end{array}$$
$$\sum_{s=t}^{N}\sum_{r=k}^{M}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\geqslant 0 \quad \begin{array}{l}k=m+1,\dots,M\\ t=n+1,\dots,N.\end{array}$$

Proof. The proof is similar to the proof of Theorem 2.2. □

REMARK 3.3. The case when $f(x_i,y_j)=a_{ij}$ and $m=n=1$ was considered in [5]. The case when $f(x_i,y_j)=a_ib_j$, where (a_i) is an n-convex sequence and (b_j) is an m-convex sequence was researched in [7].

4. Integral case for a function of two variables

In this section we consider a function of two variables defined on $I\times J=[a,b]\times[c,d]$. Also, throughout this section $n,m,N,M\in\mathbb{N}\cup\{0\}$ and the notation for a partial derivative $\frac{\partial^{n+m}f}{\partial x^n\partial y^m}$ is $f_{(n,m)}$. In [5] the following theorem is given:

THEOREM 4.1. *Let $P,f:[a,b]\times[a,b]\to R$ be integrable functions, if f has the continuous partial derivatives $f_{(1,0)}$, $f_{(0,1)}$ and $f_{(1,1)}$ on $[a,b]\times[a,b]$ then*

$$\int_a^b\int_a^b P(x,y)f(x,y)dxdy=f(a,a)P_1(a,a)+\int_a^b P_1(x,a)f_{(1,0)}(x,a)dx$$
$$+\int_a^b P_1(a,y)f_{(0,1)}(a,y)dy+\int_a^b\int_a^b P_1(x,y)f_{(1,1)}(x,y)dxdy$$

where

$$P_1(x,y) = \int_x^b \int_y^b P(s,t)\,dt\,ds,$$

$$f_{(1,0)} = \frac{\partial f}{\partial x},\ f_{(0,1)} = \frac{\partial f}{\partial y}\ \text{ and }\ f_{(1,1)} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Now we give the generalization of the previous theorem using higher derivatives.

THEOREM 4.2. *Let $P, f : I \times J \to R$ be integrable functions and f has the continuous partial derivatives $f_{(i,j)}$ on $I \times J$ for $i = 0,1,...,N+1$ and $j = 0,1,...,M+1$, then we have*

$$\begin{aligned}
&\int_a^b \int_c^d P(x,y) f(x,y)\,dy\,dx \\
&= \sum_{i=0}^{N} \sum_{j=0}^{M} \int_a^b \int_c^d P(s,t) f_{(i,j)}(a,c) \frac{(s-a)^i}{i!} \frac{(t-c)^j}{j!}\,dt\,ds \\
&+ \sum_{j=0}^{M} \int_a^b \int_x^b \int_c^d P(s,t) f_{(N+1,j)}(x,c) \frac{(s-x)^N}{N!} \frac{(t-c)^j}{j!}\,dt\,ds\,dx \\
&+ \sum_{i=0}^{N} \int_c^d \int_a^b \int_y^d P(s,t) f_{(i,M+1)}(a,y) \frac{(s-a)^i}{i!} \frac{(t-y)^M}{M!}\,dt\,ds\,dy \\
&+ \int_a^b \int_c^d \int_x^b \int_y^d P(s,t) f_{(N+1,M+1)}(x,y) \frac{(s-x)^N}{N!} \frac{(t-y)^M}{M!}\,dt\,ds\,dy\,dx. \qquad (4.1)
\end{aligned}$$

Proof. Let $G(y) = f(x,y)$, i.e. we consider a function $f(x,y)$ as a function of variable y. Then a function G can be represented as

$$\begin{aligned}
f(x,y) = G(y) &= \sum_{j=0}^{M} G^{(j)}(c) \frac{(y-c)^j}{j!} + \int_c^y G^{M+1}(t) \frac{(y-t)^M}{M!}\,dt \\
&= \sum_{j=0}^{M} f_{(0,j)}(x,c) \frac{(y-c)^j}{j!} + \int_c^y f_{(0,M+1)}(x,t) \frac{(y-t)^M}{M!}\,dt,
\end{aligned}$$

where we use the facts that $G^{(j)}(c) = f_{(0,j)}(x,c)$ and $G^{(M+1)}(t) = f_{(0,M+1)}(x,t)$.

Multiply the above formula with $P(x,y)$ and integrate it over $[c,d]$ by variable y. Then we have

$$\begin{aligned}
\int_c^d P(x,y) f(x,y)\,dy = &\sum_{j=0}^{M} f_{(0,j)}(x,c) \int_c^d P(x,y) \frac{(y-c)^j}{j!}\,dy \qquad (4.2) \\
&+ \int_c^d \left(\int_c^y P(x,y) f_{(0,M+1)}(x,t) \frac{(y-t)^M}{M!}\,dt \right) dy.
\end{aligned}$$

Let us represent the functions $x \mapsto f_{(0,j)}(x,c)$ and $x \mapsto f_{(0,M+1)}(x,t)$ using Taylor expansions:

$$f_{(0,j)}(x,c) = \sum_{i=0}^{N} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!} + \int_a^x f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}ds,$$

$$f_{(0,M+1)}(x,t) = \sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!} + \int_a^x f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds.$$

Putting these two formulae in (4.2) we get

$$\begin{aligned}
&\int_c^d P(x,y)f(x,y)dy \\
&= \sum_{j=0}^{M}\left(\sum_{i=0}^{N} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!} + \int_a^x f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy \\
&\quad + \int_c^d \left(\int_c^y P(x,y)\left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right.\right. \\
&\quad \left.\left. + \int_a^x f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(y-t)^M}{M!}dt\right)dy \\
&= \sum_{j=0}^{M}\left(\sum_{i=0}^{N} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy \\
&\quad + \sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy \\
&\quad + \int_c^d\int_c^y P(x,y)\left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(y-t)^M}{M!}dt\,dy \\
&\quad + \int_c^d\int_c^y\left(\int_a^x P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(y-t)^M}{M!}dt\,dy.
\end{aligned}$$

Now, we integrate over $[a,b]$ by variable x and get:

$$\begin{aligned}
&\int_a^b\int_c^d P(x,y)f(x,y)dy\,dx \\
&= \int_a^b\left[\sum_{j=0}^{M}\left(\sum_{i=0}^{N} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx \\
&\quad + \int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx \\
&\quad + \int_a^b\left[\int_c^d\int_c^y P(x,y)\left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(y-t)^M}{M!}dt\,dy\right]dx \\
&\quad + \int_a^b\left[\int_c^d\int_c^y\left(\int_a^x P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(y-t)^M}{M!}dt\,dy\right]dx.
\end{aligned}$$

In the first summand we change the order of summation, use the linearity of the integral and get

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_c^d P(x,y)f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\frac{(y-c)^j}{j!}dy\,dx.$$

The second summand is rewriten as

$$\begin{aligned}
&\int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx\\
&=\int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x\int_c^d P(x,y)\frac{(y-c)^j}{j!}f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}dy\,ds\right)\right]dx\\
&=\sum_{j=0}^{M}\int_a^b\int_a^x\int_c^d P(x,y)f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}\frac{(y-c)^j}{j!}dy\,ds\,dx\\
&=\sum_{j=0}^{M}\int_a^b\int_s^b\int_c^d P(x,y)f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}\frac{(y-c)^j}{j!}dy\,dx\,ds,
\end{aligned}$$

where in the last equation we use the Fubini theorem for the variables s and x. Let us point out, that firstly, the variable x is changed from a to b while the variable s is changed from a to x. After changing the order of integration we have that variable s is changed from a to b while the variable x is changed from s to b.

Similarly, the third summand is rewriten as:

$$\begin{aligned}
&\int_a^b\left[\int_c^d\int_c^y P(x,y)\left(\sum_{i=0}^{N}f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(y-t)^M}{M!}dt\,dy\right]dx\\
&=\sum_{i=0}^{N}\int_a^b\int_c^d\int_c^y P(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(y-t)^M}{M!}dt\,dy\,dx\\
&=\sum_{i=0}^{N}\int_a^b\int_c^d\int_t^d P(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(y-t)^M}{M!}dy\,dt\,dx\\
&=\sum_{i=0}^{N}\int_c^d\int_a^b\int_t^d P(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(y-t)^M}{M!}dy\,dx\,dt,
\end{aligned}$$

where we use the Fubini theorem twice, firstly for changing t and y, and then for t and x.

The fourth summand is rewriten as:

$$\begin{aligned}
&\int_a^b\left[\int_c^d\int_c^y\left(\int_a^x P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(y-t)^M}{M!}dt\,dy\right]dx\\
&=\int_a^b\int_c^d\int_c^y\int_a^x P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}\frac{(y-t)^M}{M!}ds\,dt\,dy\,dx\\
&=\int_a^b\int_c^d\int_s^b\int_t^d P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}\frac{(y-t)^M}{M!}dy\,dx\,dt\,ds,
\end{aligned}$$

where we use the Fubini theorem several times. Firstly, we change t and y, then y and s, then s and t, then s and x, then t and x.

Using all these results we get

$$\begin{aligned}
&\int_a^b \int_c^d P(x,y)f(x,y)dy\,dx \\
&= \sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b \int_c^d P(x,y)f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\frac{(y-c)^j}{j!}dy\,dx \\
&\quad + \sum_{j=0}^{M}\int_a^b \int_s^b \int_c^d P(x,y)f_{(N+1,j)}(s,c)\frac{(x-s)^N}{N!}\frac{(y-c)^j}{j!}dy\,dx\,ds \\
&\quad + \sum_{i=0}^{N}\int_c^d \int_a^b \int_t^d P(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(y-t)^M}{M!}dy\,dx\,dt \\
&\quad + \int_a^b \int_c^d \int_s^b \int_t^d P(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}\frac{(y-t)^M}{M!}dy\,dx\,dt\,ds.
\end{aligned}$$

It is, in fact, the statement of Theorem 4.2 when we change the names of variables on the right side: $x \leftrightarrow s$, $y \leftrightarrow t$. □

Using results of the previous theorem we obtain necessary and sufficient conditions that inequality $\Lambda(f) \geqslant 0$ holds for every (n,m)-convex two-variables function.

THEOREM 4.3. *The inequality*

$$\Lambda(f) = \int_a^b \int_c^d P(x,y)f(x,y)dy\,dx \geqslant 0 \tag{4.3}$$

holds for every function whose continuous partial derivative $f_{(N+1,M+1)} \geqslant 0$ *on* $I \times J$ *iff*

$$\int_a^b \int_c^d P(s,t)\frac{(s-a)^i}{i!}\frac{(t-c)^j}{j!}dt\,ds = 0, \quad i = 0,\dots,N;\ j = 0,\dots,M \tag{4.4}$$

$$\int_x^b \int_c^d P(s,t)\frac{(s-x)^N}{N!}\frac{(t-c)^j}{j!}dt\,ds = 0, \quad j = 0,\dots,M;\ \forall\, x \in [a,b] \tag{4.5}$$

$$\int_a^b \int_y^d P(s,t)\frac{(s-a)^i}{i!}\frac{(t-y)^M}{M!}dt\,ds = 0, \quad i = 0,\dots,N;\ \forall\, y \in [c,d] \tag{4.6}$$

$$\int_x^b \int_y^d P(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds \geqslant 0,\ \forall\, x \in [a,b];\ \forall\, y \in [c,d]. \tag{4.7}$$

Proof. If (4.4), (4.5) and (4.6) hold then the first three sums are zero in (4.1) and the required inequality (4.3) holds by using (4.7).

Conversely, if we consider in (4.3) the following functions

$$f^{(1)}(s,t) = \frac{(s-a)^n}{n!}\frac{(t-c)^m}{m!}$$

$$f^{(2)}(s,t) = -\frac{(s-a)^n}{n!}\frac{(t-c)^m}{m!}$$

for $0 \leqslant n \leqslant N$ and $0 \leqslant m \leqslant M$ such that $f^{(k)}_{(N+1,M+1)}(s,t) \geqslant 0,\ k = 1,2$; then we get the required equality (4.4) i.e.

$$\int_a^b \int_c^d P(s,t)\frac{(s-a)^n}{n!}\frac{(t-c)^m}{m!}dt\,ds = 0, \quad 0 \leqslant n \leqslant N; \quad 0 \leqslant m \leqslant M.$$

In the same way if we consider in (4.3) the following functions for $0 \leqslant m \leqslant M$, $\forall x \in [a,b]$ and $t \in [c,d]$

$$f^{(3)}(s,t) = \begin{cases} \frac{(s-x)^N}{N!}\frac{(t-c)^m}{m!}, & x < s \\ 0, & x \geqslant s \end{cases}$$

$$f^{(4)}(s,t) = \begin{cases} -\frac{(s-x)^N}{N!}\frac{(t-c)^m}{m!}, & x < s \\ 0, & x \geqslant s \end{cases}$$

such that $f^{(k)}_{(N+1,M+1)}(s,t) \geqslant 0, \quad k = 3,4$, then we get the required equality (4.5) i.e.

$$\int_x^b \int_c^d P(s,t)\frac{(s-x)^N}{N!}\frac{(t-c)^m}{m!}dtds = 0, \quad 0 \leqslant m \leqslant M;\ \forall\ x \in [a,b].$$

Similarly, if we consider in (4.3) the following functions for $0 \leqslant n \leqslant N, \quad \forall\ y \in [c,d]$ and $s \in [a,b]$

$$f^{(5)}(s,t) = \begin{cases} \frac{(s-a)^n}{n!}\frac{(t-y)^M}{M!}, & y < t \\ 0, & y \geqslant t \end{cases}$$

$$f^{(6)}(s,t) = \begin{cases} -\frac{(s-a)^n}{n!}\frac{(t-y)^M}{M!}, & y < t \\ 0, & y \geqslant t \end{cases}$$

such that $f^{(k)}_{(N+1,M+1)}(s,t) \geqslant 0,\ k = 5,6$, then we get the required equality (4.6) i.e.

$$\int_a^b \int_y^d P(s,t)\frac{(s-a)^n}{n!}\frac{(t-y)^M}{M!}dtds = 0, \quad 0 \leqslant n \leqslant N;\ \forall\ y \in [c,d].$$

The last inequality (4.7) is followed by considering the following function in (4.3) for $x \in [a,b]$, $y \in [c,d]$

$$f(s,t) = \begin{cases} \frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}, & x < s, \quad y < t \\ 0, & x \geqslant s \ \text{ or } \ y \geqslant t. \end{cases} \qquad \square$$

THEOREM 4.4. *Let $f \in C^{(n+1,m+1)}(I \times J)$ and $P : I \times J \to R$ be an integrable function. Let $\Lambda : C(I \times J) \to \mathbb{R}$ be a linear functional defined in (4.3) and let the conditions (4.4), (4.5), (4.6), and (4.7) of Theorem 4.3 for function P be satisfied. Then there exists $(\xi,\eta) \in I \times J$ such that*

$$\Lambda(f) = f_{(n+1,m+1)}(\xi,\eta)\Lambda(G_0) \tag{4.8}$$

where $G_0(x,y) = \frac{x^{n+1}}{(n+1)!}\frac{y^{m+1}}{(m+1)!}$.

Proof. Let $L = \min_{(x,y)\in I\times J} f_{(n+1,m+1)}(x,y)$, $U = \max_{(x,y)\in I\times J} f_{(n+1,m+1)}(x,y)$. Then the function

$$G(x,y) = U\frac{x^{n+1}}{(n+1)!}\frac{y^{m+1}}{(m+1)!} - f(x,y) = UG_0(x,y) - f(x,y)$$

gives us

$$G_{(n+1,m+1)}(x,y) = U - f_{(n+1,m+1)}(x,y) \geqslant 0$$

i.e. G is $(n+1,m+1)$-convex function. Hence $\Lambda(G)\geqslant 0$ by Theorem 4.3 and we conclude that

$$\Lambda(f) \leqslant U\Lambda(G_0).$$

Similarly

$$L\Lambda(G_0) \leqslant \Lambda(f).$$

Combining the two inequalities we get

$$L\Lambda(G_0) \leqslant \Lambda(f) \leqslant U\Lambda(G_0)$$

which gives us (4.8). □

THEOREM 4.5. *Let $f,g \in C^{(n+1,m+1)}(I\times J)$ and let $\Lambda : C(I\times J)\to\mathbb{R}$ be a linear functional as defined in (4.3) and let the conditions (4.4), (4.5), (4.6), and (4.7) of Theorem 4.3 for function P be satisfied. Then there exists $(\xi,\eta)\in I\times J$ such that*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(n+1,m+1)}(\xi,\eta)}{g_{(n+1,m+1)}(\xi,\eta)}$$

assuming that both the denominators are non-zero.

Proof. Let $h\in C^{(n+1,m+1)}(I\times J)$ be defined as

$$h = \Lambda(g)f - \Lambda(f)g.$$

Using Theorem 4.4 there exists (ξ,η) such that

$$0 = \Lambda(h) = h_{(n+1,m+1)}(\xi,\eta)\Lambda(G_0)$$

or

$$[\Lambda(g)f_{(n+1,m+1)}(\xi,\eta) - \Lambda(f)g_{(n+1,m+1)}(\xi,\eta)]\Lambda(G_0) = 0$$

which gives us required result. □

REMARK 4.6. The case when $n = m = 1$ was considered in the paper [5].

COROLLARY 4.7. *Let $\Lambda : C(I\times J) \to \mathbb{R}$ be a linear functional as defined in (4.3) and let the conditions (4.4), (4.5), (4.6), and (4.7) of Theorem 4.3 for function P be satisfied with $M = N = n$. Then there exists $(\xi,\eta) \in I\times J$ such that*

$$(\xi\eta)^{p-q} = \frac{[(q+1)q...(q-n+1)]^2\Lambda((xy)^{p+1})}{[(p+1)p...(p-n+1)]^2\Lambda((xy)^{q+1})}$$

for $-\infty < p \neq q < +\infty$ and $p,q \notin \{-1,0,1,2,...,n-1\}$.

Proof. If we put $f(x,y) = \dfrac{(xy)^{p+1}}{[(p+1)!]^2}$ and $g(x,y) = \dfrac{(xy)^{q+1}}{[(q+1)!]^2}$ in Theorem 4.5 then we get the required result. □

5. Exponential convexity

Here J stands for an open interval of $\mathbb{R}$.

DEFINITION 5.1. [2] A function $\psi : J \to \mathbb{R}$ is exponentially convex on J if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i\xi_j\,\psi(x_i + x_j) \geqslant 0$$

$\forall n \in \mathbb{N}$ and all choices $\xi_i,\xi_j \in \mathbb{R}$; $i,j = 1,\dots,n$ such that $x_i + x_j \in J$; $1 \leqslant i,j \leqslant n$.

EXAMPLE 5.2. For constant $c \geqslant 0$ and $k \in \mathbb{R}$; $x \mapsto ce^{kx}$, is an example of exponentially convex function.

The following proposition is given in [1]:

PROPOSITION 5.3. *Let $\psi : J \to \mathbb{R}$, the following propositions are equivalent:*

(i) *ψ is exponentially convex on J.*

(ii) *ψ is continuous and*

$$\sum_{i,j=1}^{n} \xi_i\xi_j\,\psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0,$$

$\forall \xi_i,\xi_j \in \mathbb{R}$ and every $x_i,x_j \in J$; $1 \leqslant i,j \leqslant n$.

COROLLARY 5.4. *If ψ is exponentially convex function on J, then the matrix*

$$\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

is a positive semi-definite matrix. Particularly

$$det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0,$$

$\forall n\in\mathbb{N},\ x_i,x_j\in J;\ i,j=1,\ldots,n.$

COROLLARY 5.5. *If $\psi : J \to (0,\infty)$ is exponentially convex function, then ψ is a* $\log$*-convex function i.e. for every $x,y\in J$ and every $\lambda\in[0,1]$, we have*

$$\psi(\lambda x+(1-\lambda)y)\leqslant \psi^{\lambda}(x)\psi^{1-\lambda}(y).$$

Let

$$D=\{\varphi^{(p)}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}:p\in\mathbb{R}\}$$

be a family of functions defined as:

$$\varphi^{(p)}(x,y) = \begin{cases} \dfrac{(xy)^p}{[p(p-1)...(p-n)]^2}, & p\notin\{0,1,2,...,n\}; \\ \dfrac{(xy)^p[\log(xy)]^2}{2[p!(n-p)!]^2}, & p\in\{0,1,2,...,n\}. \end{cases}$$

Clearly $\varphi^{(p)}_{(n+1,n+1)}(x,y) = (xy)^{p-n-1} = e^{(p-n-1)\log(xy)}$ for $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ so $\varphi^{(p)}$ is $(n+1,n+1)$-convex function and $p\mapsto\varphi^{(p)}_{(n+1,n+1)}(x,y)$ is exponentially convex function on $\mathbb{R}$. From Corollary 5.5 we know that every exponentially convex function is log convex. So, now we are in the position to state our next theorem.

THEOREM 5.6. *Let $\Lambda : C(\mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}$ be a linear functional as defined in (4.3) and let the conditions (4.4), (4.5), (4.6), and (4.7) of Theorem 4.3 for function P be satisfied and $\varphi^{(p)}$ be a function defined above. Then the following statements hold:*

(i) $p\mapsto\Lambda(\varphi^{(p)})$ is continuous on $\mathbb{R}$.

(ii) $p\mapsto\Lambda(\varphi^{(p)})$ is exponentially convex function on $\mathbb{R}$.

(iii) If $p\mapsto\Lambda(\varphi^{(p)})$ is positive function on $\mathbb{R}$, then $p\mapsto\Lambda(\varphi^{(p)})$ is log-convex function on $\mathbb{R}$.

(iv) For every $k\in\mathbb{N}$ and $p_1,...,p_k\in\mathbb{R}$, the matrix $\left[\Lambda(\varphi^{(\frac{p_i+p_j}{2})})\right]_{i,j=1}^{k}$ is a positive semi-definite. Particularly

$$\det\left[\Lambda(\varphi^{(\frac{p_i+p_j}{2})})\right]_{i,j=1}^{k} \geqslant 0.$$

(v) If $p \mapsto \Lambda(\varphi^{(p)})$ *is differentiable on* $\mathbb{R}$*. Then for every* $s,t,u,v \in \mathbb{R}$*, such that* $s \leqslant u$ *and* $t \leqslant v$*, we have*

$$\mathfrak{M}_{s,t}(x,y) \leqslant \mathfrak{M}_{u,v}(x,y) \tag{5.1}$$

where

$$\mathfrak{M}_{s,t}(x,y) = \begin{cases} \left(\dfrac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})} \right)^{\frac{1}{s-t}} & , \ s \neq t, \\ \exp\left(\dfrac{\frac{d}{ds}\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})} \right), & s = t \end{cases}$$

for $\varphi^{(s)}, \varphi^{(t)} \in D$*.*

Proof. (i) For fixed $n \in \mathbb{N} \cup \{0\}$, using L'Hôpital rule twice and applying limit, we get

$$\lim_{p \to 0} \Lambda(\varphi^{(p)}) = \lim_{p \to 0} \frac{\int_a^b \int_a^b P(x,y)(xy)^p dydx}{[p(p-1)...(p-n)]^2} = \frac{\int_a^b \int_a^b P(x,y)[\log(xy)]^2 dydx}{2[n!]^2} = \Lambda(\varphi^{(0)}).$$

In the same way we can get

$$\lim_{p \to k} \Lambda(\varphi^{(p)}) = \Lambda(\varphi^{(k)}) \quad k = 1,2,...,n.$$

(ii) Let us define the function

$$\omega(x,y) = \sum_{i,j=1}^{k} \alpha_i \alpha_j \varphi^{(\frac{p_i+p_j}{2})}(x,y),$$

$p_i \in \mathbb{R}, \alpha_i \in \mathbb{R}, \ i = 1,2,...,k$.

Since the function $p \mapsto \varphi^{(p)}_{(n+1,n+1)}$ is exponentially convex, we have

$$\omega_{(n+1,n+1)} = \sum_{i,j=1}^{k} \alpha_i \alpha_j \varphi^{(\frac{p_i+p_j}{2})}_{(n+1,n+1)} \geqslant 0,$$

which implies that ω is $(n+1,n+1)$-convex function on $\mathbb{R}_+ \times \mathbb{R}_+$ and therefore we have $\Lambda(\omega) \geqslant 0$. Hence

$$\sum_{i,j=1}^{k} \alpha_i \alpha_j \Lambda(\varphi^{(\frac{p_i+p_j}{2})}) \geqslant 0.$$

We conclude that the function $p \to \Lambda(\varphi^{(p)})$ is an exponentially convex function on $\mathbb{R}$.

(iii) It is direct consequence of (ii).

(iv) This is consequence of Corollary 5.4.

(v) From the definition of convex function ϕ, we have the following inequality [8, p. 2]

$$\frac{\phi(s) - \phi(t)}{s - t} \leqslant \frac{\phi(u) - \phi(v)}{u - v}, \tag{5.2}$$

$\forall s,t,u,v \in J \subset \mathbb{R}$ such that $s \leqslant u$, $t \leqslant v$, $s \neq t$, $u \neq v$.

Since by (iii), $\Lambda(\varphi^{(p)})$ is log-convex, so set $\phi(x) = \log \Lambda(\varphi^{(x)})$ in (5.2) we have

$$\frac{\log \Lambda(\varphi^{(s)}) - \log \Lambda(\varphi^{(t)})}{s-t} \leqslant \frac{\log \Lambda(\varphi^{(u)}) - \log \Lambda(\varphi^{(v)})}{u-v} \tag{5.3}$$

for $s \leqslant u, t \leqslant v, s \neq t, u \neq v$, which is equivalent to (5.1). The cases for $s = t$, and / or $u = v$ are easily followed from (5.3) by taking respective limits. □

REFERENCES

[1] M. ANWAR, J. JAKŠETIĆ, J. PEČARIĆ AND ATIQ UR REHMAN, *Exponential convexity, positive semi-definite matrices and fundamental inequalities*, J. Math. Inequal. **4**, 2 (2010), 171–189.

[2] S. N. BERNSTIEN, *Sur les fonctions absolument monotones*, Acta Math. **52** (1929), 1–66.

[3] I. Ž. MILOVANOVIĆ AND J. E. PEČARIĆ, *On some inequalities for ∇-convex sequences of higher order*, Period. Math. Hung. **17** (1986), 21–24.

[4] J. E. PEČARIĆ, *An inequality for m-convex sequences*, Mat. Vesnik **5** (18)(33) (1981), 201–203.

[5] J. E. PEČARIĆ, *Some further remarks on the Ostrowski Generalization of Čebyšev's inequality*, J. Math. Anal. Appl. **123** (1987), 18–33.

[6] J. E. PEČARIĆ, *Convex functions: Inequalities*, Naučna knjiga, Beograd, 1987.

[7] J. E. PEČARIĆ, B. A. MESIHOVIĆ, I. Ž. MILOVANOVIĆ AND N. STOJANOVIĆ, *On some inequalities for convex and ∇-convex sequences of higher order II*, Period. Math. Hung. **17**, 4 (1986), 313–320.

[8] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[9] T. POPOVICIU, *Introduction à la théorie des différences divisées*, Bull Math Soc Roumaine des Sciences **42**, 1 (1940), 65–78.

(Received March 5, 2012)

Asif R. Khan
Department of Mathematical Sciences
University of Karachi, Karachi, Pakistan
and
Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan
e-mail: asif_rizkhan@yahoo.com

Josip Pečarić
Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan
and
University Of Zagreb, Faculty Of Textile Technology
Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr

Sanja Varošanec
Department of Mathematics, University Of Zagreb
10000 Zagreb, Croatia
e-mail: varosans@math.hr