Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 213–225 doi:10.7153/jmi-07-20

INEQUALITIES OF J–P–S–F TYPE

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(*Communicated by J. Pečarić*)

Abstract. By means of the theory of majorization and under the proper hypotheses, the following inequalities of Jensen-Pečarić-Svrtan-Fan (Abbreviated as J-P-S-F) type are established:

$$\frac{f(A(x))}{g(A(x))} \leqslant \cdots \leqslant \frac{f_{k+1,n}(x)}{g_{k+1,n}(x)} \leqslant \frac{f_{k,n}(x)}{g_{k,n}(x)} \leqslant \cdots \leqslant \frac{A(f(x))}{A(g(x))},$$

where

$$f_{k,n}(x) := \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right), \quad x \in [a,b]^n.$$

1. Introduction and main results

The following notation and hypotheses in [1, 2, 3, 4] will be used throughout the paper:

$$x := (x_1, \ldots, x_n); \qquad I^n := \{x | x_i \in I, i = 1, \ldots, n\};$$

$$A(x) := \frac{x_1 + \cdots + x_n}{n}; \qquad G(x) := \sqrt[n]{x_1 \cdots x_n};$$

$$f(x) := (f(x_1), \ldots, f(x_n)); \qquad g(x) := (g(x_1), \ldots, g(x_n));$$

$$f_{k,n}(x) \equiv f_{k,n} := \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

Here $I \subset \mathbb{R}$ is an interval, and $n \geqslant 2$.

The well-known Jensen's inequality [6, 7, 8] with equal weights can be stated as

$$f(A(x)) = f_{n,n}(x) \leqslant f_{1,n}(x) = A(f(x)), \tag{1}$$

where $f : I \to \mathbb{R}$ is a convex function, and $x \in I^n$. The inequality is clearly reversed if $f : I \to \mathbb{R}$ is a concave function.

Since 1990s of the last century, combinatorial improvements of Jensen's inequality have been still the heating point of research, and some investigators have enjoyed considerable success (e.g., [9, 10, 15, 16] and the references cited therein). For example, we can easily discover the following inequalities (2–6) are more useful and more interesting than those in [7] if we compare them with [7]. And many refinements of

Mathematics subject classification (2010): 26D15, 26E60.
Keywords and phrases: Jensen's inequality, Pečarić-Svrtan's inequality, Fan's inequality, majorization.

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the well-known and new inequalities can be deduced from the above (see [7, 8]). In a word, various further refinements of Jensen's inequality have been obtained by many mathematicians. For example, those in [8, 9] are precisely some graceful chains of inequalities. Apart from a few papers [3, 4, 5] introduced below, we shall give the reader a brief introduction about Chinese authors' works of which seem to be more difficult to know these better.

Pečarić, Svrtan and Volenec [3, 4, 5] established one of many interesting results is: If $f : I \to \mathbb{R}$ is a mid-convex function, and $x_i \in I$, $i = 1, \ldots, n$, then for all $k = 1, \ldots, n-1$, the following refinement of Jensen inequality holds:

$$f(A(x)) = f_{n,n} \leqslant \cdots \leqslant f_{k+1,n} \leqslant f_{k,n} \leqslant \cdots \leqslant f_{1,n} = A(f(x)), \tag{2}$$

In 2003, Tang and Wen [11] obtained the following inequalities: For all r, j, s, i : $1 \leqslant r \leqslant j \leqslant s \leqslant i \leqslant n$, the following refinement holds:

$$f_{r,s,n} \geqslant \cdots \geqslant f_{r,s,i} \geqslant \cdots \geqslant f_{r,s,s} \geqslant \cdots \geqslant f_{r,j,j} \geqslant \cdots \geqslant f_{r,r,r} = 0, \tag{3}$$

where

$$f_{r,s,n} := \binom{n}{r}\binom{n}{s}(f_{r,n} - f_{s,n}).$$

The equality conditions are also considered.

In 2008, Gao and Wen [12] obtained the following results in this direction:

$$\frac{f(A(a))}{f(A(b))} \leqslant \cdots \leqslant \frac{f_{k+1,n}(a)}{f_{k+1,n}(b)} \leqslant \frac{f_{k,n}(a)}{f_{k,n}(b)} \leqslant \cdots \leqslant \frac{A(f(a))}{A(f(b))}, \tag{4}$$

where

$$a, b \in I^n, \quad a_1 \leqslant \cdots \leqslant a_n \leqslant b_n \leqslant \cdots \leqslant b_1, \quad a_1 + b_1 \leqslant \cdots \leqslant a_n + b_n,$$

$$f(t) > 0, \quad f'(t) > 0, \quad f''(t) > 0, \quad f'''(t) < 0, \ \forall t \in I, \quad 1 \leqslant k \leqslant n-1.$$

The inequalities are reversed for

$$f(t) > 0, \quad f'(t) > 0, \quad f''(t) < 0, \quad f'''(t) > 0, \ \forall t \in I.$$

Moreover, Wen and Wang [13] considered some inequalities for linear combinations involving $f_{k,n}$.

Another type of generalization is due to Wen [14]: Let $f : I \to \mathbb{R}$ be a twice-differentiable function, and let whose second derivative f'' be a continuous, convex function. Then, for any $x \in I^n$, we have

$$f''(D(x)) \leqslant \frac{2J[f(x)]}{J[x^2]} \leqslant \frac{1}{3}\Big[\max_{1\leqslant i\leqslant n}\{f''(x_i)\} + A(f''(x)) + f''(A(x))\Big], \tag{5}$$

where

$$D(x) = \frac{1}{3}\frac{A(x^3) - A^3(x)}{A(x^2) - A^2(x)},$$

$$J[f(x)] = A(f(x)) - f(A(x)), \qquad J[x^2] = A(x^2) - A^2(x).$$

A review is presented on recent progress in these researches as fellows: In 2011, Horvath [20] proposed a new method to refine the discrete Jensen's inequality for convex and mid-convex functions. In fact, this is a new parameter-dependent refinement. In the same year, Horvath and Pečarić [21] established a new refinement for these functions. In 2012, Horvath, Khan and Pečarić [22] obtained the related results for operator convex functions on a Hilbert space.

In this paper we study a kind of interesting inequalities centering about the topic of refinements involving two functions. Our main result is:

THEOREM 1. (Inequalities of Jensen-Pečarić-Svrtan-Fan type) *Let two functions*

$$f : [a,b] \to (0,\infty), \qquad g : [a,b] \to (0,\infty)$$

satisfy

$$\sup_{t\in[a,b]} \left\{ \left| \frac{g''(t)}{f''(t)} \right| \right\} < \inf_{t\in[a,b]} \left\{ \frac{g(t)}{f(t)} \right\}.$$

If $f''(t) > 0$, $\forall t \in [a,b]$, *then for any* $x \in [a,b]^n$, *we have the following inequalities:*

$$\frac{f(A(x))}{g(A(x))} \leqslant \cdots \leqslant \frac{f_{k+1,n}(x)}{g_{k+1,n}(x)} \leqslant \frac{f_{k,n}(x)}{g_{k,n}(x)} \leqslant \cdots \leqslant \frac{A(f(x))}{A(g(x))}, \tag{6}$$

where $1 \leqslant k \leqslant n-1$. *If* $f''(t) < 0$, $\forall t \in [a,b]$, *then the above inequalities are reversed. In each case, the sign of the equality holding throughout if and only if* $x_1 = \cdots = x_n$.

2. Proof of Theorem 1

In this section, in order to simplify some expressions, let us set

$$\alpha := (\alpha_1, \cdots, \alpha_n); \qquad \Omega_n := \{\alpha \in [0,1]^n | \alpha_1 + \cdots + \alpha_n = 1\};$$

$$S_f(\alpha, x) := \frac{1}{n!} \sum_{i_1 i_2 \cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}); \qquad F(\alpha) := \log \frac{S_f(\alpha, x)}{S_g(\alpha, x)};$$

$$u_i(x) := \alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \sum_{j=3}^{n} \alpha_j x_{i_j}; \qquad v_i(x) := \alpha_1 x_{i_2} + \alpha_2 x_{i_1} + \sum_{j=3}^{n} \alpha_j x_{i_j}.$$

Here and in the sequel $x \in [a,b]^n$, $\alpha \in \Omega_n$, $i = (i_1, \cdots, i_n)$, and we let $i_1 \cdots i_n$ and $i_3 \ldots i_n$ denote the possible permutations of $\mathbb{N}_n = \{1, \ldots, n\}$ and the possible permutations of $\mathbb{N}_n \backslash \{i_1, i_2\}$, respectively.

LEMMA 1. *Under the hypotheses of Theorem 1, there exist ξ_i and ξ_i^* between $u_i(x)$ and $v_i(x)$ such that*

$$(\alpha_1-\alpha_2)\left(\frac{\partial F}{\partial \alpha_1}-\frac{\partial F}{\partial \alpha_2}\right)=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{f''(\xi_i)(u_i(x)-v_i(x))^2}{S_f(\alpha,x)} \times\left(1-\frac{g''(\xi_i^*)}{f''(\xi_i^*)}\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}\right). \tag{7}$$

Proof. Note the following identities:

$$\begin{aligned}
S_f(\alpha,x)&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1\neq i_2\leqslant n} f(\alpha_1 x_{i_1}+\cdots+\alpha_n x_{i_n})\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}[f(u_i(x))+f(v_i(x))];\\
S_g(\alpha,x)&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}[g(u_i(x))+g(v_i(x))];
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial\alpha_1}[f(u_i(x))+f(v_i(x))]-\frac{\partial}{\partial\alpha_2}[f(u_i(x))+f(v_i(x))]\\
&=[x_{i_1}f'(u_i(x))+x_{i_2}f'(v_i(x))]-[x_{i_2}f'(u_i(x))+x_{i_1}f'(v_i(x))]\\
&=[f'(u_i(x))-f'(v_i(x))](x_{i_1}-x_{i_2});\\
&\frac{\partial}{\partial\alpha_1}[g(u_i(x))+g(v_i(x))]-\frac{\partial}{\partial\alpha_2}[g(u_i(x))+g(v_i(x))]\\
&=[g'(u_i(x))-g'(v_i(x))](x_{i_1}-x_{i_2}).
\end{aligned}$$

Thus

$$\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial S_f(\alpha,x)}{\partial\alpha_1}-\frac{\partial S_f(\alpha,x)}{\partial\alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}[f'(u_i(x))-f'(v_i(x))](\alpha_1-\alpha_2)(x_{i_1}-x_{i_2})\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}[f'(u_i(x))-f'(v_i(x))](u_i(x)-v_i(x));\\
&(\alpha_1-\alpha_2)\left(\frac{\partial S_g(\alpha,x)}{\partial\alpha_1}-\frac{\partial S_g(\alpha,x)}{\partial\alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}[g'(u_i(x))-g'(v_i(x))](u_i(x)-v_i(x)).
\end{aligned}$$

Based on the above facts, we have

$$
\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial F}{\partial \alpha_1}-\frac{\partial F}{\partial \alpha_2}\right)\\
&=(\alpha_1-\alpha_2)\left(\frac{\frac{\partial S_f(\alpha,x)}{\partial \alpha_1}-\frac{\partial S_f(\alpha,x)}{\partial \alpha_2}}{S_f(\alpha,x)}-\frac{\frac{\partial S_g(\alpha,x)}{\partial \alpha_1}-\frac{\partial S_g(\alpha,x)}{\partial \alpha_2}}{S_g(\alpha,x)}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}\left\{\frac{[f'(u_i(x))-f'(v_i(x))](u_i(x)-v_i(x))}{S_f(\alpha,x)}\right.\\
&\qquad\left.-\frac{[g'(u_i(x))-g'(v_i(x))](u_i(x)-v_i(x))}{S_g(\alpha,x)}\right\}\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{[f'(u_i(x))-f'(v_i(x))](u_i(x)-v_i(x))}{S_f(\alpha,x)}\\
&\qquad\times\left(1-\frac{g'(u_i(x))-g'(v_i(x))}{f'(u_i(x))-f'(v_i(x))}\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}\right).
\end{aligned}
$$

By Lagrange's mean-value theorem, there exists ξ_i between $u_i(x)$ and $v_i(x)$ such that

$$f'(u_i(x))-f'(v_i(x))=f''(\xi_i)(u_i(x)-v_i(x)).$$

By Cauchy's mean-value theorem, there exists ξ_i^* between $u_i(x)$ and $v_i(x)$ such that

$$\frac{g'(u_i(x))-g'(v_i(x))}{f'(u_i(x))-f'(v_i(x))}=\frac{g''(\xi_i^*)}{f''(\xi_i^*)}.$$

Finally one has

$$
\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial F}{\partial \alpha_1}-\frac{\partial F}{\partial \alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{[f'(u_i(x))-f'(v_i(x))](u_i(x)-v_i(x))}{S_f(\alpha,x)}\\
&\qquad\times\left(1-\frac{g'(u_i(x))-g'(v_i(x))}{f'(u_i(x))-f'(v_i(x))}\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\leqslant i_1<i_2\leqslant n}\frac{f''(\xi_i)\,(u_i(x)-v_i(x))^2}{S_f(\alpha,x)}\left(1-\frac{g''(\xi_i^*)}{f''(\xi_i^*)}\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}\right).
\end{aligned}
$$

The proof of Lemma 1 has been finished. □

LEMMA 2. *Let the conditions of Theorem 1 be satisfied.*
(I) *If* $f''(t)>0,\ \forall t\in[a,b]$, *then* $F(\alpha)$ *is a Schur-convex function on* Ω_n.
(II) *If* $f''(t)<0,\ \forall t\in[a,b]$, *then* $F(\alpha)$ *is a Schur-concave function on* Ω_n.

Proof. We first affirm that Case (I) is true as follow.

One can easily see that Ω_n is a symmetric convex set, and $F(\alpha)$ is a symmetric function on Ω_n and it has continuous partial derivatives. By [1, 2], we need to prove that F satisfies the Schur condition:

$$(\alpha_1-\alpha_2)\left(\frac{\partial F}{\partial \alpha_1}-\frac{\partial F}{\partial \alpha_2}\right)\geqslant 0. \tag{8}$$

Equality is valid if and only if $\alpha_1=\alpha_2$ or $x_1=\cdots=x_n$.

In the following, we shall apply the identity (7) in Lemma 1.

Note that $x\in[a,b]^n, \alpha\in\Omega_n$, for any $i=(i_1,\ldots,i_n)$, we have

$$u_i(x)=\alpha_1x_{i_1}+\cdots+\alpha_nx_{i_n}\in[a,b];$$

$$\frac{f(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})}{g(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})}=\frac{f(u_i(x))}{g(u_i(x))}\leqslant\sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\};$$

$$\begin{aligned}
S_f(\alpha,x)&=\frac{1}{n!}\sum_{i_1\cdots i_n}f(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})\\
&=\frac{1}{n!}\sum_{i_1\cdots i_n}\frac{f(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})}{g(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})}\cdot g(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})\\
&\leqslant\frac{1}{n!}\sum_{i_1\cdots i_n}\sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\}g(\alpha_1x_{i_1}+\ldots+\alpha_nx_{i_n})\\
&=\sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\}S_g(\alpha,x),
\end{aligned}$$

or, equivalently,

$$\frac{S_g(\alpha,x)}{S_f(\alpha,x)}\geqslant\left[\sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\}\right]^{-1}=\inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}. \tag{9}$$

Combining (9) with the following inequality

$$0\leqslant\left|\frac{g_i''(\xi_i^*)}{f_i''(\xi_i^*)}\right|\leqslant\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} \tag{10}$$

and the hypotheses of Theorem 1, we obtain that

$$\begin{aligned}
1-\frac{g''(\xi_i^*)}{f''(\xi_i^*)}\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}&\geqslant 1-\left|\frac{g''(\xi_i^*)}{f''(\xi_i^*)}\right|\cdot\frac{S_f(\alpha,x)}{S_g(\alpha,x)}\\
&\geqslant 1-\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\}\Big/\frac{S_g(\alpha,x)}{S_f(\alpha,x)}\\
&\geqslant 1-\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\}\Big/\inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}\\
&>0.
\end{aligned} \tag{11}$$

By the identity (7), the inequality (11) and given $f''(t) > 0$, $\forall t \in [a,b]$, the Schur condition (8) can be satisfied. So $F(\alpha)$ is a Schur-convex function on Ω_n.

Let us now turn to the conclusion (II) of our lemma.

From above argument for (I) we know that the inequalities (9–11) hold still. Using (7), (11) and $f''(t) < 0$, $\forall t \in [a,b]$, the converse of (8) can be obtained. Thus, $F(\alpha)$ is a Schur-concave function on Ω_n. From the argument, we obtain that equality is valid if and only if $\alpha_1 = \alpha_2$ or $x_1 = \cdots = x_n$.

This completes the proof of Lemma 2. $\square$

Proof of Theorem 1. We only prove the first assertion, that is, the inequalities (6) hold for $f''(t) > 0$, $\forall t \in [a,b]$, because we also prove the second assertion for $f''(t) < 0$, $\forall t \in [a,b]$ by an analogous procedure. Define

$$\alpha[k] := (\underbrace{k^{-1},\ldots,k^{-1}}_{k},\underbrace{0,\ldots,0}_{n-k}), \quad k = 1,\ldots,n.$$

Clearly, $\alpha[k] \in \Omega_n$, $k = 1,\ldots,n$, and

$$\alpha[k+1] \prec \alpha[k], \quad k = 1,\ldots,n-1.$$

By Lemma 2, for any $x \in [a,b]^n$, $F(\alpha)$ is a Schur-convex function on Ω_n (see [1, 2]). Using the definition of Schur-convex function, we have

$$F(\alpha[k+1]) \leqslant F(\alpha[k]), \quad k = 1,\ldots,n-1.$$

Combining this result with the definition of $F(\alpha)$, it follows that the inequalities (6) hold. By the argument of Lemma 2 and the fact of which $\alpha[k]$ strictly majorizes $\alpha[k+1]$, the sign of equality holding throughout if and only if $x_1 = \cdots = x_n$.

Theorem 1 is thus proved. $\square$

3. Applications

Let $x \in (0,\infty)^n$. The Dresher mean of order k of x, where $1 \leqslant k \leqslant n$, is defined by

$$[D_{p,q}(x)]_{k,n} := \begin{cases} \dfrac{1}{k}\left[\dfrac{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(\sum_{j=1}^{k} x_{i_j}\right)^p}{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(\sum_{j=1}^{k} x_{i_j}\right)^q}\right]^{1/(p-q)}, & \text{if } p \neq q, \\[2ex] \dfrac{1}{k}\exp\left[\dfrac{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(\sum_{j=1}^{k} x_{i_j}\right)^p \log\left(\sum_{j=1}^{k} x_{i_j}\right)}{\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\left(\sum_{j=1}^{k} x_{i_j}\right)^p}\right], & \text{if } p = q. \end{cases}$$

Especially,

$$D_{p,q}(x) := [D_{p,q}(x)]_{1,n}$$

is the Dresher mean of x (see [18]), and

$$[D_{0,0}(x)]_{k,n} = \left(\prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)^{1/\binom{n}{k}} = [GA;x]_{k,n},$$

$$[D_{1,1}(x)]_{k,n} = \left[\prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)^{\frac{x_{i_1}+\cdots+x_{i_k}}{k}}\right]^{1/\left[\binom{n}{k}A(x)\right]},$$

$$[D_{1,1}(x)]_{1,n} = (x_1^{x_1}\cdots x_n^{x_n})^{1/(x_1+\cdots+x_n)},$$
$$[D_{0,0}(x)]_{1,n} = G(x),$$
$$[D_{p,q}(x)]_{n,n} = A(x).$$

Write

$$D(p,q) := \begin{cases} \left[\frac{p(1-p)}{q(1-q)}\right]^{1/(p-q)}, & \text{if } p \neq q, \\ \exp \frac{1-2p}{p(1-p)}, & \text{if } p = q. \end{cases}$$

Theorem 1 implies the following three corollaries.

COROLLARY 1. (Inequalities of Pečarić-Svrtan-Dresher type, see [5], [19]) *Let*

$$x \in (0,\infty)^n, \quad \frac{\max\{x\}}{\min\{x\}} < D(p,q).$$

(I) *If* $p>0,\ q>0,\ p+q<1$, *then*

$$\begin{aligned} A(x) = [D_{p,q}(x)]_{n,n} &\geqslant \cdots \geqslant [D_{p,q}(x)]_{k+1,n} \geqslant [D_{p,q}(x)]_{k,n} \\ &\geqslant \cdots \geqslant [D_{p,q}(x)]_{1,n} = D_{p,q}(x) \geqslant G(x). \end{aligned} \tag{12}$$

(II) *If* $p>1,\ q>1$, *then*

$$\begin{aligned} A(x) = [D_{p,q}(x)]_{n,n} &\leqslant \cdots \leqslant [D_{p,q}(x)]_{k+1,n} \leqslant [D_{p,q}(x)]_{k,n} \\ &\leqslant \cdots \leqslant [D_{p,q}(x)]_{1,n} = D_{p,q}(x). \end{aligned} \tag{13}$$

In each case, the sign of equality holds throughout if and only if $x_1 = \cdots = x_n$.

Proof. We only prove the case (I), that is, the inequalities (13) hold, because we also prove the case (II) with the same method. Since

$$[D_{p,q}(x)]_{k,n} = [D_{q,p}(x)]_{k,n}$$

is continuous of (p,q), we can assume that $0<q<p<1$. Now we take

$$[a,b] = [\min\{x\},\max\{x\}], \quad f:[a,b]\to(0,\infty), \quad f(t)=t^p,$$

and

$$g:[a,b]\to(0,\infty), \quad g(t)=t^q.$$

We verify that the conditions of Theorem 1 can be satisfied below.

Firstly, we notice that

$$\begin{aligned}\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} &= \sup_{t\in[a,b]}\left\{\left|\frac{q(q-1)t^{q-2}}{p(p-1)t^{p-2}}\right|\right\}\\ &= \sup_{t\in[a,b]}\left\{\frac{q(1-q)}{p(1-p)}t^{q-p}\right\}\\ &= \frac{q(1-q)}{p(1-p)}a^{q-p},\end{aligned}$$

$$\inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\} = \inf_{t\in[a,b]}\{t^{q-p}\} = b^{q-p}.$$

By

$$0<q<p<1, \quad p+q<1, \quad p(1-p)-q(1-q)=(p-q)(1-p-q)>0,$$

we have

$$D(p,q)=\left[\frac{p(1-p)}{q(1-q)}\right]^{1/(p-q)}>1,$$

$$\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} = \frac{q(1-q)}{p(1-p)}a^{q-p} < b^{q-p} = \inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\} \Leftrightarrow \frac{\max\{x\}}{\min\{x\}} < D\{p,q\},$$

$$f''(t)=p(p-1)t^{p-2}<0,\ \forall t\in[a,b].$$

Thus, by Theorem 1, the reverse (6) holds. In other words, we have

$$\begin{aligned}A(x) = [D_{p,q}(x)]_{n,n} \geqslant \cdots \geqslant [D_{p,q}(x)]_{k+1,n} \geqslant [D_{p,q}(x)]_{k,n}\\ \geqslant \cdots \geqslant [D_{p,q}(x)]_{1,n} = D_{p,q}(x).\end{aligned} \tag{14}$$

Secondly, using the results of [19]:

$$D_{p,q}(x) \geqslant D_{r,s}(x) \Leftrightarrow \max\{p,q\} \geqslant \max\{r,s\} \text{ and } \min\{p,q\} \geqslant \min\{r,s\},$$

and $p>0,\ q>0,$ we get

$$D_{p,q}(x) \geqslant D_{0,0}(x) = G(x). \tag{15}$$

From (14) and (15) we get (12).

The proof is therefore complete. □

REMARK 1. From Corollary 1 and

$$\lim_{p\to 0^+, q\to 0^+} D(p,q) = \lim_{p\to 1^+, q\to 1^+} D(p,q) = \infty,$$

we can obtain some interesting inequalities (see [5]) : If $x\in(0,\infty)^n$, then

$$A(x) \geqslant \cdots \geqslant [GA;x]_{k+1,n} \geqslant [GA;x]_{k,n} \geqslant \cdots \geqslant G(x), \tag{16}$$

and

$$A(x) \leqslant \cdots \leqslant [D_{1,1}(x)]_{k+1,n} \leqslant [D_{1,1}(x)]_{k,n} \leqslant \cdots \leqslant [D_{1,1}(x)]_{1,n}. \tag{17}$$

The sign of equality holds throughout if and only if $x_1 = \cdots = x_n$.

REMARK 2. Since (12) implies the following inequality

$$A(x) \geqslant D_{p,q}(x) = \left[\frac{A(x^p)}{A(x^q)}\right]^{1/(p-q)} \geqslant G(x),\ \forall p,q:\ p>0,\ q>0,\ p+q<1,$$

by Corollary 1 and the definition of Riemann integral, we know that: If $p>0$, $q>0$, $p \neq q$, $p+q<1$, the function $f:[\alpha,\beta] \to (0,\infty)$ is continuous, and it satisfies the condition

$$\frac{\max_{t\in[\alpha,\beta]}\{f(t)\}}{\min_{t\in[\alpha,\beta]}\{f(t)\}} < D(p,q),$$

then we have

$$\frac{\int_\alpha^\beta f\mathrm{d}t}{\beta-\alpha} \geqslant \left(\frac{\int_\alpha^\beta f^p\mathrm{d}t}{\int_\alpha^\beta f^q\mathrm{d}t}\right)^{1/(p-q)} \geqslant \exp\left(\frac{\int_\alpha^\beta \ln f\mathrm{d}t}{\beta-\alpha}\right). \tag{18}$$

One of the integral analogues of the inequalities (6) is the following inequality (19).

COROLLARY 2. *Under the hypotheses of Theorem 1, let $E \subset \mathbb{R}^m$ be a bounded closed domain with measure (m-dimensional volume) $|E|=1$, and let $\phi: E \to [a,b]$ be a Riemann integrable function. If $f''(t)>0$, $\forall t \in [a,b]$, then*

$$\frac{f\left(\int_E \phi\right)}{g\left(\int_E \phi\right)} \leqslant \frac{\int_E f\circ\phi}{\int_E g\circ\phi}, \tag{19}$$

where $f\circ\phi = f(\phi)$, $g\circ\phi = g(\phi)$, and $\int_E$ is Riemann integral. If $f''(t)<0$, $\forall t\in[a,b]$, then the inequality (19) is reversed.

Proof. On the one hand, the hypotheses of Corollary 2 imply that the functions

$$\phi: E \to \mathbb{R}, \quad f\circ\phi: E\to\mathbb{R}, \quad g\circ\phi: E\to\mathbb{R}$$

are integrable. On the other hand, Theorem 1 implies the inequality

$$\frac{f(A(x,w))}{g(A(x,w))} \leqslant \frac{A(f(x),w)}{A(g(x),w)},\ \ \forall x\in[a,b]^n, \tag{20}$$

where

$$w\in(0,1)^n, \quad \sum_{i=1}^n w_i = 1, A(x,w) = \sum_{i=1}^n w_i x_i.$$

Let

$$T = \{\Delta E_1, \cdots, \Delta E_n\}$$

be a partition of E, and let

$$\|T\| = \max_{1 \leqslant i \leqslant n} \max_{X,Y \in \Delta E_i} \{\| X - Y \|\}$$

be the 'norm' of the partition T, where $\|X-Y\|$ is the length of the vector $X-Y$. Pick any

$$\xi \in \Delta E_1 \times \cdots \times \Delta E_n,$$

by (20) we get

$$\frac{f\left(\int_E \phi\right)}{g\left(\int_E \phi\right)} = \lim_{\|T\| \to 0} \frac{f(A(\phi(\xi),w))}{g(A(\phi(\xi),w))} \leqslant \lim_{\|T\| \to 0} \frac{A(f(\phi(\xi)),w)}{A(g(\phi(\xi)),w)} = \frac{\int_E f \circ \phi}{\int_E g \circ \phi}, \tag{21}$$

where

$$w = (|\Delta E_1|, \cdots, |\Delta E_n|) \in (0,1)^n, \quad \sum_{i=1}^{n} |\Delta E_i| = 1, \quad \phi(\xi) \in [a,b]^n.$$

Therefore the inequality (16) holds from (21). This ends the proof. $\square$

COROLLARY 3. (Inequalities of Fan type, see[8]) *If* $x \in \left(0, \frac{1}{2}\right]^n$, *then*

$$\frac{A(x)}{A(1-x)} \geqslant \cdots \geqslant \frac{[GA;x]_{k+1,n}}{[GA;1-x]_{k+1,n}} \geqslant \frac{[GA;x]_{k,n}}{[GA;1-x]_{k,n}} \geqslant \cdots \geqslant \frac{G(x)}{G(1-x)}, \tag{22}$$

where

$$1 - x = (1-x_1, \ldots, 1-x_n), \quad 1 \leqslant k \leqslant n-1,$$

and the sign of equality holding throughout if and only if $x_1 = \cdots = x_n$.

Proof. It goes without saying that, for each $x \in \left(0, \frac{1}{2}\right]^n$, we can always find $a \in \left(0, \frac{1}{2}\right)$ such that $x \in \left[a, \frac{1}{2}\right]^n$. In Theorem 1, we take

$$f : \left[a, \frac{1}{2}\right] \to (0,\infty), \quad f(t) = t^{\gamma}, \quad 0 < \gamma < 1;$$

$$g : \left[a, \frac{1}{2}\right] \to (0,\infty), \quad g(t) = (1-t)^{\gamma}, \quad 0 < \gamma < 1.$$

We verify that the conditions of Theorem 1 can be satisfied as follows.

$$\begin{aligned} \sup_{t \in [a,1/2]} \left\{ \left| \frac{g''(t)}{f''(t)} \right| \right\} &= \sup_{t \in [a,1/2]} \left\{ \left| \frac{\gamma(\gamma-1)(1-t)^{\gamma-2}}{\gamma(\gamma-1)t^{\gamma-2}} \right| \right\} \\ &= \sup_{t \in [a,1/2]} \left\{ \left(\frac{1}{t} - 1\right)^{\gamma-2} \right\} \\ &= 1, \end{aligned}$$

$$\inf_{t\in[a,1/2]}\left\{\frac{g(t)}{f(t)}\right\} = \inf_{t\in[a,1/2]}\left\{\left(\frac{1}{t}-1\right)^{\gamma}\right\} = 1.$$

From the above we have

$$\sup_{t\in[a,1/2]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} \leqslant \inf_{t\in[a,1/2]}\left\{\frac{g(t)}{f(t)}\right\}.$$

It is easy to see that

$$f''(t) = \gamma(\gamma-1)t^{\gamma-2} < 0, \quad \forall t \in \left[a, \frac{1}{2}\right].$$

By now, our verification procedure has been finished. Thus the inverse inequalities (6) are true, that is, we have

$$\left[\frac{f_{k+1,n}(x)}{f_{k+1,n}(1-x)}\right]^{1/\gamma} \geqslant \left[\frac{f_{k,n}(x)}{f_{k,n}(1-x)}\right]^{1/\gamma}, \quad k = 1,\dots,n-1. \tag{23}$$

Passing the limit as $\gamma \to 0$ in (23), we can obtain (22). By the same argument as in Theorem 1, we can derive the sign of equality in (22) holding throughout if and only if $x_1 = \cdots = x_n$.

This completes the proof of Corollary 3. $\square$

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(Received March 18, 2012)

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