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# NEW REFINEMENTS OF TWO INEQUALITIES FOR MEANS

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*Abstract.* In paper [2] H. Alzer proved that the logarithmic mean of two distinct positive real numbers lies between the geometric mean and the arithmetic mean of the geometric and identric means of these numbers. Refinements of these inequalities were provided in [11]. In this note we offer refinements of a new type.

## 1. Introduction

The logarithmic and identric means of two positive numbers $a$ and $b$ are defined by

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a} \quad (a \neq b); \qquad L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad (a \neq b); \qquad I(a,a) = a,$$

respectively.

Let $A = A(a,b) := \frac{a+b}{2}$ and $G = G(a,b) := \sqrt{ab}$ denote the arithmetic and geometric means of $a$ and $b$, respectively. For these means many interesting inequalities have been proved. For a survey of results, see [1], [3], [7], [11], [12]. It may be surprising that the means of two arguments have applications in physics, economics, statistics, or even in meteorology. See e.g. [3], [6] and the references therein. For connections of such means with Ky Fan, or Huygens type inequalities; or with Seiffert and Gini type means, we quote papers [13] and [14]; as well as [5], [8], or [12].

In what follows we shall assume that $a \neq b$. In paper [2] H. Alzer proved that

$$\sqrt{GI} < L < \frac{G+I}{2} \tag{1}$$

and in [1] he proved that

$$AG < LI \quad \text{and} \quad L + I < A + G. \tag{2}$$

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In paper [8] the author proved that the first inequality of (2) is weaker than the left side of (1), while the second inequality of (2) is stronger than the right side of (1). In fact, these statements are consequences of

$$I > \sqrt[3]{A^2G} \tag{3}$$

and

$$I > \frac{2A+G}{3}. \tag{4}$$

Clearly, by the weighted arithmetic-geometric mean inequality, (4) implies (3), but one can obtain different methods of proof for these results (see [8]). In [7] J. Sándor has proved that

$$\ln\frac{I}{L} > 1 - \frac{G}{L} \tag{5}$$

and this was used in [11] to obtain the following refinement of right side of (1):

$$L < \frac{I+aG}{1+a} < \frac{I+G}{2}, \tag{6}$$

where $a = \sqrt{I}/\sqrt{L} > 1$.

In paper [11] the following refinements of left side of (1) has been also proved:

$$\sqrt{IG} < \frac{I-G}{A-L}\cdot L < L. \tag{7}$$

The aim of this paper is to offer certain new refinements of other type for inequalities (1).

## 2. Main results

The main result of this paper is contained in the following:

THEOREM. *One has*

$$L < \sqrt{\frac{(A+G)(L+G)}{4}} < \frac{A+L+2G}{4} < \frac{I+G}{2} \tag{8}$$

*and*

$$L > \sqrt[3]{G\left(\frac{A+G}{2}\right)^2} > \sqrt{GI}. \tag{9}$$

*Proof.* First we note that the second inequality of (8) follows by $\sqrt{xy} < \dfrac{x+y}{2}$, applied to $x := \dfrac{A+G}{2}$ and $y := \dfrac{L+G}{2}$, while the last inequality can be written as

$$I > \frac{A+L}{2}. \tag{10}$$

This appears in [7], but we note that follows also by (4) and

$$\frac{2A+G}{3} > \frac{A+L}{2}, \tag{11}$$

which can be written equivalently as

$$L < \frac{2G+A}{3}, \tag{12}$$

due to B. C. Carlson [4] and G. Pólya and G. Szegö [6].

Thus we have to prove only the first inequality of (8).

For this purpose, we shall use inequality (5) combined with the identity

$$\ln\frac{I}{G} = \frac{A-L}{L}, \tag{13}$$

due to H. J. Seiffert [15]. See also [9] for this and related identities.

Since $\ln x > \dfrac{2(x-1)}{x+1}$ for $x > 1$ (equivalent in fact with the classical inequality $L(x,1) < A(x,1)$), by letting $x = \dfrac{L}{G}$, and by

$$\ln\frac{I}{L} = \ln\frac{I}{G} - \ln\frac{L}{G}, \qquad \ln\frac{L}{G} > 2\cdot\frac{L-G}{L+G}$$

and (13) combined with (5) gives the following inequality:

$$2\cdot\frac{L-G}{L+G} < \frac{A+G}{L} - 2, \tag{14}$$

which after some elementary computations gives the first inequality of (8). $\square$

REMARK. The first and third term of (8) is exactly inequality (12). Therefore, the first two inequalities provide also a refinement of (12).

Now, for the proof of relation (9) remark first that the first inequality has been proved by the author in paper [10]. The second inequality will be reduced to an inequality involving hyperbolic functions. Put $a = e^x G$, $b = e^{-x}G$, where $x > 0$ (for this method see e.g. [1]). Then the inequality to be proved becomes equivalent to

$$\ln\left(\frac{\cosh x+1}{2}\right) > \frac{3}{4}\left(\frac{x\cosh x - \sinh x}{\sinh x}\right). \tag{15}$$

Let us introduce the function

$$f(x) = 4\ln\left(\frac{\cosh x+1}{2}\right) - 3x\coth x + 3, \quad x > 0.$$

An immediate computation gives

$$(\cosh x+1)\sinh^2 x\cdot f'(x) = \sinh^3 x - 3\sinh x + 3x\cosh x + 3x - 3\sinh x\cosh x = g(x).$$

One has

$$g'(x) = 3\sinh x(\sinh x\cosh x + x - 2\sinh x).$$

Now, as it is well known that $\sinh x < x\cosh x$, we can remark that

$$\sinh x < \sqrt{x\sinh x\cosh x} \leqslant \frac{x+\sinh x\cosh x}{2} \quad \text{by} \quad \sqrt{uv} \leqslant \frac{u+v}{2}.$$

This in turn implies $g'(x) > 0$, and as $g(x)$ can be defined for $x \geqslant 0$ and $g(0) = 0$, we get $g(x) \geqslant 0$, and $g(x) > 0$ for $x > 0$. Thus $f'(x) > 0$ for $x > 0$, so $f$ is strictly increasing and as $\lim_{x\to 0} f(x) = 0$, inequality (15) follows.

This finishes the proof of the Theorem.

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