A HILBERT–TYPE INTEGRAL INEQUALITY IN THE
WHOLE PLANE WITH THE HOMOGENEOUS KERNEL

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Abstract. By using the way of weight functions and the technique of real analysis, a new Hilbert-type integral inequality with the homogeneous kernel of degree 0 in the whole plane with the best constant factor is given. As applications, the equivalent inequalities with the best constant factors, the reverses and some particular cases are obtained.

1. Introduction

If $f(x), g(x) \geqslant 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is well known as Hilbert's integral inequality, which is important in Mathematical Analysis and its applications [2]. In recent years, by using the way of weight functions, a number of extensions of (1) were given by Yang [3]. Noticing that inequality (1) is a homogenous kernel of degree -1, in 2009, a survey of the study of Hilbert-type inequalities with the homogenous kernels of degree negative numbers and some parameters were given by [4]. Recently, some inequalities with the homogenous kernel of degree 0 and non-homogenous kernels have been studied [5]–[9], a unified treatment of Hilbert-type inequalities has been obtained by Krnic [10] and the best constants for a wide class of homogeneous kernels in R^+ have been given by Peric [11]. All of the above inequalities are built in the quarter plane. In 2006, Yang [12] built a new Hilbert-type integral inequality in the whole plane as follow:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{1+e^{x+y}}dxdy < \pi \left(\int_{-\infty}^\infty e^{-x}f^2(x)dx \int_{-\infty}^\infty e^{-x}g^2(x)dx \right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor π is the best possible. In 2010, Xie et al. [13], [14] and [15] also gave some new inequalities in the whole plane.

In this paper, by using the way of weight functions and the technique of real analysis, a new Hilbert-type integral inequality in the whole plane with the homogenous kernel of degree 0 and a best constant factor is built. As applications, the equivalent forms, the reverses and some particular cases are obtained.

2. Some lemmas

LEMMA 1. *If $0 < \alpha_1 < \alpha_2 < \pi$, define the weight functions $\omega(y)$ and $\tilde{\omega}(x)$ as follow:*

$$\omega(y) := \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{1}{|x|} dx, \qquad (y \in (-\infty,\infty)), \quad (3)$$

$$\tilde{\omega}(x) := \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{1}{|y|} dy, \qquad (x \in (-\infty,\infty)). \quad (4)$$

Then we have $\omega(y) = \tilde{\omega}(x) = k \ (x,y \neq 0)$, where

$$k := 2\ln\left[\left(1 + \sec\frac{\alpha_1}{2}\right)\left(1 + \csc\frac{\alpha_2}{2}\right)\right]. \quad (5)$$

Proof. Setting $u = \frac{x}{|y|}$ in (3), we find

$$\omega(y) = \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|u|,1\}}{\sqrt{u^2 + 2u(y/|y|)\cos\alpha_i + 1}} \right\} \frac{1}{|u|} du. \quad (6)$$

For $y \in (0,\infty)$, we have

$$\omega(y) = \int_{-\infty}^{-1} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}} \frac{-1}{u} du + \int_{-1}^{0} \frac{du}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}}$$
$$+ \int_{0}^{1} \frac{du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \int_{1}^{\infty} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \frac{du}{u}. \quad (7)$$

Setting

$$\omega_1 := \int_{-\infty}^{-1} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}} \frac{-1}{u} du,$$

$$\omega_2 := \int_{-1}^{0} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_2 + 1}} du,$$

$$\omega_3 := \int_{0}^{1} \frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du,$$

$$\omega_4 := \int_1^\infty \frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \frac{1}{u} du,$$

we find

$$\omega_1 \overset{v=-u}{=} \int_1^\infty \frac{dv}{v\sqrt{v^2 + 2v\cos(\pi - \alpha_2) + 1}}$$

$$\overset{z=1/v}{=} \int_0^1 \frac{dz}{\sqrt{z^2 + 2z\cos(\pi - \alpha_2) + 1}},$$

$$\omega_2 \overset{v=-u}{=} \int_0^1 \frac{dv}{\sqrt{v^2 + 2v\cos(\pi - \alpha_2) + 1}} = \omega_1,$$

$$\omega_4 \overset{z=1/u}{=} \int_0^1 \frac{dz}{\sqrt{z^2 + 2z\cos\alpha_1 + 1}} = \omega_3.$$

Then, we have

$$\omega(y) = 2(\omega_1 + \omega_3)$$

$$= 2\left\{ \int_0^1 \frac{du}{\sqrt{u^2 + 2u\cos(\pi - \alpha_2) + 1}} + \int_0^1 \frac{du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \right\}$$

$$= 2\left\{ \int_0^1 \frac{1}{\sqrt{[u + \cos(\pi - \alpha_2)]^2 + \sin^2(\pi - \alpha_2)}} du + \int_0^1 \frac{1}{\sqrt{(u + \cos\alpha_1)^2 + \sin^2\alpha_1}} du \right\}$$

$$= 2\left\{ \ln\left(1 + \csc\frac{\alpha_2}{2}\right) + \ln\left(1 + \sec\frac{\alpha_1}{2}\right) \right\}$$

$$= 2\ln\left[\left(1 + \sec\frac{\alpha_1}{2}\right)\left(1 + \csc\frac{\alpha_2}{2}\right) \right] = k.$$

For $y \in (-\infty, 0)$, we still have

$$\omega(y) = \int_{-\infty}^{-1} \frac{1}{\sqrt{u^2 - 2u\cos\alpha_1 + 1}} \frac{-du}{u} + \int_{-1}^0 \frac{du}{\sqrt{u^2 - 2u\cos\alpha_1 + 1}}$$

$$+ \int_0^1 \frac{du}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} + \int_1^\infty \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \frac{du}{u}$$

$$= 2(\omega_1 + \omega_3) = k.$$

By the same way, we still can find that $\tilde{\omega}(x) = \omega(y) = k$ $(x, y \neq 0)$. The lemma is proved. $\quad\square$

LEMMA 2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f(x)$ is a nonnegative measurable function in $(-\infty, \infty)$, then we have*

$$J := \int_{-\infty}^\infty |y|^{-1} \left[\int_{-\infty}^\infty \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} f(x)dx \right]^p dy$$

$$\leqslant k^p \int_{-\infty}^\infty |x|^{p-1} f^p(x)dx. \tag{8}$$

Proof. By Lemma 1 and Hölder's inequality [16], we have

$$
\left(\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} f(x) dx \right)^p
$$

$$
= \left(\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \left[\frac{|x|^{1/q}}{|y|^{1/p}} f(x) \right] \left[\frac{|y|^{1/p}}{|x|^{1/q}} \right] dx \right)^p
$$

$$
\leqslant \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{|x|^{p-1}}{|y|} f^p(x) dx \right]
$$

$$
\times \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{|y|^{q-1}}{|x|} dx \right]^{p-1}
$$

$$
= \omega(y)^{p-1} |y| \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{|x|^{p-1}}{|y|} f^p(x) dx \right]. \tag{9}
$$

Then by (4), (9) and Fubini theorem [17], it follows

$$
J \leqslant k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{|x|^{p-1}}{|y|} f^p(x) dx \right] dy
$$

$$
= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \frac{1}{|y|} dy \right] |x|^{p-1} f^p(x) dx
$$

$$
= k^{p-1} \int_{-\infty}^{\infty} \tilde{\omega}(x) |x|^{p-1} f^p(x) dx
$$

$$
= k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx.
$$

The lemma is proved. □

3. Main results and applications

THEOREM 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f(x)$, $g(x) \geqslant 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx < \infty$ and $0 < \int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy < \infty$, then we have*

$$
I := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} f(x) g(y) dx dy
$$

$$
< k \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy \right)^{\frac{1}{q}}, \tag{10}
$$

$$
J = \int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|,|y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} f(x) dx \right]^p dy
$$

$$
< k^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx, \tag{11}
$$

where the constant factor k and k^p are the best possible (k is defined by (5)). *Inequality* (10) *and* (11) *are equivalent.*

Proof. If (9) takes the form of equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then there exists constants M and N, such that they are not all zero, and

$$M \frac{|x|^{p/q}}{|y|} f^p(x) = N \frac{|y|^{q/p}}{|x|} \quad a.e. \text{ in } (-\infty, \infty).$$

Hence, there exists a constant C, such that

$$M |x|^p f^p(x) = N |y|^q = C \quad a.e. \text{ in } (-\infty, \infty).$$

We suppose $M \neq 0$ (otherwise $N = M = 0$). Then it follows

$$|x|^{p-1} f^p(x) = \frac{C}{M |x|} \quad a.e. \text{ in } (-\infty, \infty),$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx < \infty$. Hence (9) takes the form of strict inequality; so does (8), and we have (11).

By the Hölder's inequality [16], we have

$$I = \int_{-\infty}^{\infty} \left[|y|^{\frac{-1}{p}} \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy \cos \alpha_i + y^2}} \right\} f(x) dx \right] [|y|^{\frac{1}{p}} g(y) dy]$$

$$\leqslant J^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy \right)^{\frac{1}{q}}. \tag{12}$$

By (11), we have (10). On the other hand, suppose that (10) is valid. Setting

$$g(y) = |y|^{-1} \left[\int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy \cos \alpha_i + y^2}} \right\} f(x) dx \right]^{p-1},$$

then $J = \int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy$. By (8), it follows $J < \infty$. If $J = 0$, then (11) is naturally valid. Assuming that $0 < J < \infty$, by (10), we obtain

$$\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy = J = I < k \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy \right)^{\frac{1}{q}}, \tag{13}$$

$$J^{\frac{1}{p}} = \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy \right)^{\frac{1}{p}} < k \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx \right)^{\frac{1}{p}}. \tag{14}$$

Hence we have (11), which is equivalent to (10).

If the constant factor k in (10) is not the best possible, then there exists a positive constant K with $K < k$, such that (10) is still valid as we replace k by K, then we have

$$I < K \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y) dy \right)^{\frac{1}{q}}. \tag{15}$$

For $\varepsilon > 0$, define functions $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} x^{-\frac{2\varepsilon}{p}-1}, & x \in (1, \infty), \\ 0, & x \in [-1, 1], \\ (-x)^{-\frac{2\varepsilon}{p}-1}, & x \in (-\infty, -1), \end{cases}$$

$$\tilde{g}(y) := \begin{cases} y^{-\frac{2\varepsilon}{q}-1}, & y \in (1, \infty), \\ 0, & y \in [-1, 1], \\ (-y)^{-\frac{2\varepsilon}{q}-1}, & y \in (-\infty, -1). \end{cases}$$

Replacing $f(x), g(y)$ by $\tilde{f}(x), \tilde{g}(y)$ in (15), we obtain

$$\tilde{I} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \tilde{f}(x)\tilde{g}(y)dxdy$$

$$< K \left(\int_{-\infty}^{\infty} |x|^{p-1} \tilde{f}^p(x)dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} \tilde{g}^q(y)dy \right)^{\frac{1}{q}}$$

$$= \frac{K}{\varepsilon}, \tag{16}$$

$$\tilde{I} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} \tilde{f}(x)\tilde{g}(y)dxdy = \sum_{i=1}^{4} I_i, \tag{17}$$

where,

$$I_1 := \int_{-\infty}^{-1} (-y)^{-\frac{2\varepsilon}{q}-1} \left[\int_{-\infty}^{-1} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} (-x)^{-\frac{2\varepsilon}{p}-1}dx \right] dy,$$

$$I_2 := \int_{-\infty}^{-1} (-y)^{-\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} x^{-\frac{2\varepsilon}{p}-1}dx \right] dy,$$

$$I_3 := \int_{1}^{\infty} y^{-\frac{2\varepsilon}{q}-1} \left[\int_{-\infty}^{-1} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} (-x)^{-\frac{2\varepsilon}{p}-1}dx \right] dy,$$

$$I_4 := \int_{1}^{\infty} y^{-\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \min_{i \in \{1,2\}} \left\{ \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha_i + y^2}} \right\} x^{-\frac{2\varepsilon}{p}-1}dx \right] dy.$$

By Fubini theorem [17], we obtain

$$I_1 = I_4 = \int_{1}^{\infty} y^{-\frac{2\varepsilon}{q}-1} \left[\int_{1}^{\infty} \frac{\min\{x, y\}}{\sqrt{x^2 + 2xy\cos\alpha_1 + y^2}} x^{-\frac{2\varepsilon}{p}-1}dx \right] dy$$

$$\overset{u=x/y}{=} \int_{1}^{\infty} y^{-2\varepsilon-1} \left[\int_{\frac{1}{y}}^{\infty} \frac{\min\{u, 1\}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} u^{-\frac{2\varepsilon}{p}-1}du \right] dy$$

$$
= \int_1^\infty y^{-2\varepsilon-1} \left[\int_{\frac{1}{y}}^1 \frac{u^{-\frac{2\varepsilon}{p}}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du \right] dy
$$

$$
+ \int_1^\infty y^{-2\varepsilon-1} \left[\int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du \right] dy
$$

$$
= \int_0^1 \left(\int_{\frac{1}{u}}^\infty y^{-2\varepsilon-1} dy \right) \frac{u^{-\frac{2\varepsilon}{p}} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{2\varepsilon} \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}}
$$

$$
= \frac{1}{2\varepsilon} \left[\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du \right],
$$

$$
I_2 = I_3 = \int_1^\infty y^{-\frac{2\varepsilon}{q}-1} \left[\int_1^\infty \frac{\min\{x,y\}}{\sqrt{x^2 - 2xy\cos\alpha_2 + y^2}} x^{-\frac{2\varepsilon}{p}-1} dx \right] dy
$$

$$
= \frac{1}{2\varepsilon} \left[\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} du + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} du \right].
$$

In view of the above results and by using (16) and (17), it follows

$$
\int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du
$$

$$
+ \int_0^1 \frac{u^{\frac{2\varepsilon}{q}}}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} du + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1}}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} du
$$

$$
= \varepsilon \tilde{I} < \varepsilon \cdot \frac{K}{\varepsilon} = K. \tag{18}
$$

By Fatou lemma [17] and (18), we find

$$
k = \omega(y) = \int_0^\infty \frac{\min\{u,1\}}{u\sqrt{u^2 + 2u\cos\alpha_1 + 1}} du + \int_0^\infty \frac{\min\{u,1\}}{u\sqrt{u^2 - 2u\cos\alpha_2 + 1}} du
$$

$$
= \int_0^1 \lim_{\varepsilon \to 0^+} \frac{u^{\frac{2\varepsilon}{q}} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \int_1^\infty \lim_{\varepsilon \to 0^+} \frac{u^{-\frac{2\varepsilon}{p}-1} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}}
$$

$$
+ \int_0^1 \lim_{\varepsilon \to 0^+} \frac{u^{\frac{2\varepsilon}{q}} du}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} + \int_1^\infty \lim_{\varepsilon \to 0^+} \frac{u^{-\frac{2\varepsilon}{p}-1} du}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}}
$$

$$
\leqslant \underline{\lim}_{\varepsilon \to 0^+} \left[\int_0^1 \frac{u^{\frac{2\varepsilon}{q}} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1} du}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} \right.
$$

$$
\left. + \int_0^1 \frac{u^{\frac{2\varepsilon}{q}} du}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} + \int_1^\infty \frac{u^{-\frac{2\varepsilon}{p}-1} du}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] \leqslant K, \tag{19}
$$

which contradicts the fact that $K < k$. Hence the constant factor k in (10) is the best possible. If the constant factor in (11) is not the best possible, then by (12), we may get

a contradiction that the constant factor in (10) is not the best possible. Thus the theorem is proved. $\square$

THEOREM 2. *As the assumptions of Theorem 1, replacing $p > 1$ by $0 < p < 1$, we have the equivalent reverse of* (10) *and* (11) *with the best constant factors.*

Proof. The way of proving of Theorem 2 is similar to Theorem 1. By the reverse Hölder's inequality [16], we have the reverse of (8) and (12). It is easy to obtain the reverse of (11). In view of the reverses of (11) and (12), we obtain the reverse of (10). On the other hand, suppose that the reverse of (10) is valid. Setting the same $g(y)$ as theorem 1, by the reverse of (8), we have $J > 0$. If $J = \infty$, then the reverse of (11) is obvious value; if $J < \infty$, then by the reverse of (10), we obtain the reverses of (13) and (14). Hence we have the reverse of (11), which is equivalent to the reverse of (10).

If the constant factor k in the reverse of (10) is not the best possible, then there exists a positive constant $\tilde{K}$ (with $\tilde{K} > k$), such that the reverse of (10) is still valid as we replace k by $\tilde{K}$. By the reverse of (18), we have

$$\int_0^1 \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{\frac{2\varepsilon}{q}} du$$

$$+ \int_1^\infty \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{-\frac{2\varepsilon}{p} - 1} du$$

$$> \tilde{K}. \tag{20}$$

For $0 < \varepsilon_0 < \frac{|q|}{2}$, we have $\frac{2\varepsilon_0}{q} > -1$. For $0 < \varepsilon \leqslant \varepsilon_0$, we obtain $u^{\frac{2\varepsilon}{q}} \leqslant u^{\frac{2\varepsilon_0}{q}}$ ($u \in (0,1]$) and

$$\int_0^1 \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{\frac{2\varepsilon_0}{q}} du$$

$$\leqslant \left(\frac{1}{\sin\alpha_1} + \frac{1}{\sin\alpha_2} \right) \int_0^1 u^{\frac{2\varepsilon_0}{q}} du = \left(\frac{1}{\sin\alpha_1} + \frac{1}{\sin\alpha_2} \right) \cdot \frac{1}{1 + (2\varepsilon_0)/q} < \infty.$$

Then by Lebesgue control convergence theorem [17], we have for $\varepsilon \to 0^+$ that

$$\int_0^1 \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{\frac{2\varepsilon}{q}} du$$

$$= \int_0^1 \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] du + o(1). \tag{21}$$

By the Levi's theorem [17], we find for $\varepsilon \to 0^+$ that

$$\int_1^\infty \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{-\frac{2\varepsilon}{p} - 1} du$$

$$= \int_1^\infty \left[\frac{1}{\sqrt{u^2 + 2u\cos\alpha_1 + 1}} + \frac{1}{\sqrt{u^2 - 2u\cos\alpha_2 + 1}} \right] u^{-1} du + \tilde{o}(1). \tag{22}$$

By (20), (21) and (22), for $\varepsilon \to 0^+$ in (20), we have $k \geqslant \tilde{K}$, which contradicts the fact that $k < \tilde{K}$. Hence the constant factor k in the reverse of (10) is the best possible. If the constant factor in reverse of (11) is not the best possible, then by the reverse of (12), we may get a contradiction that the constant factor in the reverse of (10) is not the best possible. Thus the theorem is proved. $\square$

REMARK 1. For $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$ in (10) and (11), we have the following equivalent inequalities:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha + y^2}} f(x)g(y)dxdy$$
$$< k_0 \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)dy \right)^{\frac{1}{q}}, \tag{23}$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy\cos\alpha + y^2}} f(x)dx \right]^p dy$$
$$< k_0^p \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)dx, \tag{24}$$

where $k_0 := 2\ln[(1 + \sec\frac{\alpha}{2})(1 + \csc\frac{\alpha}{2})]$ and k_0^p are the best possible.

REMARK 2. For $\alpha_1 = \alpha_2 = \frac{\pi}{3}$, $p = q = 2$ in (10) and (11), we have the following equivalent inequalities:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + xy + y^2}} f(x)g(y)dxdy$$
$$< 2\ln(3 + 2\sqrt{3}) \left(\int_{-\infty}^{\infty} |x| f^2(x)dx \int_{-\infty}^{\infty} |y| g^2(y)dy \right)^{\frac{1}{2}}, \tag{25}$$

$$\int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + xy + y^2}} f(x)dx \right]^2 dy$$
$$< 2^2 [\ln(3 + 2\sqrt{3})]^2 \int_{-\infty}^{\infty} |x| f^2(x)dx. \tag{26}$$

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