MINKOWSKI AND BECKENBACH—DRESHER
INEQUALITIES AND FUNCTIONALS ON TIME SCALES

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Abstract. We obtain integral forms of the Minkowski inequality and Beckenbach–Dresher inequality on time scales. Also, we investigate a converse of Minkowski’s inequality and several functionals arising from the Minkowski inequality and the Beckenbach–Dresher inequality.

1. Introduction and preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Stefan Hilger [6] in order to unify the theory of difference equations and the theory of differential equations. For an introduction to the theory of dynamic equations on time scales, we refer to [2, 7]. Martin Bohner and Gusein Sh. Guseinov [3, 4] defined the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue $\Delta$-integral with the Riemann $\Delta$-integral.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i, 1 \leq i \leq n\}$$

an $n$-dimensional time scale. Let $\mu_{\Delta}$ be the $\sigma$-additive Lebesgue $\Delta$-measure on $\Lambda^n$ and $\mathcal{F}$ be the family of $\Delta$-measurable subsets of $\Lambda^n$. Let $E \in \mathcal{F}$ and $(E, \mathcal{F}, \mu_{\Delta})$ be a time scale measure space. Then for a $\Delta$-measurable function $f : E \to \mathbb{R}$, the corresponding $\Delta$-integral of $f$ over $E$ will be denoted according to [4, (3.18)] by

$$\int_E f(t_1, \ldots, t_n) \Delta t_1 \ldots \Delta t_n, \quad \int_E f(t) \Delta t, \quad \int_E f \mu_{\Delta}, \text{ or } \int_E f(t) \mu_{\Delta}(t).$$

By [4, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue $\Delta$-integrals on $\Lambda^n$. Here we state Fubini’s theorem for integrals on time scales. It is used in the proofs of our main results.


Keywords and phrases: Integration, time scale, Minkowski inequality, Beckenbach–Dresher inequality, subadditive, superadditive, Minkowski functional, Beckenbach–Dresher functional.

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THEOREM 1.1. (Fubini’s theorem) Let $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite-dimensional time scale measure spaces. If $f : X \times Y \to \mathbb{R}$ is a $\Delta$-integrable function and if we define the functions

$$\varphi(y) = \int_X f(x,y) d\mu_\Delta(x) \quad \text{for a.e. } y \in Y$$

and

$$\psi(x) = \int_Y f(x,y) d\nu_\Delta(y) \quad \text{for a.e. } x \in X,$$

then $\varphi$ is $\Delta$-integrable on $Y$ and $\psi$ is $\Delta$-integrable on $X$ and

$$\int_X \mu_\Delta(x) \int_Y f(x,y) d\nu_\Delta(y) = \int_Y \nu_\Delta(y) \int_X f(x,y) d\mu_\Delta(x). \quad (1.1)$$

Some classical inequalities, including Jensen’s inequality, Hölder’s inequality, Minkowski’s inequality and their converses for multiple integration on time scales were investigated in [1]. These inequalities hold for both Riemann integrals and Lebesgue integrals on time scales. For completeness, let us recall these inequalities from [1].

THEOREM 1.2. (Jensen’s inequality [1, Theorem 4.2]) Assume $\Phi \in C(I, \mathbb{R})$ is convex, where $I \subseteq \mathbb{R}$ is an interval. Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space and suppose $f$ is $\Delta$-integrable on $E$ such that $\int f(E) = I$. Moreover, let $h : E \to \mathbb{R}$ be nonnegative $\Delta$-integrable such that $\int_E h d\mu_\Delta > 0$. Then

$$\Phi \left( \frac{\int_E f(t)h(t) d\mu_\Delta(t)}{\int_E h(t) d\mu_\Delta(t)} \right) \leq \frac{\int_E \Phi(f(t))h(t) d\mu_\Delta(t)}{\int_E h(t) d\mu_\Delta(t)}. \quad (1.2)$$

If $\Phi$ is concave, then (1.2) is reversed.

THEOREM 1.3. (Hölder’s inequality [1, Theorem 6.2]) For $p \neq 1$, define $q = p/(p-1)$. Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Assume $w, f, g$ are nonnegative functions such that $w f^p, w g^q, wfg$ are $\Delta$-integrable on $E$. If $p > 1$, then

$$\int_E w(t)f(t)g(t) d\mu_\Delta(t) \leq \left( \int_E w(t)f^p(t) d\mu_\Delta(t) \right)^{1/p} \left( \int_E w(t)g^q(t) d\mu_\Delta(t) \right)^{1/q}. \quad (1.3)$$

If $0 < p < 1$ and $\int_E w^q d\mu_\Delta > 0$, or if $p < 0$ and $\int_E w f^p d\mu_\Delta > 0$, then (1.3) is reversed.

THEOREM 1.4. (Minkowski’s inequality [1, Theorem 7.2]) Let $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. For $p \in \mathbb{R}$, assume $w, f, g$, are nonnegative functions such that $w f^p, w g^q, w(f + g)^p$ are $\Delta$-integrable on $E$. If $p \geq 1$, then

$$\left( \int_E w(t)(f(t) + g(t))^p d\mu_\Delta(t) \right)^{1/p} \leq \left( \int_E w(t)f^p(t) d\mu_\Delta(t) \right)^{1/p} + \left( \int_E w(t)g^q(t) d\mu_\Delta(t) \right)^{1/p}. \quad (1.4)$$
If \( 0 < p < 1 \) or \( p < 0 \), then (1.4) is reversed provided each of the two terms on the right-hand side is positive.

**Theorem 1.5.** (Converse of Hölder’s inequality [1, Theorem 11.3]) For \( p \neq 1 \), define \( q = p / (p - 1) \). Let \((E, \mathcal{F}, \mu_\Delta)\) be a time scale measure space. Assume \( w, f, g \) are nonnegative functions such that \( wf^p, wg^q, wfg \) are \( \Delta \)-integrable on \( E \). Suppose

\[
0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all } t \in E.
\]

If \( p > 1 \), then

\[
\int_E w(t)f(t)g(t)d\mu_\Delta(t) \geq K(p,m,M) \left( \int_E w(t)f^p(t)d\mu_\Delta(t) \right)^{1/p} \times \left( \int_E w(t)g^q(t)d\mu_\Delta(t) \right)^{1/q},
\]

where

\[
K(p,m,M) = |p|^{1/p}|q|^{1/q} \frac{(M-m)^{1/p}|mM^p-Mm^p|^{1/q}}{|M^p-m^p|}.
\]

If \( 0 < p < 1 \) or \( p < 0 \), then (1.5) is reversed provided either \( \int_E wg^q d\mu_\Delta > 0 \) or \( \int_E wf^p d\mu_\Delta > 0 \).

**2. Minkowski inequalities**

Theorem 1.4 also holds if we have a finite number of functions. The next theorem gives an inequality of Minkowski type for infinitely many functions. In the sequel, we assume that all occurring integrals are finite.

**Theorem 2.1.** (Integral Minkowski inequality) Let \((X, \mathcal{M}, \mu_\Delta)\) and \((Y, \mathcal{L}, \nu_\Delta)\) be time scale measure spaces and let \( u, v, \) and \( f \) be nonnegative functions on \( X, Y, \) and \( X \times Y \), respectively. If \( p \geq 1 \), then

\[
\left[ \int_X \left( \int_Y f(x,y)v(y)dv_\Delta(y) \right)^p u(x)d\mu_\Delta(x) \right]^{1/p} \leq \int_Y \left( \int_X f^p(x,y)u(x)d\mu_\Delta(x) \right)^{1/p} v(y)dv_\Delta(y)
\]

holds provided all integrals in (2.1) exists. If \( 0 < p < 1 \) and

\[
\int_X \left( \int_Y f^p v dv_\Delta \right)^{1/p} ud\mu_\Delta > 0, \quad \int_Y f^p v dv_\Delta > 0
\]

holds, then (2.1) is reversed. If \( p < 0 \) and (2.2) and

\[
\int_X f^p ud\mu_\Delta > 0,
\]

hold, then (2.1) is reversed as well.
Proof. Let $p \geq 1$. Put 

$$H(x) = \int_Y f(x,y)v(y)d\nu_{\Delta}(y).$$

Now, by using Fubini’s theorem (Theorem 1.1) and Hölder’s inequality (Theorem 1.3) on time scales, we have

$$\int_X H^p(x)u(x)d\mu_{\Delta}(x) = \int_X H(x)H^{p-1}(x)u(x)d\mu_{\Delta}(x)$$

$$= \int_X \left( \int_Y f(x,y)v(y)d\nu_{\Delta}(y) \right) H^{p-1}(x)u(x)d\mu_{\Delta}(x)$$

$$= \int_Y \left( \int_X f(x,y)H^{p-1}(x)u(x)d\mu_{\Delta}(x) \right) v(y)d\nu_{\Delta}(y)$$

$$\leq \int_Y \left( \int_X f^p(x,y)u(x)d\mu_{\Delta}(x) \right)^{\frac{1}{p}} \left( \int_X H^p(x)u(x)d\mu_{\Delta}(x) \right)^{\frac{p-1}{p}} v(y)d\nu_{\Delta}(y)$$

$$= \int_Y \left( \int_X f^p(x,y)u(x)d\mu_{\Delta}(x) \right)^{\frac{1}{p}} v(y)d\nu_{\Delta}(y) \left( \int_X H^p(x)u(x)d\mu_{\Delta}(x) \right)^{\frac{p-1}{p}}$$

and hence

$$\left( \int_X H^p(x)u(x)d\mu_{\Delta}(x) \right)^{\frac{1}{p}} \leq \int_Y \left( \int_X f^p(x,y)u(x)d\mu_{\Delta}(x) \right)^{\frac{1}{p}} v(y)d\nu_{\Delta}(y).$$

For $p < 0$ and $0 < p < 1$, the corresponding results can be obtained similarly. □

THEOREM 2.2. (Converse of integral Minkowski inequality) Let $(X, \mathcal{M}, \mu_{\Delta})$ and $(Y, \mathcal{L}, \nu_{\Delta})$ be time scale measure spaces and let $u$, $v$, and $f$ be nonnegative functions on $X$, $Y$, and $X \times Y$, respectively. Suppose

$$0 < m \leq \frac{f(x,y)}{\int_Y f(x,y)v(y)d\nu_{\Delta}(y)} \leq M \quad \text{for all} \quad x \in X, y \in Y.$$

If $p \geq 1$, then

$$\left[ \int_X \left( \int_Y f(x,y)v(y)d\nu_{\Delta}(y) \right)^p u(x)d\mu_{\Delta}(x) \right]^{\frac{1}{p}} \geq K(p,m,M) \int_Y \left( \int_X f^p(x,y)u(x)d\mu_{\Delta}(x) \right)^{\frac{1}{p}} v(y)d\nu_{\Delta}(y)$$

(2.4)

provided all integrals in (2.4) exist, where $K(p,m,M)$ is defined by (1.6). If $0 < p < 1$ and (2.2) holds, then (2.4) is reversed. If $p < 0$ and (2.2) and (2.3) hold, then (2.4) is reversed as well.
Proof. Let \( p \geq 1 \). Put
\[
H(x) = \int_Y f(x,y) v(y) \, dv_\Delta(y).
\]
Then by using Fubini’s theorem (Theorem 1.1) and the converse Hölder inequality (Theorem 1.5) on time scales, we get
\[
\int_X H^p(x) u(x) \, d\mu_\Delta(x) = \int_Y \left( \int_X f(x,y) v(y) \, dv_\Delta(y) \right) H^{p-1}(x) u(x) \, d\mu_\Delta(x)
\geq K(p,m,M) \int_Y \left( \int_X f^p(x,y) u(x) \, d\mu_\Delta(x) \right)^{1/p} \left( \int_X H^p(x) u(x) \, d\mu_\Delta(x) \right)^{p-1/p} \, v(y) \, dv_\Delta(y).
\]
Dividing both sides by \( (\int_X H^p(x) u(x) \, d\mu_\Delta(x) )^{p-1/p} \), we obtain (2.4). For \( 0 < p < 1 \) and \( p < 0 \), the corresponding results can be obtained similarly. \( \square \)

Let the functions \( f, u, v \) be defined as in Theorem 2.1. Now we define the \( r \)th power mean \( M^{[r]}(f, \mu_\Delta) \) of the function \( f \) with respect to the measure \( \mu_\Delta \) by
\[
M^{[r]}(f, \mu_\Delta) = \begin{cases} 
\left( \frac{\int_X f^r(x,y) u(x) \, d\mu_\Delta(x)}{\int_X u(x) \, d\mu_\Delta(x)} \right)^{1/r} & \text{if } r \neq 0, \\
\exp \left( \frac{\int_X \log f(x,y) u(x) \, d\mu_\Delta(x)}{\int_X u(x) \, d\mu_\Delta(x)} \right) & \text{if } r = 0,
\end{cases}
\]
where \( \int_X u \, d\mu_\Delta > 0 \).

**Corollary 2.3.** Let \( 0 < s \leq r \). Then
\[
M^{[r]}(M^{[s]}(f, dv_\Delta), dv_\Delta) \geq K \left( \frac{r}{s}, m, M \right) M^{[s]}(M^{[r]}(f, d\mu_\Delta), dv_\Delta).
\]

**Proof.** By putting \( p = r/s \) and replacing \( f \) by \( f^s \) in (2.4), raising to the power of \( \frac{1}{s} \) and dividing by
\[
\left( \int_X u(x) \, d\mu_\Delta(x) \right)^{\frac{1}{s}} \left( \int_Y v(y) \, dv_\Delta(y) \right)^{\frac{1}{s}},
\]
we get the above result. \( \square \)
3. Minkowski functionals

In this section, we will consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [8].

Let \( f \) and \( v \) be fixed functions satisfying the assumptions of Theorem 2.1. Let us consider the functional \( M_1 \) defined by

\[
M_1(u) = \left[ \int_Y \left( \int_X f^p(x,y)u(x)d\mu_\Delta(x) \right)^\frac{1}{p} v(y)d\nu_\Delta(y) \right]^p - \int_X \left( \int_Y f(x,y)v(y)d\nu_\Delta(y) \right)^p u(x)d\mu_\Delta(x),
\]

where \( u \) is a nonnegative function on \( X \) such that all occurring integrals exist. Also, if we fix the functions \( f \) and \( u \), then we can consider the functional

\[
M_2(v) = \int_Y \left( \int_X f^p(x,y)u(x)d\mu_\Delta(x) \right)^\frac{1}{p} v(y)d\nu_\Delta(y) - \left[ \int_X \left( \int_Y f(x,y)v(y)d\nu_\Delta(y) \right)^p u(x)d\mu_\Delta(x) \right]^\frac{1}{p},
\]

where \( v \) is a nonnegative function on \( Y \) such that all occurring integrals exist.

**Remark 3.1.**
(i) It is obvious that \( M_1 \) and \( M_2 \) are positive homogeneous, i.e., \( M_1(au) = aM_1(u) \), and \( M_2(av) = aM_2(v) \), for any \( a > 0 \).

(ii) If \( p \geq 1 \) or \( p < 0 \), then \( M_1(u) \geq 0 \), and if \( 0 < p < 1 \), then \( M_1(u) \leq 0 \).

(iii) If \( p \geq 1 \), then \( M_2(v) \geq 0 \), and if \( p < 1 \) and \( p \neq 0 \), then \( M_2(v) \leq 0 \).

**Theorem 3.2.**
(i) If \( p \geq 1 \) or \( p < 0 \), then \( M_1 \) is superadditive. If \( 0 < p < 1 \), then \( M_1 \) is subadditive.

(ii) If \( p \geq 1 \), then \( M_2 \) is superadditive. If \( p < 1 \) and \( p \neq 0 \), then \( M_2 \) is subadditive.

(iii) Suppose \( u_1 \) and \( u_2 \) are nonnegative functions such that \( u_2 \geq u_1 \). If \( p \geq 1 \) or \( p < 0 \), then

\[
0 \leq M_1(u_1) \leq M_1(u_2),
\]

and if \( 0 < p < 1 \), then (3.1) is reversed.

(iv) Suppose \( v_1 \) and \( v_2 \) are nonnegative functions such that \( v_2 \geq v_1 \). If \( p \geq 1 \), then

\[
0 \leq M_2(v_1) \leq M_2(v_2),
\]

and if \( p < 1 \) and \( p \neq 0 \), then (3.2) is reversed.
Proof. First we show (i). We have
\[
M_1(u_1 + u_2) - M_1(u_1) - M_1(u_2) = \left[ \int_Y \left( \int_X f^p(x,y) (u_1 + u_2)(x) \, d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) \, dv_\Delta(y) \right]^p \\
- \int_X \left( \int_Y f(x,y) v(y) \, dv_\Delta(y) \right)^p (u_1 + u_2)(x) \, d\mu_\Delta(x) \\
- \left[ \int_Y \left( \int_X f^p(x,y) u_1(x) \, d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) \, dv_\Delta(y) \right]^p \\
+ \int_X \left( \int_Y f(x,y) v(y) \, dv_\Delta(y) \right)^p u_1(x) \, d\mu_\Delta(x) \\
- \left[ \int_Y \left( \int_X f^p(x,y) u_2(x) \, d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) \, dv_\Delta(y) \right]^p \\
+ \int_X \left( \int_Y f(x,y) v(y) \, dv_\Delta(y) \right)^p u_2(x) \, d\mu_\Delta(x)
\]

Using the Minkowski inequality (1.4) for integrals (Theorem 1.4) with \( p \) replaced by \( \frac{1}{p} \), we have
\[
M_1(u_1 + u_2) - M_1(u_1) - M_1(u_2) \begin{cases} 
\geq 0 & \text{if } p \geq 1 \text{ or } p < 0, \\
\leq 0 & \text{if } 0 < p \leq 1. 
\end{cases} \tag{3.3}
\]

So, \( M_1 \) is superadditive for \( p \geq 1 \) or \( p < 0 \), and it is subadditive for \( 0 < p \leq 1 \). The proof of (ii) is similar: After a simple calculation, we have
\[
M_2(v_1 + v_2) - M_2(v_1) - M_2(v_2)
\]
\[
= \left[ \int_X \left( \int_Y f(x,y) v_1(y) \, dv_\Delta(y) \right)^p u(x) \, d\mu_\Delta(x) \right]^{\frac{1}{p}} \\
+ \left[ \int_X \left( \int_Y f(x,y) v_2(y) \, dv_\Delta(y) \right)^p u(x) \, d\mu_\Delta(x) \right]^{\frac{1}{p}} \\
- \left[ \int_X \left( \int_Y f(x,y) (v_1 + v_2)(y) \, dv_\Delta(y) \right)^p u(x) \, d\mu_\Delta(x) \right]^{\frac{1}{p}}.
\]
Using the Minkowski inequality (2.1) for integrals (Theorem 2.1), we have that this is nonnegative for \( p \geq 1 \) and nonpositive for \( p < 1 \) and \( p \neq 0 \). Now we show (iii). If \( p \geq 1 \) or \( p < 0 \), then using superadditivity and positivity of \( M_1, u_2 \geq u_1 \) implies
\[
M_1(u_2) = M_1(u_1 + (u_2 - u_1)) \geq M_1(u_1) + M_1(u_2 - u_1) \geq M_1(u_1),
\]
and the proof of (3.1) is established. If \( 0 < p < 1 \), then using subadditivity and negativity of \( M_1 \)
\[
M_1(u_2) \leq M_1(u_1) + M_1(u_2 - u_1) \leq M_1(u_1).
\]
The proof of (iv) is similar. □

REMARK 3.3. From Theorem 3.2, we obtain a refinement of the discrete Minkowski inequality given in [8]. Namely, put \( X, Y \subseteq \mathbb{N} \) and let \( u \) be \( \Delta \)-measurable on \( X \) and \( v_1, v_2 \) be \( \Delta \)-measurable on \( Y \) such that \( u(i) = u_i \geq 0, \ i \in X, \ v_1(j) = n_j \geq 0, \ v_2(j) = p_j \geq 0, \ j \in Y \). Then, for fixed \( f \) and \( u \), the function \( M_2 \) has the form
\[
M_2(v_1) = \sum_{j \in Y} n_j \left( \sum_{i \in X} u_i a_{ij}^p \right)^{1/p} - \left( \sum_{i \in X} u_i \left( \sum_{j \in Y} n_j a_{ij} \right)^p \right)^{1/p},
\]
where \( f(i, j) = a_{ij} \geq 0 \). If \( p \geq 1 \), then the mapping \( M_2 \) is superadditive, and \( p_j \geq n_j \) for all \( j \in Y \) implies
\[
0 \leq \sum_{j \in Y} n_j \left( \sum_{i \in X} u_i a_{ij}^p \right)^{1/p} - \left( \sum_{i \in X} u_i \left( \sum_{j \in Y} n_j a_{ij} \right)^p \right)^{1/p}
\]
\[
\leq \sum_{j \in Y} p_j \left( \sum_{i \in X} u_i a_{ij}^p \right)^{1/p} - \left( \sum_{i \in X} u_i \left( \sum_{j \in Y} p_j a_{ij} \right)^p \right)^{1/p}
\]
provided all occurring sums are finite.

COROLLARY 3.4. (i) Suppose \( u_1 \) and \( u_2 \) are nonnegative functions such that \( C u_2 \geq u_1 \geq c u_2 \), where \( c, C \geq 0 \). If \( p \geq 1 \) or \( p < 0 \), then
\[
c M_1(u_2) \leq M_1(u_1) \leq C M_1(u_2),
\]
and if \( 0 < p < 1 \), then the above inequality is reversed.

(ii) Suppose \( v_1 \) and \( v_2 \) are nonnegative functions such that \( C v_2 \geq v_1 \geq c v_2 \), where \( c, C \geq 0 \). If \( p \geq 1 \), then
\[
c M_2(v_2) \leq M_2(v_1) \leq C M_2(v_2),
\]
and if \( p < 1 \) and \( p \neq 0 \), then the above inequality is reversed.
COROLLARY 3.5. If $v_1$ and $v_2$ are nonnegative functions such that $v_2 \geq v_1$, then

$$M^{[0]} \left( \int_Y f(x,y)v_1(y)dv_\Delta(y), \mu_\Delta \right) - \int_Y M^{[0]}(f, \mu_\Delta)v_1(y)dv_\Delta(y) \leq M^{[0]} \left( \int_Y f(x,y)v_2(y)dv_\Delta(y), \mu_\Delta \right) - \int_Y M^{[0]}(f, \mu_\Delta)v_2(y)dv_\Delta(y),$$

where $M^{[0]}(f, \mu_\Delta)$ is defined in (2.5).

REMARK 3.6. If the measures are discrete, then from Corollary 3.5, we get the following result: Let $u_j, v_i, w_i, a_{ij} > 0$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, k$. Put $U = \sum_{j=1}^k u_j$. If $v_i \leq w_i$ for all $i = 1, \ldots, n$, then

$$\prod_{j=1}^k \left( \sum_{i=1}^n v_i a_{ij} \right)^{\frac{n_j}{u_j}} - \sum_{i=1}^n v_i \left( \prod_{j=1}^k a_{ij}^{\frac{n_j}{u_j}} \right) \leq \prod_{j=1}^k \left( \sum_{i=1}^n w_i a_{ij} \right)^{\frac{n_j}{u_j}} - \sum_{i=1}^n w_i \left( \prod_{j=1}^k a_{ij}^{\frac{n_j}{u_j}} \right).$$

This inequality is a refinement of the discrete Hölder inequality

$$\prod_{j=1}^k \left( \sum_{i=1}^n w_i a_{ij} \right)^{\frac{n_j}{u_j}} \geq \sum_{i=1}^n w_i \left( \prod_{j=1}^k a_{ij}^{\frac{n_j}{u_j}} \right).$$

The next result gives another property of $M_1$, but a similar result can also be stated for $M_2$.

THEOREM 3.7. Let $\varphi : [0, \infty) \to [0, \infty)$ be a concave function. Suppose $u_1$ and $u_2$ are nonnegative functions such that

$$\varphi \circ u_1, \quad \varphi \circ u_2, \quad \varphi \circ (\alpha u_1 + (1-\alpha)u_2)$$

are $\Delta$-integrable for $\alpha \in [0,1]$. If $p \geq 1$, then

$$M_1(\varphi \circ (\alpha u_1 + (1-\alpha)u_2)) \geq \alpha M_1(\varphi \circ u_1) + (1-\alpha)M_1(\varphi \circ u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

Proof. We show this only for $p \geq 1$ as the other case follows similarly. Since $\varphi$ is concave, we have

$$\varphi(\alpha u_1 + (1-\alpha)u_2) \geq \alpha \varphi(u_1) + (1-\alpha)\varphi(u_2).$$

Now, from (3.1) and (3.3), we have

$$M_1(\varphi \circ (\alpha u_1 + (1-\alpha)u_2)) \geq M_1(\alpha(\varphi \circ u_1) + (1-\alpha)(\varphi \circ u_2))$$

$$\geq M_1(\alpha(\varphi \circ u_1)) + M_1((1-\alpha)(\varphi \circ u_2))$$

$$\geq \alpha M_1(\varphi \circ u_1) + (1-\alpha)M_1(\varphi \circ u_2),$$
and the proof is established. □

Let $f$, $u$ and $v$ be fixed functions satisfying the assumptions of Theorem 2.1. Let us define functionals $M_3$ and $M_4$ by

$$M_3(A) = \left[ \int_Y \left( \int_A f^p(x,y)u(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y)d\nu_\Delta(y) \right]^p - \int_A \left( \int_Y f(x,y)v(y)d\nu_\Delta(y) \right)^p u(x)d\mu_\Delta(x)$$

and

$$M_4(B) = \int_B \left( \int_X f^p(x,y)u(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y)d\nu_\Delta(y) - \left[ \int_X \left( \int_B f(x,y)v(y)d\nu_\Delta(y) \right)^p u(x)d\mu_\Delta(x) \right]^{\frac{1}{p}},$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings $M_3$ and $M_4$.

**Theorem 3.8.** (i) Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then

$$M_3(A_1 \cup A_2) \geq M_3(A_1) + M_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) Suppose $A_1, A_2 \subseteq X$ and $A_1 \subseteq A_2$. If $p \geq 1$ or $p < 0$, then

$$M_3(A_1) \leq M_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(iii) Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then

$$M_4(B_1 \cup B_2) \geq M_4(B_1) + M_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

(iv) Suppose $B_1, B_2 \subseteq Y$ and $B_1 \subseteq B_2$. If $p \geq 1$, then

$$M_4(B_1) \leq M_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

The proof of Theorem 3.8 is omitted as it is similar to the proof of Theorem 3.2.

**Remark 3.9.** For $p \geq 1$, if $S_m$ is a subset of $Y$ with $m$ elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then we have

$$M_4(S_m) \geq M_4(S_{m-1}) \geq \ldots \geq M_4(S_2) \geq 0$$

and

$$M_4(S_m) \geq \max\{M_4(S_2) : S_2 \text{ is any subset of } S_m \text{ with 2 elements} \}.$$
4. Beckenbach–Dresher inequalities

**Theorem 4.1.** Let \((X, \mathcal{M}, \mu_\Delta), (X, \mathcal{M}, \lambda_\Delta)\) and \((Y, \mathcal{L}, \nu_\Delta)\) be time scale measure spaces. Suppose \(u\) and \(w\) are nonnegative functions on \(X\), \(v\) is a nonnegative function on \(Y\), \(f\) is a nonnegative function on \(X \times Y\) with respect to the measure \((\mu_\Delta \times \nu_\Delta)\), and \(g\) is a nonnegative function on \(X \times Y\) with respect to the measure \((\lambda_\Delta \times \nu_\Delta)\). If

\[
 s \geq 1, \quad q \leq 1 \leq p, \quad \text{and} \quad q \neq 0 \tag{4.1}
\]

or

\[
 s < 0, \quad p \leq 1 \leq q, \quad \text{and} \quad p \neq 0, \tag{4.2}
\]

then

\[
 \frac{\left[ \int_X \left( \int_Y f(x,y) v(y) d\nu_\Delta(y) \right)^p u(x) d\mu_\Delta(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y) v(y) d\nu_\Delta(y) \right)^q w(x) d\lambda_\Delta(x) \right]^{\frac{1}{q}}} \leq \int_Y \left( \frac{\left( \int_X f^p(x,y) u(x) d\mu_\Delta(x) \right)^{\frac{1}{p}}}{\left( \int_Y \left( \int_X g^q(x,y) w(x) d\lambda_\Delta(x) \right)^{\frac{1}{q}} v(y) d\nu_\Delta(y) \right)^{\frac{1}{q}}} v(y) d\nu_\Delta(y) \right) \tag{4.3}
\]

provided all occurring integrals in (4.3) exist. If

\[
 0 < s \leq 1, \quad p \leq 1, \quad q \leq 1, \quad \text{and} \quad q \neq 0, \tag{4.4}
\]

then (4.3) is reversed.

**Proof.** Assume (4.1) or (4.2). By using the integral Minkowski inequality (2.1) and Hölder’s inequality (1.3), we have

\[
 \frac{\left[ \int_X \left( \int_Y f(x,y) v(y) d\nu_\Delta(y) \right)^p u(x) d\mu_\Delta(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y) v(y) d\nu_\Delta(y) \right)^q w(x) d\lambda_\Delta(x) \right]^{\frac{1}{q}}} \leq \left[ \int_Y \left( \int_X f^p(x,y) u(x) d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) d\nu_\Delta(y) \right]^s \quad \left[ \int_Y \left( \int_X g^q(x,y) w(x) d\lambda_\Delta(x) \right)^{\frac{1}{q}} v(y) d\nu_\Delta(y) \right]^{1-s}
\]

\[
 = \left[ \int_Y \left( \left( \int_X f^p(x,y) u(x) d\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) d\nu_\Delta(y) \right)^s \right] \left[ \int_Y \left( \left( \int_X g^q(x,y) w(x) d\lambda_\Delta(x) \right)^{\frac{1}{q}} v(y) d\nu_\Delta(y) \right)^{1-s} \right]
\]

\[
 \leq \int_Y \left( \int_X f^p(x,y) u(x) d\mu_\Delta(x) \right)^{\frac{1}{p}} \left( \int_X g^q(x,y) w(x) d\lambda_\Delta(x) \right)^{\frac{1}{q}} v(y) d\nu_\Delta(y). \nonumber
\]

If (4.4) holds, then the reversed inequality in (4.3) can be proved in a similar way. \(\square\)
REMARK 4.2. If \( X, Y \subseteq \mathbb{R}^n \), then Theorem 4.1 is a generalization of the well-known Beckenbach–Dresher inequality which states that for nonnegative real functions \( f, g \) and for \( p \geq 1 \geq q \geq 0 \), we have

\[
\left( \frac{\int_E (f + g)^p \, d\Phi}{\int_E (f + g)^q \, d\Phi} \right)^{\frac{1}{p}} \leq \left( \frac{\int_E f^p \, d\Phi}{\int_E f^q \, d\Phi} \right)^{\frac{1}{p}} + \left( \frac{\int_E g^p \, d\Phi}{\int_E g^q \, d\Phi} \right)^{\frac{1}{p}}. \tag{4.5}
\]

Some historical facts about (4.5) and new results which generalize (4.5) are given in [5, 9]. For a time scale analogue of (4.5), see [1, Theorem 8.2].

5. Beckenbach–Dresher functionals

Let \( f, g, u, w \) be fixed functions satisfying the assumptions of Theorem 4.1. We define the Beckenbach–Dresher functional \( BD(v) \) by

\[
BD(v) = \int_Y \frac{\left( \int_X f^p(x,y)u(x)\, d\mu(x) \right)^{\frac{1}{p}} v(y)\, d\nu(y)}{\left( \int_X f^q(x,y)w(x)\, d\lambda(x) \right)^{\frac{1}{q}}} v(y)\, d\nu(y)
- \frac{\left[ \int_X \left( \int_Y f(x,y) v(y)\, d\nu(y) \right)^p u(x)\, d\mu(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y) v(y)\, d\nu(y) \right)^q w(x)\, d\lambda(x) \right]^{\frac{1}{q}}}
\]

where we suppose that all occurring integrals exist.

THEOREM 5.1. If (4.1) or (4.2) holds, then

\[
BD(v_1 + v_2) \geq BD(v_1) + BD(v_2). \tag{5.1}
\]

If \( v_2 \geq v_1 \), then

\[
BD(v_1) \leq BD(v_2). \tag{5.2}
\]

If \( C, c \geq 0 \) and \( Cv_2 \geq v_1 \geq cv_2 \), then

\[
CBD(v_2) \geq BD(v_1) \geq cBD(v_1). \tag{5.3}
\]

If (4.4) holds, then (5.1), (5.2) and (5.3) are reversed.

Proof. Assume (4.1) or (4.2). Then we have

\[
BD(v_1 + v_2) - BD(v_1) - BD(v_2)
= \frac{\left[ \int_X \left( \int_Y f(x,y) v_1(y)\, d\nu(y) \right)^p u(x)\, d\mu(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y) v_1(y)\, d\nu(y) \right)^q w(x)\, d\lambda(x) \right]^{\frac{1}{q}}}
+ \frac{\left[ \int_X \left( \int_Y f(x,y) v_2(y)\, d\nu(y) \right)^p u(x)\, d\mu(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y) v_2(y)\, d\nu(y) \right)^q w(x)\, d\lambda(x) \right]^{\frac{1}{q}}}
\]
\[
\frac{\left[ \int_X \left( \int_Y f(x,y)v_1(y) d\nu\Delta(y) + \int_Y f(x,y)v_2(y) d\nu\Delta(y) \right)^p u(x) d\mu\Delta(x) \right]^{\frac{1}{p}}}{\left[ \int_X \left( \int_Y g(x,y)v_1(y) d\nu\Delta(y) + \int_Y g(x,y)v_2(y) d\nu\Delta(y) \right)^q w(x) d\lambda\Delta(x) \right]^{\frac{1}{q}}} \geq 0,
\]

where in the last inequality we used (4.3) from Theorem 4.1. Using Theorem 4.1 again, \( v_2 \geq v_1 \) implies

\[
\text{BD}(v_2) = \text{BD}(v_1 + (v_2 - v_1)) \geq \text{BD}(v_1) + \text{BD}(v_2 - v_1) \geq \text{BD}(v_1).
\]

The proof of (5.3) is similar. If (4.4) holds, then the reversed inequalities of (5.1), (5.2) and (5.3) can be proved in a similar way. \( \Box \)

Let \( f, g, u, v, w \) be fixed functions. We define a functional \( \text{BD}_1 \) by

\[
\text{BD}_1(A) = \int_A \left( \frac{\left( \int_X f^p(x,y) u(x) d\mu\Delta(x) \right)^{\frac{1}{p}}}{{\left( \int_X g^q(x,y) w(x) d\lambda\Delta(x) \right)^{\frac{1}{q}}}^2} v(y) d\nu\Delta(y) - \frac{\left[ \int_X \left( \int_Y f(x,y)v(y) d\nu\Delta(y) \right)^p u(x) d\mu\Delta(x) \right]^{\frac{1}{p}}}{{\left[ \int_X \left( \int_Y g(x,y)v(y) d\nu\Delta(y) \right)^q w(x) d\lambda\Delta(x) \right]^{\frac{1}{q}}}^2},
\]

where \( A \subseteq Y \).

For \( \text{BD}_1 \), the following result holds.

**Theorem 5.2.** (i) Suppose \( A_1, A_2 \subseteq Y \) and \( A_1 \cap A_2 = \emptyset \). If (4.1) or (4.2) holds, then

\[
\text{BD}_1(A_1 \cup A_2) \geq \text{BD}_1(A_1) + \text{BD}_1(A_2),
\]

and if (4.4) holds, then the above inequality is reversed.

(ii) Suppose \( A_1, A_2 \subseteq Y \) and \( A_1 \subseteq A_2 \). If (4.1) or (4.2) holds, then

\[
\text{BD}_1(A_1) \leq \text{BD}_1(A_2),
\]

and if (4.4) holds, then the above inequality is reversed.

The proof of Theorem 5.2 is omitted as it is similar to the proof of Theorem 5.1.

**Remark 5.3.** If \( S_k \subseteq Y \) has \( k \) elements and if \( S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2 \), then (4.1) or (4.2) implies

\[
\text{BD}_1(S_m) \geq \text{BD}_1(S_{m-1}) \geq \ldots \geq \text{BD}_1(S_2) \geq 0
\]

and \( \text{BD}_1(S_m) \geq \max \{ \text{BD}_1(S_2) : S_2 \text{ is any subset of } S_m \text{ with 2 elements} \} \), while (4.4) implies the reversed inequalities with max replaced by min.
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