Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 313–335 doi:10.7153/jmi-07-29

n–EXPONENTIAL CONVEXITY FOR JENSEN–TYPE INEQUALITIES

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(*Communicated by A. Vukelić*)

Abstract. Starting from the results given in [14] where the uniform treatment of the Jensen type inequalities and its converses is given, we investigate the exponential convexity of differences of the left-hand and the right-hand side of these inequalities. Using these differences, we produce new exponentially convex functions. Finally, we give several examples of the families of functions for which the obtained results can be applied, and we get some generalized Cauchy means. Results from this paper present the generalization of the results from [2].

1. Introduction

Starting from the discrete Jensen inequality, A. McD. Mercer gave in [8] and [9] two mean-value theorems, of the Lagrange and of the Cauchy type. Having in mind the integral Jensen inequality, the authors in [13] gave similar results in integral form.

The generalization of these results, for the real Stieltjes measure, is given in [14] using the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \text{for } \alpha\leqslant s\leqslant t,\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} & \text{for } t\leqslant s\leqslant \beta.\end{cases} \tag{1.1}$$

The function G is convex and continuous with respect to both s and t.

For any function $\varphi:[\alpha,\beta]\to\mathbb{R}$, $\varphi\in C^2([\alpha,\beta])$, it can be easily shown by integrating by parts that the following is valid

$$\varphi(x)=\frac{\beta-x}{\beta-\alpha}\varphi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\varphi(\beta)+\int_\alpha^\beta G(x,s)\varphi''(s)ds,$$

where the function G is defined as above in (1.1) (see also [16]). Using this, several interesting results concerning the Jensen type inequalities are derived in [14].

First of all, the following theorem gave the conditions on the real Stieltjes measure $d\lambda$ (not necessarily positive!), such that $\lambda(a)\neq\lambda(b)$, under which for continuous convex function φ the Jensen inequality holds.

Mathematics subject classification (2010): 26D15.

Keywords and phrases: Green function, Jensen inequality, converse of the Jensen inequality, Cauchy-type mean value theorems, $n-$exponential convexity, exponential convexity, log-convexity, means.

THEOREM 1.1. [14] *Let $g:[a,b]\to\mathbb{R}$ be continuous function and $[\alpha,\beta]$ interval such that the image of g is a subset of $[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a)\neq\lambda(b)$ and*

$$\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}\in[\alpha,\beta].$$

Then the following two statements are equivalent:

(1) *For every continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$*

$$\varphi\left(\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}\right)\leqslant\frac{\int_a^b \varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)} \tag{1.2}$$

holds.

(2) *For all $s\in[\alpha,\beta]$*

$$G\left(\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)},s\right)\leqslant\frac{\int_a^b G(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)} \tag{1.3}$$

holds, where the function $G:[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$ is defined in (1.1).

Furthermore, the statements (1) and (2) are also equivalent if we change the sign of inequality in both (1.2) and (1.3). Also note that for every continuous concave function $\varphi:[\alpha,\beta]\to\mathbb{R}$ the inequality (1.2) is reversed.

and

$$\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}\in[\alpha,\beta].$$

REMARK 1.1. For the case of positive measure $d\lambda$, we get the well known results. If the function λ is increasing and bounded, with $\lambda(a)\neq\lambda(b)$, then inequality (1.2) becomes Jensen's integral inequality. On the other hand, if the function g is continuous and monotonic, and λ is either continuous or of bounded variation, satisfying

$$\lambda(a)\leqslant\lambda(x)\leqslant\lambda(b)\quad\text{for all } x\in[a,b],\ \text{and}\ \lambda(a)<\lambda(b),$$

then inequality (1.2) becomes the Jensen-Steffensen inequality given by Boas in [3] (see also [15, p. 59]). Several other theorems when inequality (1.2) or the reverse inequality in (1.2) holds, can be found in [15].

2. n-exponential convexity for Jensen-type inequalities

Throughout this paper we shall use the notation

$$\overline{g}=\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}.$$

Motivated by the inequality (1.2), for continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$ we define the functional $\Phi_1(g,\lambda,\varphi)$ by

$$\Phi_1(g,\lambda,\varphi)=\frac{\int_a^b\varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)}-\varphi(\overline{g}) \tag{2.1}$$

where $g:[a,b]\to\mathbb{R}$ is continuous function, the image of g is a subset of $[\alpha,\beta]$, $\lambda:[a,b]\to\mathbb{R}$ is continuous function or the function of bounded variation, such that $\lambda(a)\neq\lambda(b)$, and $\overline{g}\in[\alpha,\beta]$.

Using this, we define the functional $A_1(g,\lambda,\varphi)$ by

$$A_1(g,\lambda,\varphi)=\begin{cases}\Phi_1(g,\lambda,\varphi), & \text{if for all } s\in[\alpha,\beta] \text{ inequality (1.3) holds,}\\ -\Phi_1(g,\lambda,\varphi), & \text{if for all } s\in[\alpha,\beta] \text{ the reverse inequality in (1.3) holds.}\end{cases} \tag{2.2}$$

Now, for our functional A_1 we have that whenever it is defined, for every continuous convex function φ, $A_1(g,\lambda,\varphi)\geqslant 0$ holds.

REMARK 2.1. For the functions g and λ with some specific properties we already know that for all $s\in[\alpha,\beta]$ inequality (1.3) or the reverse inequality in (1.3) holds.

If the function $g:[a,b]\to\mathbb{R}$ is continuous, and the function $\lambda:[a,b]\to\mathbb{R}$ is increasing, bounded and such that $\lambda(a)\neq\lambda(b)$, by the integral Jensen inequality, we have that for all $s\in[\alpha,\beta]$ inequality (1.3) holds.

If $g:[a,b]\to\mathbb{R}$ is continuous and monotonic function, $\lambda:[a,b]\to\mathbb{R}$ either continuous function or the function of bounded variation, such that $\lambda(a)\leqslant\lambda(x)\leqslant\lambda(b)$ for all $x\in[a,b]$, and $\lambda(b)>\lambda(a)$, by the integral Jensen-Steffensen inequality (see [15, p. 59]) we have that for all $s\in[\alpha,\beta]$ the inequality (1.3) holds.

Analogous result is obtained under assumptions on the functions g and λ as given in the Boas generalization of the Jensen-Steffensen inequality (the Jensen-Boas inequality, see [3] or [15, p. 59]), the Brunk generalization of the Jensen-Steffensen inequality (the Jensen-Brunk inequality, see [4] or [15, p. 60]) or the generalization of the Jensen-Steffensen inequality (see [10] or [15, p. 62]).

On the other hand, if g is continuous function, and λ is the function of bounded variation, decreasing on the intervals $[a,c]$ and $(c,b]$, and such that $\lambda(b)>\lambda(a)$, by the reverse Jensen inequality (see [11] or [15, p. 84]), we have that for all $s\in[\alpha,\beta]$ the reverse inequality in (1.3) holds. Analogous result is obtained under assumptions on the functions g and λ as given in the reverse Jensen-Steffensen inequality (see [11] or [15, p. 84]), or the reverse Jensen-Brunk inequality (see [11] or [15, p. 85]), or the reverse Jensen-Boas inequality (see [11] or [15, p. 86]).

Now we can reformulate Theorem 2.3. from [14] in which there are given the conditions on the real Stieltjes measure $d\lambda$, with $\lambda(a)\neq\lambda(b)$, so that for the functions of the class C^2, the Cauchy-type mean value theorem holds.

THEOREM 2.1. *Let $g:[a,b]\to\mathbb{R}$ be a continuous function such that the image of g is a subset of $[\alpha,\beta]$, and $\varphi,\psi:[\alpha,\beta]\to\mathbb{R}$, $\varphi,\psi\in C^2([\alpha,\beta])$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation such that $\lambda(a)\neq\lambda(b)$, and $\overline{g}\in[\alpha,\beta]$, and let A_1 be the functional defined in (2.2). Then there exists some $\xi\in[\alpha,\beta]$ such that*

$$\frac{A_1(g,\lambda,\varphi)}{A_1(g,\lambda,\psi)}=\frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{2.3}$$

provided that the denominator of the left-hand side is nonzero.

REMARK 2.2. If the inverse of the function φ''/ψ'' exists, then (2.3) gives

$$\xi=\left(\frac{\varphi''}{\psi''}\right)^{-1}\left(\frac{A_1(g,\lambda,\varphi)}{A_1(g,\lambda,\psi)}\right). \tag{2.4}$$

Now, let us recall some definitions and facts about exponentially convex functions (see [7]).

DEFINITION 2.1. A function $f:I\to\mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i+x_j}{2}\right)\geqslant 0$$

holds for all $p_i\in\mathbb{R}$ and $x_i\in I$, $i=1,\dots,n$.

A function $f:I\to\mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 2.3. We can see from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k\in\mathbb{N}$, $k\leqslant n$.

By definition of the positive semi-definite matrices and some basic linear algebra, we have the following result.

LEMMA 2.1. *If f is an n-exponentially convex function in the Jensen sense, then the matrix $\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite for all $k\in\mathbb{N}, k\leqslant n$. Particularly,* $\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}\geqslant 0$ *for all $k\in\mathbb{N}$, $k\leqslant n$.*

DEFINITION 2.2. A function $f:I\to\mathbb{R}$ is exponentially convex in the Jensen sense on I, if it is n-exponentially convex in the Jensen sense for all $n\in\mathbb{N}$.

A function $f:I\to\mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 2.4. Some examples of exponentially convex functions are (see [6]):

(i) $f : I \to \mathbb{R}$ defined by $f(x) = ce^{kx}$, where $c \geqslant 0$ and $k \in \mathbb{R}$.

(ii) $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^{-k}$, where $k > 0$.

(iii) $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = e^{-k\sqrt{x}}$, where $k > 0$.

REMARK 2.5. It is known that a function $f : I \to \mathbb{R}^+$ is log-convex in the Jensen sense on I if and only if the relation

$$\alpha^2 f(x) + 2\alpha\beta f\left(\frac{x+y}{2}\right) + \beta^2 f(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic theory of convex functions, it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see [15, p. 2]).

LEMMA 2.2. *If $f : I \to \mathbb{R}$ is a convex function and $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{2.5}$$

If the function f is concave, then the reverse inequality in (2.5) holds.

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

DEFINITION 2.3. The second order divided difference of a function $f : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$\begin{aligned} [y_i]f &= f(y_i), \qquad i = 0,1,2 \\ [y_i, y_{i+1}]f &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \qquad i = 0,1 \\ [y_0, y_1, y_2]f &= \frac{[y_1, y_2]f - [y_0, y_1]f}{y_2 - y_0}. \end{aligned} \tag{2.6}$$

REMARK 2.6. The value $[y_0, y_1, y_2]f$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide (see [15, p. 16]). Taking the limit $y_1 \to y_0$ in (2.6), we get

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2]f = [y_0, y_0, y_2]f = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0$$

provided that f' exists. Furthermore, taking the limits $y_i \to y_0,\, i = 1,2$ in (2.6), we get

$$\lim_{y_2\to y_0}\lim_{y_1\to y_0}[y_0,y_1,y_2]f = [y_0,y_0,y_0]f = \frac{f''(y_0)}{2}$$

provided that f'' exists.

A function $f : I \to \mathbb{R}$ is convex if and only if for every choice of three mutually different points $y_0,y_1,y_2 \in I$ $[y_0,y_1,y_2]f \geqslant 0$ holds.

Now, we use an idea from [6] to give an elegant method of producing $n-$exponentially convex functions and exponentially convex functions, applying the functional A_1 on a given family of functions with the same property.

THEOREM 2.2. *Let $g : [a,b] \to \mathbb{R}$ be continuous function such that the image of g is a subset of $[\alpha,\beta]$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha,\beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha,\beta])$, such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is n-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$, and $\overline{g} \in [\alpha,\beta]$, and let A_1 be the linear functional defined in (2.2). Then the function $p \mapsto A_1(g,\lambda,\varphi_p)$ is n-exponentially convex in the Jensen sense on I. If the function $p \mapsto A_1(g,\lambda,\varphi_p)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. For $q_i \in \mathbb{R}$ $(i = 1,\dots,n)$ we define the function

$$h(x) = \sum_{i,j=1}^{n} q_i q_j \varphi_{\frac{p_i+p_j}{2}}(x),$$

where $p_i, p_j \in I$, $1 \leqslant i,\ j \leqslant n$ and $\varphi_{\frac{p_i+p_j}{2}} \in \Omega$. For every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$ we have

$$[y_0,y_1,y_2]\,h = \sum_{i,j=1}^{n} q_i q_j [y_0,y_1,y_2]\,\varphi_{\frac{p_i+p_j}{2}} \geqslant 0,$$

since $p \to [y_0,y_1,y_2]\,\varphi_p$ is n-exponentially convex in the Jensen sense by assumption. It follows that h is convex (and continuous) function on I, so it is

$$A_1\,(g,\lambda,h) \geqslant 0,$$

hence

$$\sum_{i,j=1}^{n} q_i q_j A_1\left(g,\lambda,\varphi_{\frac{p_i+p_j}{2}}\right) \geqslant 0.$$

We conclude that the function $p \mapsto A_1\,(g,\lambda,\varphi_p)$ is n-exponentially convex on I in the Jensen sense.

If the function $p \mapsto A_1\,(g,\lambda,\varphi_p)$ is also continuous on I, then it is n-exponentially convex by definition. $\square$

COROLLARY 2.1. *Let* $g:[a,b]\to\mathbb{R}$ *be continuous function such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $\Omega=\{\varphi_p : p\in I\}$ *(where* I *is an interval in* $\mathbb{R}$*) be a family of functions* $\varphi_p:[\alpha,\beta]\to\mathbb{R}$, $\varphi_p\in C([\alpha,\beta])$, *such that the function* $p\mapsto[y_0,y_1,y_2]\varphi_p$ *is exponentially convex in the Jensen sense on* I *for every three mutually different points* $y_0,y_1,y_2\in[\alpha,\beta]$. *Let* $\lambda:[a,b]\to\mathbb{R}$ *be continuous function or the function of bounded variation, such that* $\lambda(a)\neq\lambda(b)$, *and* $\overline{g}\in[\alpha,\beta]$, *and let* A_1 *be the linear functional defined in* (2.2). *Then the function* $p\mapsto A_1(g,\lambda,\varphi_p)$ *is exponentially convex in the Jensen sense on* I. *If the function* $p\mapsto A_1(g,\lambda,\varphi_p)$ *is continuous on* I, *then it is exponentially convex on* I.

COROLLARY 2.2. *Let* $g:[a,b]\to\mathbb{R}$ *be continuous function such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $\Omega=\{\varphi_p : p\in I\}$ *(where* I *is an interval in* $\mathbb{R}$*) be a family of functions* $\varphi_p:[\alpha,\beta]\to\mathbb{R}$, $\varphi_p\in C([\alpha,\beta])$, *such that the function* $p\mapsto[y_0,y_1,y_2]\varphi_p$ *is* 2*-exponentially convex in the Jensen sense on* I *for every three mutually different points* $y_0,y_1,y_2\in[\alpha,\beta]$. *Let* $\lambda:[a,b]\to\mathbb{R}$ *be continuous function or the function of bounded variation, such that* $\lambda(a)\neq\lambda(b)$, *and* $\overline{g}\in[\alpha,\beta]$. *Let* A_1 *be the linear functional defined in* (2.2). *Then the following statements hold:*

(i) *If the function* $p\mapsto A_1(g,\lambda,\varphi_p)$ *is continuous on I, then it is* 2*-exponentially convex on* I. *If* $p\mapsto A_1(g,\lambda,\varphi_p)$ *is additionally strictly positive, then it is also log-convex on I.*

(ii) *If the function* $p\mapsto A_1(g,\lambda,\varphi_p)$ *is strictly positive and differentiable on I, then for every* $p,q,u,v\in I$ *such that* $p\leqslant u$ *and* $q\leqslant v$, *we have*

$$\mu_{p,q}(g,A_1,\Omega)\leqslant\mu_{u,v}(g,A_1,\Omega) \tag{2.7}$$

where

$$\mu_{p,q}(g,A_1,\Omega)=\begin{cases}\left(\dfrac{A_1(g,\lambda,\varphi_p)}{A_1(g,\lambda,\varphi_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left(\dfrac{\frac{d}{dp}A_1(g,\lambda,\varphi_p)}{A_1(g,\lambda,\varphi_p)}\right), & p=q\end{cases} \tag{2.8}$$

for $\varphi_p,\varphi_q\in\Omega$.

Proof. (i) This is an immediate consequence of Theorem 2.2 and Remark 2.5.

(ii) Since by (i) the function $p\mapsto A_1(g,\lambda,\varphi_p)$ is log-convex on I, that is, the function $p\mapsto\log A_1(g,\lambda,\varphi_p)$ is convex on I, applying Lemma 2.2 for $p\leqslant u$, $q\leqslant v$, $p\neq q$, $u\neq v$, we get

$$\frac{\log A_1(g,\lambda,\varphi_p)-\log A_1(g,\lambda,\varphi_q)}{p-q}\leqslant\frac{\log A_1(g,\lambda,\varphi_u)-\log A_1(g,\lambda,\varphi_v)}{u-v}, \tag{2.9}$$

and therefore conclude that

$$\mu_{p,q}(g,A_1,\Omega)\leqslant\mu_{u,v}(g,A_1,\Omega).$$

The cases $p=q$ and $u=v$ follow from (2.9) as limit cases. □

REMARK 2.7. Note that the results from Theorem 2.2, Corollary 2.1 and Corollary 2.2 still hold when two of the points $y_0, y_1, y_2 \in [\alpha, \beta]$ coincide (say $y_1 = y_0$), for a family of differentiable functions φ_p such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, these results still hold when all three points coincide for a family of twice differentiable functions with the above mentioned properties. The proofs are obtained by recalling Remark 2.6 and suitable characterization of convexity.

3. n-exponential convexity for discrete Jensen-type inequalities

The well known discrete Jensen's inequality asserts that for convex function φ on interval $I \subseteq \mathbb{R}$

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) \tag{3.1}$$

holds, where p_i are positive real numbers and $x_i \in I$ $(i = 1, \ldots, n)$, while $P_n = \sum_{i=1}^{n} p_i$.

In [14] the generalization of above result is given, allowing that p_i can also be negative, with the sum different from 0, but with a supplementary demand on p_i, x_i given using the Green function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ defined in (1.1).

For p_i, x_i $(i = 1, \ldots, n)$ we shall use the common notation: $P_k = \sum_{i=1}^{k} p_i$, $\overline{P_k} = P_n - P_{k-1}$ $(k = 1, \ldots, n)$, and $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$.

In [14] the following result is derived.

THEOREM 3.1. [14] *Let $x_i \in [a,b] \subseteq [\alpha, \beta]$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha, \beta]$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$*

$$\varphi(\overline{x}) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) \tag{3.2}$$

holds.

(2) *For all $s \in [\alpha, \beta]$*

$$G(\overline{x}, s) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i, s) \tag{3.3}$$

holds, where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (1.1).

Moreover, the statements (1) and (2) are also equivalent if we change the sign of inequality in both (3.2) and (3.3).

REMARK 3.1. Note that in the case when all $p_i > 0$ $(i = 1, \ldots, n)$, inequality (3.2) becomes discrete Jensen's inequality (3.1).

Motivated by the inequality (3.2), for continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$ we define the functional $\Phi_2(\mathbf{x},\mathbf{p},\varphi)$ by

$$\Phi_2(\mathbf{x},\mathbf{p},\varphi) = \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi(x_i) - \varphi(\overline{x}), \tag{3.4}$$

where $\mathbf{x} = (x_1,x_2,...,x_n)$, $\mathbf{p} = (p_1,p_2,...,p_n)$, $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$ are such that $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$.

Using this, we define the functional $A_2(\mathbf{x},\mathbf{p},\varphi)$ by

$$A_2(\mathbf{x},\mathbf{p},\varphi) = \begin{cases} \Phi_2(\mathbf{x},\mathbf{p},\varphi), & \text{if for all } s \in [\alpha,\beta] \text{ inequality (3.3) holds,} \\ -\Phi_2(\mathbf{x},\mathbf{p},\varphi), & \text{if for all } s \in [\alpha,\beta] \text{ the reverse inequality in (3.3) holds.} \end{cases} \tag{3.5}$$

Now, for our functional A_2 we have that whenever it is defined, for every continuous convex function φ, $A_2(\mathbf{x},\mathbf{p},\varphi) \geqslant 0$ holds.

Now we can reformulate the discrete Cauchy mean-value theorem given in [14].

THEOREM 3.2. *Let $\mathbf{x} = (x_1,x_2,...,x_n)$, $\mathbf{p} = (p_1,p_2,...,p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$. Let $\varphi,\psi : [\alpha,\beta] \to \mathbb{R}$, $\varphi,\psi \in C^2([\alpha,\beta])$, and let A_2 be the functional defined in (3.5). Then there exists some $\xi \in [\alpha,\beta]$ such that*

$$\frac{A_2(\mathbf{x},\mathbf{p},\varphi)}{A_2(\mathbf{x},\mathbf{p},\psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)} \tag{3.6}$$

provided that the denominator of the left-hand side is nonzero.

REMARK 3.2. If the inverse of the function φ''/ψ'' exists, then (3.6) gives

$$\xi = \left(\frac{\varphi''}{\psi''}\right)^{-1}\left(\frac{A_2(\mathbf{x},\mathbf{p},\varphi)}{A_2(\mathbf{x},\mathbf{p},\psi)}\right). \tag{3.7}$$

Let us now consider the $n-$exponential convexity and exponential convexity. The proofs are similar to those in the integral case given in the previous section, so we give these results here without proofs.

THEOREM 3.3. *Let $\mathbf{x} = (x_1,x_2,...,x_n)$, $\mathbf{p} = (p_1,p_2,...,p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha,\beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha,\beta])$, such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is n-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$, and let A_2 be the linear functional defined in (3.5). Then $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is a n-exponentially convex function in the Jensen sense on I. If the function $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is continuous on I, then it is n-exponentially convex on I.*

COROLLARY 3.1. *Let $\mathbf{x} = (x_1,x_2,...,x_n)$, $\mathbf{p} = (p_1,p_2,...,p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$. Let $\Omega = \{\varphi_p : p \in I\}$*

(where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha,\beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha,\beta])$, such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$, and let A_2 be the linear functional defined in (3.5). Then $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is an exponentially convex function in the Jensen sense on I. If the function $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is continuous on I, then it is exponentially convex on I.

COROLLARY 3.2. *Let $\mathbf{x} = (x_1,x_2,...,x_n)$, $\mathbf{p} = (p_1,p_2,...,p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha,\beta]$, $p_i \in \mathbb{R}$ $(i = 1,...,n)$, $P_n \neq 0$ and $\overline{x} \in [\alpha,\beta]$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha,\beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha,\beta])$, such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$. Let A_2 be the linear functional defined in (3.5).*

Then the following statements hold:

- (i) *If the function $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is continuous on I, then it is 2-exponentially convex on I. If $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is additionally strictly positive, then it is also log-convex on I.*
- (ii) *If the function $p \mapsto A_2(\mathbf{x},\mathbf{p},\varphi_p)$ is strictly positive and differentiable on I, then for every $p,q,u,v \in I$ such that $p \leqslant u$ and $q \leqslant v$, we have*

$$\mu_{p,q}(\mathbf{x},A_2,\Omega) \leqslant \mu_{u,v}(\mathbf{x},A_2,\Omega) \tag{3.8}$$

where

$$\mu_{p,q}(\mathbf{x},A_2,\Omega) = \begin{cases} \left(\dfrac{A_2(\mathbf{x},\mathbf{p},\varphi_p)}{A_2(\mathbf{x},\mathbf{p},\varphi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\dfrac{\frac{d}{dp}A_2(\mathbf{x},\mathbf{p},\varphi_p)}{A_2(\mathbf{x},\mathbf{p},\varphi_p)}\right), & p = q \end{cases} \tag{3.9}$$

for $\varphi_p, \varphi_q \in \Omega$.

REMARK 3.3. Note that the results from Theorem 3.3, Corollary 3.1 and Corollary 3.2 still hold when two of the points $y_0,y_1,y_2 \in [\alpha,\beta]$ coincide, for a family of differentiable functions φ_p such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, these results still hold when all three points coincide for a family of twice differentiable functions with the above mentioned properties.

REMARK 3.4. For n-tuples $\mathbf{x}$ and $\mathbf{p}$ with some specific properties, we already know that for all $s \in [\alpha,\beta]$ inequality (3.3) or the reverse inequality in (3.3) holds.

In the case when all $p_i > 0$ $(i = 1,...,n)$ (or that all $p_i \geqslant 0$, $i = 1,...,n$, and $P_n > 0$), by the discrete Jensen inequality we have that for all $s \in [\alpha,\beta]$ the inequality (3.3) holds.

If $\mathbf{x} = (x_1,...,x_n)$ is monotonous $n-$tuple (i.e. if it holds $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ or $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$) and if $0 \leqslant P_k \leqslant P_n$, for $k = 1,...,n-1$, and $P_n > 0$, by the discrete Jensen-Steffensen inequality (see [15, p. 57]) we also have that for all $s \in [\alpha,\beta]$ inequality (3.3) holds.

On the other hand, if $\mathbf{p} = (p_1,...,p_n)$ is such that $p_1 > 0$, $p_2,...,p_n \leqslant 0$ and $P_n > 0$, then by the reverse Jensen inequality (see [5, p. 45]) we have that for all $s \in [\alpha,\beta]$ the reverse inequality in (3.3) holds. If $\mathbf{x} = (x_1,...,x_n)$ is monotonous $n-$tuple and $\mathbf{p} = (p_1,...,p_n)$ such that there exists $m \in \{1,...,n\}$ so that $P_k \leqslant 0$ for $k < m$ and $\overline{P_k} \leqslant 0$ for $k > m$, and that it is $P_n > 0$, then by the reverse Jensen-Steffensen inequality (see [15, p. 83]) we have that for all $s \in [\alpha,\beta]$ the reverse inequality in (3.3) holds.

REMARK 3.5. Results for the Jensen-Steffensen inequality regarding exponential convexity, which are a special case of some of the results given here, were given in [1].

4. n-exponential convexity for converse Jensen-type inequalities

The following theorem from [14] gave the conditions on the real Stieltjes measure $d\lambda$ (not necessarily positive!), such that $\lambda(a) \neq \lambda(b)$, under which for continuous convex function φ the converse of the Jensen inequality holds.

THEOREM 4.1. [14] *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $m, M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leqslant g(t) \leqslant M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$*

$$\frac{\int_a^b \varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)} \leqslant \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) \tag{4.1}$$

holds.

(2) *For all $s \in [\alpha,\beta]$*

$$\frac{\int_a^b G(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)} \leqslant \frac{M-\overline{g}}{M-m}G(m,s) + \frac{\overline{g}-m}{M-m}G(M,s) \tag{4.2}$$

holds, where the function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ is defined in (1.1).

Furthermore, the statements (1) and (2) are also equivalent if we change the sign of inequality in both (4.1) and (4.2).

REMARK 4.1. If we set in Theorem 4.1 $m = \alpha$ and $M = \beta$, inequality (4.2) transforms into (see also [14])

$$\frac{\int_a^b G(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)} \leqslant 0.$$

Motivated by inequality (4.1), for continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$ we define the functional $\Phi_3(g,\lambda,\varphi)$ by

$$\Phi_3(g,\lambda,\varphi) = \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_a^b \varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)}, \tag{4.3}$$

where $g : [a,b] \to \mathbb{R}$ is continuous function, the image of g is a subset of $[\alpha,\beta]$, $m,M \in [\alpha,\beta]$ $(m \neq M)$ such that $m \leqslant g(t) \leqslant M$ for all $t \in [a,b]$, $\lambda : [a,b] \to \mathbb{R}$ is continuous function or the function of bounded variation such that $\lambda(a) \neq \lambda(b)$.

Using this, we define the functional $A_3(g,\lambda,\varphi)$ by

$$A_3(g,\lambda,\varphi) = \begin{cases} \Phi_3(g,\lambda,\varphi), \text{ if for all } s \in [\alpha,\beta] \text{ inequality (4.2) holds,} \\ -\Phi_3(g,\lambda,\varphi), \text{ if for all } s \in [\alpha,\beta] \text{ the reverse inequality in (4.2) holds.} \end{cases} \tag{4.4}$$

Now, for our functional A_3 we have that whenever it is defined, for every continuous convex function φ, $A_3(g,\lambda,\varphi) \geqslant 0$ holds.

Now we can reformulate the adequate mean-value theorem given in [14].

THEOREM 4.2. *Let $g : [a,b] \to \mathbb{R}$ be continuous function such that the image of g is a subset of $[\alpha,\beta]$, and $\varphi,\psi : [\alpha,\beta] \to \mathbb{R}$, $\varphi,\psi \in C^2([\alpha,\beta])$. Let $m,M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leqslant g(t) \leqslant M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$, and let A_3 be the functional defined in (4.4). Then there exists some $\xi \in [\alpha,\beta]$ such that the following holds*

$$\frac{A_3(g,\lambda,\varphi)}{A_3(g,\lambda,\psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{4.5}$$

provided that the denominator of the left-hand side of (4.5) is nonzero.

REMARK 4.2. If the inverse of the function φ''/ψ'' exists, then (4.5) gives

$$\xi = \left(\frac{\varphi''}{\psi''}\right)^{-1}\left(\frac{A_3(g,\lambda,\varphi)}{A_3(g,\lambda,\psi)}\right). \tag{4.6}$$

We now consider the $n-$ exponential convexity and exponential convexity, and get the following results.

THEOREM 4.3. *Let $g : [a,b] \to \mathbb{R}$ be continuous function such that the image of g is a subset of $[\alpha,\beta]$. Let $m,M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \leqslant g(t) \leqslant M$ for all $t \in [a,b]$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha,\beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha,\beta])$, such that the function $p \mapsto [y_0,y_1,y_2]\varphi_p$ is n-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2 \in [\alpha,\beta]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$, and let A_3 be the linear functional defined in (4.4). Then the function $p \mapsto A_3(g,\lambda,\varphi_p)$ is n-exponentially convex in the Jensen sense on I. If the function $p \mapsto A_3(g,\lambda,\varphi_p)$ is continuous on I, then it is n-exponentially convex on I.*

COROLLARY 4.1. *Let $g:[a,b]\to\mathbb{R}$ be continuous function such that the image of g is a subset of $[\alpha,\beta]$. Let $m,M\in[\alpha,\beta]$ $(m\neq M)$ be such that $m\leqslant g(t)\leqslant M$ for all $t\in[a,b]$. Let $\Omega=\{\varphi_p : p\in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p:[\alpha,\beta]\to\mathbb{R}$, $\varphi_p\in C([\alpha,\beta])$, such that the function $p\mapsto[y_0,y_1,y_2]\varphi_p$ is exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2\in[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a)\neq\lambda(b)$, and let A_3 be the linear functional defined in (4.4). Then the function $p\mapsto A_3(g,\lambda,\varphi_p)$ is exponentially convex in the Jensen sense on I. If the function $p\mapsto A_3(g,\lambda,\varphi_p)$ is continuous on I, then it is exponentially convex on I.*

COROLLARY 4.2. *Let $g:[a,b]\to\mathbb{R}$ be continuous function such that the image of g is a subset of $[\alpha,\beta]$. Let $m,M\in[\alpha,\beta]$ $(m\neq M)$ be such that $m\leqslant g(t)\leqslant M$ for all $t\in[a,b]$. Let $\Omega=\{\varphi_p : p\in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p:[\alpha,\beta]\to\mathbb{R}$, $\varphi_p\in C([\alpha,\beta])$, such that the function $p\mapsto[y_0,y_1,y_2]\varphi_p$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0,y_1,y_2\in[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a)\neq\lambda(b)$. Let A_3 be the linear functional defined in* (4.4).

Then the following statements hold:

(*i*) *If the function $p\mapsto A_3(g,\lambda,\varphi_p)$ is continuous on I, then it is 2-exponentially convex on I. If $p\mapsto A_3(g,\lambda,\varphi_p)$ is additionally strictly positive, then it is also log-convex on I.*

(*ii*) *If the function $p\mapsto A_3(g,\lambda,\varphi_p)$ is strictly positive and differentiable on I, then for every $p,q,u,v\in I$ such that $p\leqslant u$ and $q\leqslant v$, we have*

$$\mu_{p,q}(g,A_3,\Omega)\leqslant\mu_{u,v}(g,A_3,\Omega) \tag{4.7}$$

where

$$\mu_{p,q}(g,A_3,\Omega)=\begin{cases}\left(\dfrac{A_3(g,\lambda,\varphi_p)}{A_3(g,\lambda,\varphi_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left(\dfrac{\frac{d}{dp}A_3(g,\lambda,\varphi_p)}{A_3(g,\lambda,\varphi_p)}\right), & p=q\end{cases} \tag{4.8}$$

for $\varphi_p,\varphi_q\in\Omega$.

REMARK 4.3. Note that the results from Theorem 4.3, Corollary 4.1 and Corollary 4.2 still hold when two of the points $y_0,y_1,y_2\in[\alpha,\beta]$ coincide, for a family of differentiable functions φ_p such that the function $p\mapsto[y_0,y_1,y_2]\varphi_p$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, these results still hold when all three points coincide for a family of twice differentiable functions with the above mentioned properties.

5. n-exponential convexity for discrete converse Jensen-type inequalities

The similar results can also be derived for the converse of the Jensen inequality in discrete case.

We have the following result from [14].

THEOREM 5.1. [14] *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i=1,\dots,n)$ be such that $P_n \neq 0$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) \leqslant \frac{b-\overline{x}}{b-a}\varphi(a) + \frac{\overline{x}-a}{b-a}\varphi(b) \tag{5.1}$$

holds.

(2) *For all $s \in [\alpha,\beta]$*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i,s) \leqslant \frac{b-\overline{x}}{b-a}G(a,s) + \frac{\overline{x}-a}{b-a}G(b,s) \tag{5.2}$$

holds, where the function $G : [\alpha,\beta]\times[\alpha,\beta] \to \mathbb{R}$ is defined in (1.1).

Moreover, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (5.1) *and* (5.2).

REMARK 5.1. If we set that all $p_i \in \mathbb{R}$ $(i=1,\dots,n)$ are positive, then (5.1) becomes classical converse of the Jensen inequality (see for example [12, p. 48]).

REMARK 5.2. If we set in Theorem 5.1 that $a=\alpha$ and $b=\beta$, inequality (5.2) transforms into (see also [14])

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i,s) \leqslant 0.$$

Motivated by inequality (5.1), for continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$ we define the functional $\Phi_4(\mathbf{x},\mathbf{p},\varphi)$ by

$$\Phi_4(\mathbf{x},\mathbf{p},\varphi) = \frac{b-\overline{x}}{b-a}\varphi(a) + \frac{\overline{x}-a}{b-a}\varphi(b) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i), \tag{5.3}$$

where $\mathbf{x} = (x_1,x_2,\dots,x_n)$, $\mathbf{p} = (p_1,p_2,\dots,p_n)$, $x_i \in [a,b] \subseteq [\alpha,\beta]$ $(a \neq b)$ and $p_i \in \mathbb{R}$ $(i=1,\dots,n)$ are such that $P_n \neq 0$.

Using this, we define the functional $A_4(\mathbf{x},\mathbf{p},\varphi)$ by

$$A_4(\mathbf{x},\mathbf{p},\varphi) = \begin{cases} \Phi_4(\mathbf{x},\mathbf{p},\varphi), & \text{if for all } s \in [\alpha,\beta] \text{ inequality (5.2) holds,} \\ -\Phi_4(\mathbf{x},\mathbf{p},\varphi), & \text{if for all } s \in [\alpha,\beta] \text{ the reverse inequality in (5.2) holds.} \end{cases} \tag{5.4}$$

Now, for our functional A_4 we have that whenever it is defined, for every continuous convex function φ, $A_4(\mathbf{x},\mathbf{p},\varphi) \geqslant 0$ holds.

REMARK 5.3. For some n-tuples $\mathbf{p}$ with some specific properties, we already know that for all $s \in [\alpha, \beta]$ hold (5.2) or the reverse inequality in (5.2).

If all p_i $(i = 1, \ldots, n)$ are positive, then by the discrete form of the converse of the Jensen inequality (see for example [12, p. 48]) we have that for all $s \in [\alpha, \beta]$ inequality (5.2) holds.

Now we can reformulate the discrete Cauchy mean-value theorem given in [14].

THEOREM 5.2. *Let $x_i \in [a,b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ be such that $P_n \neq 0$, let $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$, $\varphi, \psi \in C^2([\alpha, \beta])$, and let A_4 be the functional defined in (5.4). Then there exists some $\xi \in [\alpha, \beta]$ such that the following is valid*

$$\frac{A_4(\mathbf{x}, \mathbf{p}, \varphi)}{A_4(\mathbf{x}, \mathbf{p}, \psi)} = \frac{\varphi''(\xi)}{\psi''(\xi)} \tag{5.5}$$

provided that the denominator of the left-hand side is nonzero.

REMARK 5.4. If the inverse of the function φ''/ψ'' exists, then (5.5) gives

$$\xi = \left(\frac{\varphi''}{\psi''}\right)^{-1} \left(\frac{A_4(\mathbf{x}, \mathbf{p}, \varphi)}{A_4(\mathbf{x}, \mathbf{p}, \psi)}\right). \tag{5.6}$$

Concluding as before, we get our results concerning the $n-$exponential convexity and exponential convexity for our functional A_4.

THEOREM 5.3. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha, \beta]$ $(a \neq b)$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $P_n \neq 0$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha, \beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha, \beta])$, such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is n-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in [\alpha, \beta]$, and let A_4 be the linear functional defined in (5.4). Then the function $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ is n-exponentially convex in the Jensen sense on I. If the function $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ is continuous on I, then it is n-exponentially convex on I.*

COROLLARY 5.1. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha, \beta]$ $(a \neq b)$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $P_n \neq 0$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha, \beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha, \beta])$, such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in [\alpha, \beta]$, and let A_4 be the linear functional defined in (5.4). Then the function $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ is exponentially convex in the Jensen sense on I. If the function $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ is continuous on I, then it is exponentially convex on I.*

COROLLARY 5.2. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be such that $x_i \in [a,b] \subseteq [\alpha, \beta]$ $(a \neq b)$, $p_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $P_n \neq 0$. Let $\Omega = \{\varphi_p : p \in I\}$ (where I is an interval in $\mathbb{R}$) be a family of functions $\varphi_p : [\alpha, \beta] \to \mathbb{R}$, $\varphi_p \in C([\alpha, \beta])$, such* *that the function* $p \mapsto [y_0, y_1, y_2]\varphi_p$ *is* 2*-exponentially convex in the Jensen sense on* I *for every three mutually different points* $y_0, y_1, y_2 \in [\alpha, \beta]$. *Let* A_4 *be the linear functional defined in* (5.4).

Then the following statements hold:

(i) *If the function* $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ *is continuous on* I, *then it is* 2*-exponentially convex on* I. *If* $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ *is additionally strictly positive, then it is also log-convex on* I.

(ii) *If the function* $p \mapsto A_4(\mathbf{x}, \mathbf{p}, \varphi_p)$ *is strictly positive and differentiable on* I, *then for every* $p, q, u, v \in I$ *such that* $p \leqslant u$ *and* $q \leqslant v$, *we have*

$$\mu_{p,q}(\mathbf{x}, A_4, \Omega) \leqslant \mu_{u,v}(\mathbf{x}, A_4, \Omega) \tag{5.7}$$

where

$$\mu_{p,q}(\mathbf{x}, A_4, \Omega) = \begin{cases} \left(\dfrac{A_4(\mathbf{x}, \mathbf{p}, \varphi_p)}{A_4(\mathbf{x}, \mathbf{p}, \varphi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\dfrac{\frac{d}{dp}A_4(\mathbf{x}, \mathbf{p}, \varphi_p)}{A_4(\mathbf{x}, \mathbf{p}, \varphi_p)}\right), & p = q \end{cases} \tag{5.8}$$

for $\varphi_p, \varphi_q \in \Omega$.

REMARK 5.5. Note that the results from Theorem 5.3, Corollary 5.1 and Corollary 5.2 still hold when two of the points $y_0, y_1, y_2 \in [\alpha, \beta]$ coincide, for a family of differentiable functions φ_p such that the function $p \mapsto [y_0, y_1, y_2]\varphi_p$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, these results still hold when all three points coincide for a family of twice differentiable functions with the above mentioned properties.

6. Examples

In this section we will vary on choice of a family $\Omega = \{\varphi_p : p \in I\}$ in order to construct different examples of exponentially convex functions and construct some means.

EXAMPLE 6.1. Let

$$\Omega_1 = \{\psi_p : \mathbb{R} \to [0, \infty) : p \in \mathbb{R}\}$$

be a family of functions defined by

$$\psi_p(x) = \begin{cases} \frac{1}{p^2} e^{px}, & \mathrm{p} \neq 0; \\ \frac{1}{2} x^2, & \mathrm{p} = 0. \end{cases}$$

Since $\frac{d^2}{dx^2}\psi_p(x) = e^{px} > 0$ for $x \in \mathbb{R}$, ψ_p is convex function on $\mathbb{R}$ for every $p \in \mathbb{R}$. From Remark 2.4 it follows that the function $p \mapsto \frac{d^2}{dx^2}\psi_p(x)$ is exponentially convex, and from [6] we then also have that $p \mapsto [y_0, y_1, y_2]\psi_p$ is exponentially convex (and so exponentially convex in the Jensen sense). So, our family Ω_1 of functions ψ_p fulfills the condition given in Corollary 2.1, Corollary 3.1, Corollary 4.1 and Corollary 5.1, and we conclude that $p \mapsto A_k(g,\lambda,\psi_p)$ (for $k = 1,3$) and $p \mapsto A_k(\mathbf{x},\mathbf{p},\psi_p)$ (for $k = 2,4$) are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although $p \mapsto \psi_p$ is not continuous at $p = 0$), so they are exponentially convex.

For this family of functions we have the following possible cases for $\mu_{p,q}$:

– for $k = 1,3$:

$$\mu_{p,q}(g,A_k,\Omega_1) = \begin{cases} \left(\frac{A_k(g,\lambda,\psi_p)}{A_k(g,\lambda,\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{A_k(g,\lambda,id\cdot\psi_p)}{A_k(g,\lambda,\psi_p)} - \frac{2}{p}\right), & p = q \neq 0; \\ \exp\left(\frac{1}{3}\frac{A_k(g,\lambda,id\cdot\psi_0)}{A_k(g,\lambda,\psi_0)}\right), & p = q = 0; \end{cases}$$

– for $k = 2,4$:

$$\mu_{p,q}(\mathbf{x},A_k,\Omega_1) = \begin{cases} \left(\frac{A_k(\mathbf{x},\mathbf{p},\psi_p)}{A_k(\mathbf{x},\mathbf{p},\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{A_k(\mathbf{x},\mathbf{p},id\cdot\psi_p)}{A_k(\mathbf{x},\mathbf{p},\psi_p)} - \frac{2}{p}\right), & p = q \neq 0; \\ \exp\left(\frac{1}{3}\frac{A_k(\mathbf{x},\mathbf{p},id\cdot\psi_0)}{A_k(\mathbf{x},\mathbf{p},\psi_0)}\right), & p = q = 0. \end{cases}$$

For $\mu_{p,q}$ the monotonicity property holds.

If $p,q,u,v \in \mathbb{R}$ such that $p \leqslant u$, $q \leqslant v$, then by Corollary 2.2, Corollary 3.2, Corollary 4.2 and Corollary 5.2 we have

$$\mu_{p,q}(g,A_k,\Omega_1) \leqslant \mu_{u,v}(g,A_k,\Omega_1), \quad \text{for } k = 1,3, \tag{6.1}$$
$$\mu_{p,q}(\mathbf{x},A_k,\Omega_1) \leqslant \mu_{u,v}(\mathbf{x},A_k,\Omega_1), \quad \text{for } k = 2,4. \tag{6.2}$$

If A_k $(k = 1,2,3,4)$ are positive, then using Theorem 2.1, Theorem 3.2, Theorem 4.2 and Theorem 5.2 applied for $\varphi = \psi_p \in \Omega_1$ and $\psi = \psi_q \in \Omega_1$, it follows that

$$M_{p,q}(g,A_k,\Omega_1) = \log \mu_{p,q}(g,A_k,\Omega_1), \quad \text{for } k = 1,3,$$
$$M_{p,q}(\mathbf{x},A_k,\Omega_1) = \log \mu_{p,q}(\mathbf{x},A_k,\Omega_1), \quad \text{for } k = 2,4,$$

satisfy

$$\alpha \leqslant M_{p,q}(g,A_k,\Omega_1) \leqslant \beta, \quad \text{for } k = 1,3,$$
$$\alpha \leqslant M_{p,q}(\mathbf{x},A_k,\Omega_1) \leqslant \beta, \quad \text{for } k = 2,4.$$

If we set that the image of the function g is $[\alpha,\beta]$ (for $k = 1,3$), and that $\alpha = \min_{1\leqslant i\leqslant n}\{x_i\}$ and $\beta = \max_{1\leqslant i\leqslant n}\{x_i\}$ (for $k = 2,4$), then we have

$$\alpha = \min_{t\in[a,b]}\{g(t)\} \leqslant M_{p,q}(g,A_k,\Omega_1) \leqslant \max_{t\in[a,b]}\{g(t)\} = \beta, \quad \text{for } k=1,3,$$
$$\alpha = \min_{1\leqslant i\leqslant n}\{x_i\} \leqslant M_{p,q}(\mathbf{x},A_k,\Omega_1) \leqslant \max_{1\leqslant i\leqslant n}\{x_i\} = \beta, \quad \text{for } k=2,4,$$

which shows that in this case $M_{p,q}(g,A_k,\Omega_1)$ are means of $g(t)$ ($t\in[a,b]$) for $k=1,3$, and $M_{p,q}(\mathbf{x},A_k,\Omega_1)$ are means of $x_1,\dots,x_n$ for $k=2,4$. Notice that by (6.1) and (6.2) these means are monotonic.

EXAMPLE 6.2. Let

$$\Omega_2 = \{\varphi_p : \mathbb{R}^+ \to \mathbb{R} : p\in\mathbb{R}\}$$

be a family of functions defined by

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p\neq 0,1; \\ -\log x, & p=0; \\ x\log x, & p=1. \end{cases} \tag{6.3}$$

Since $\frac{d^2}{dx^2}\varphi_p(x) = x^{p-2} = e^{(p-2)\log x} > 0$, φ_p is convex function for $x>0$. From Remark 2.4 it follows that $p\mapsto \frac{d^2}{dx^2}\varphi_p(x)$ is exponentially convex, and from [6] we then also have that $p\mapsto [y_0,y_1,y_2]\varphi_p$ is exponentially convex (and so exponentially convex in the Jensen sense). So, our family Ω_2 of functions φ_p fulfills the condition given in Corollary 2.1, Corollary 3.1, Corollary 4.1 and Corollary 5.1.

In this example we assume that interval $[\alpha,\beta]$ from these corollaries is a subset of $\mathbb{R}^+$, and so for our family of functions we have the following possible cases for $\mu_{p,q}$:

– for $k=1,3$:

$$\mu_{p,q}(g,A_k,\Omega_2) = \begin{cases} \left(\frac{A_k(g,\lambda,\varphi_p)}{A_k(g,\lambda,\varphi_q)}\right)^{\frac{1}{p-q}}, & p\neq q; \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{A_k(g,\lambda,\varphi_0\varphi_p)}{A_k(g,\lambda,\varphi_p)}\right), & p=q\neq 1,0; \\ \exp\left(1-\frac{A_k(g,\lambda,\varphi_0{}^2)}{2A_k(g,\lambda,\varphi_0)}\right), & p=q=0; \\ \exp\left(-1-\frac{A_k(g,\lambda,\varphi_0\varphi_1)}{2A_k(g,\lambda,\varphi_1)}\right), & p=q=1; \end{cases}$$

– for $k=2,4$:

$$\mu_{p,q}(\mathbf{x},A_k,\Omega_2) = \begin{cases} \left(\frac{A_k(\mathbf{x},\mathbf{p},\varphi_p)}{A_k(\mathbf{x},\mathbf{p},\varphi_q)}\right)^{\frac{1}{p-q}}, & p\neq q; \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{A_k(\mathbf{x},\mathbf{p},\varphi_0\varphi_p)}{A_k(\mathbf{x},\mathbf{p},\varphi_p)}\right), & p=q\neq 1,0; \\ \exp\left(1-\frac{A_k(\mathbf{x},\mathbf{p},\varphi_0{}^2)}{2A_k(\mathbf{x},\mathbf{p},\varphi_0)}\right), & p=q=0; \\ \exp\left(-1-\frac{A_k(\mathbf{x},\mathbf{p},\varphi_0\varphi_1)}{2A_k(\mathbf{x},\mathbf{p},\varphi_1)}\right), & p=q=1; \end{cases}$$

where g is positive function, and $\overline{g},\overline{x},x_i > 0$ $(i=1,\dots,n)$.

As in the previous example, we conclude that the functions $p \mapsto A_k(g,\lambda,\varphi_p)$ (for $k=1,3$) and $p \mapsto A_k(\mathbf{x},\mathbf{p},\varphi_p)$ (for $k=2,4$) are exponentially convex, and for $\mu_{p,q}$ the monotonicity property holds.

If $p,q,u,v \in \mathbb{R}$ such that $p \leqslant u$, $q \leqslant v$, then we have

$$\mu_{p,q}(g,A_k,\Omega_2) \leqslant \mu_{u,v}(g,A_k,\Omega_2), \quad \text{for } k=1,3,$$
$$\mu_{p,q}(\mathbf{x},A_k,\Omega_2) \leqslant \mu_{u,v}(\mathbf{x},A_k,\Omega_2), \quad \text{for } k=2,4.$$

If A_k $(k=1,2,3,4)$ are positive, then Theorem 2.1, Theorem 3.2, Theorem 4.2 and Theorem 5.2 applied for $\varphi=\varphi_p \in \Omega_2$ and $\psi=\varphi_q \in \Omega_2$ yield that there exist some

$$\xi_k \in [\alpha,\beta], \quad \text{for } k=1,2,3,4,$$

such that

$$\xi_k^{p-q} = \frac{A_k(g,\lambda,\varphi_p)}{A_k(g,\lambda,\varphi_q)}, \quad \text{for } k=1,3,$$
$$\xi_k^{p-q} = \frac{A_k(\mathbf{x},\mathbf{p},\varphi_p)}{A_k(\mathbf{x},\mathbf{p},\varphi_q)}, \quad \text{for } k=2,4.$$

Since the function $\xi \mapsto \xi^{p-q}$ is invertible for $p \neq q$, we then have

$$\alpha \leqslant \left(\frac{A_k(g,\lambda,\varphi_p)}{A_k(g,\lambda,\varphi_q)}\right)^{\frac{1}{p-q}} \leqslant \beta, \quad \text{for } k=1,3, \tag{6.4}$$

$$\alpha \leqslant \left(\frac{A_k(\mathbf{x},\mathbf{p},\varphi_p)}{A_k(\mathbf{x},\mathbf{p},\varphi_q)}\right)^{\frac{1}{p-q}} \leqslant \beta, \quad \text{for } k=2,4. \tag{6.5}$$

As in the previous example, if we set that the image of the function g is $[\alpha,\beta]$ (for $k=1,3$), and that $\alpha = \min\limits_{1\leqslant i\leqslant n}\{x_i\}$ and $\beta = \max\limits_{1\leqslant i\leqslant n}\{x_i\}$ (for $k=2,4$), then we have

$$\alpha = \min_{t\in[a,b]}\{g(t)\} \leqslant \left(\frac{A_k(g,\lambda,\varphi_p)}{A_k(g,\lambda,\varphi_q)}\right)^{\frac{1}{p-q}} \leqslant \max_{t\in[a,b]}\{g(t)\} = \beta, \quad \text{for } k=1,3, \tag{6.6}$$

$$\alpha = \min_{1\leqslant i\leqslant n}\{x_i\} \leqslant \left(\frac{A_k(\mathbf{x},\mathbf{p},\varphi_p)}{A_k(\mathbf{x},\mathbf{p},\varphi_q)}\right)^{\frac{1}{p-q}} \leqslant \max_{1\leqslant i\leqslant n}\{x_i\} = \beta, \quad \text{for } k=2,4, \tag{6.7}$$

which shows that in this case $\mu_{p,q}(g,A_k,\Omega_2)$ (for $k=1,3$) and $\mu_{p,q}(\mathbf{x},A_k,\Omega_2)$ (for $k=2,4$) are means.

Now, we impose one additional parameter r. For $r \neq 0$ by substituting $g \to g^r$, $x_i \to x_i^r$, $p \to \frac{p}{r}$ and $q \to \frac{q}{r}$ in (6.6) and (6.7), we get

$$\min_{t\in[a,b]}\{(g(t))^r\} \leqslant \left(\frac{A_k(g^r,\lambda,\varphi_p)}{A_k(g^r,\lambda,\varphi_q)}\right)^{\frac{r}{p-q}} \leqslant \max_{t\in[a,b]}\{(g(t))^r\}, \quad \text{for } k=1,3, \tag{6.8}$$

and

$$\min_{1\leqslant i\leqslant n}\{x_i^r\}\leqslant\left(\frac{A_k(\mathbf{x}^r,\mathbf{p},\varphi_p)}{A_k(\mathbf{x}^r,\mathbf{p},\varphi_q)}\right)^{\frac{r}{p-q}}\leqslant\max_{1\leqslant i\leqslant n}\{x_i^r\},\quad \text{for } k=2,4, \tag{6.9}$$

where $\mathbf{x}^r=(x_1^r,x_2^r,\ldots,x_n^r)$.

We define new generalized mean as follows:

– for $k=1,3$:

$$\mu_{p,q;r}(g,A_k,\Omega_2)=\begin{cases}\left(\mu_{\frac{p}{r},\frac{q}{r}}(g^r,A_k,\Omega_2)\right)^{\frac{1}{r}}, & r\neq 0;\\ \mu_{p,q}(\log g,A_k,\Omega_1), & r=0;\end{cases} \tag{6.10}$$

– for $k=2,4$:

$$\mu_{p,q;r}(\mathbf{x},A_k,\Omega_2)=\begin{cases}\left(\mu_{\frac{p}{r},\frac{q}{r}}(\mathbf{x}^r,A_k,\Omega_2)\right)^{\frac{1}{r}}, & r\neq 0;\\ \mu_{p,q}(\log \mathbf{x},A_k,\Omega_1), & r=0;\end{cases} \tag{6.11}$$

where $\log\mathbf{x}=(\log x_1,\log x_2,\ldots,\log x_n)$.

These new generalized means are also monotonic. If $p,q,u,v\in\mathbb{R}$, $r\neq 0$ such that $p\leqslant u$, $q\leqslant v$, then we have

$$\mu_{p,q;r}(g,A_k,\Omega_2)\leqslant\mu_{u,v;r}(g,A_k,\Omega_2),\ \text{ for } k=1,3,$$
$$\mu_{p,q;r}(\mathbf{x},A_k,\Omega_2)\leqslant\mu_{u,v;r}(\mathbf{x},A_k,\Omega_2),\ \text{ for } k=2,4.$$

The above results follow from the following inequalities:

– for $k=1,3$:

$$\mu_{\frac{p}{r},\frac{q}{r}}(g^r,A_k,\Omega_2)=\left(\frac{A_k(g^r,\lambda,\varphi_{\frac{p}{r}})}{A_k(g^r,\lambda,\varphi_{\frac{q}{r}})}\right)^{\frac{r}{p-q}}\leqslant\left(\frac{A_k(g^r,\lambda,\varphi_{\frac{u}{r}})}{A_k(g^r,\lambda,\varphi_{\frac{v}{r}})}\right)^{\frac{r}{u-v}}=\mu_{\frac{u}{r},\frac{v}{r}}(g^r,A_k,\Omega_2),$$

– for $k=2,4$:

$$\mu_{\frac{p}{r},\frac{q}{r}}(\mathbf{x}^r,A_k,\Omega_2)=\left(\frac{A_k(\mathbf{x}^r,\mathbf{p},\varphi_{\frac{p}{r}})}{A_k(\mathbf{x}^r,\mathbf{p},\varphi_{\frac{q}{r}})}\right)^{\frac{r}{p-q}}\leqslant\left(\frac{A_k(\mathbf{x}^r,\mathbf{p},\varphi_{\frac{u}{r}})}{A_k(\mathbf{x}^r,\mathbf{p},\varphi_{\frac{v}{r}})}\right)^{\frac{r}{u-v}}=\mu_{\frac{u}{r},\frac{v}{r}}(\mathbf{x}^r,A_k,\Omega_2),$$

for $p,q,u,v\in\mathbb{R}$, $r\neq 0$, such that $\frac{p}{r}\leqslant\frac{u}{r}$, $\frac{q}{r}\leqslant\frac{v}{r}$, and the fact that $\mu_{p,q}(g,A_k,\Omega_2)$ for $k=1,3$, and $\mu_{p,q}(\mathbf{x},A_k,\Omega_2)$ for $k=2,4$, are monotonous in both the parameters. For $r=0$, we obtain the required result by taking the limit $r\to 0$.

EXAMPLE 6.3. Let

$$\Omega_3=\{\theta_p:\mathbb{R}^+\to\mathbb{R}^+ : p\in\mathbb{R}^+\}$$

be family of functions defined by

$$\theta_p(x) = \frac{e^{-x\sqrt{p}}}{p},$$

Since $\frac{d^2}{dx^2}\theta_p(x) = e^{-x\sqrt{p}} > 0$, θ_p is convex function for $x > 0$. From Remark 2.4 we have that $p \mapsto \frac{d^2}{dx^2}\theta_p(x)$ is exponentially convex, and from [6] we then also have that $p \mapsto [y_0, y_1, y_2]\theta_p$ is exponentially convex function. Family Ω_3 of functions θ_p fulfills the condition given in Corollary 2.1, Corollary 3.1, Corollary 4.1 and Corollary 5.1. Here in this example we again assume that interval $[\alpha, \beta]$ from these corollaries is a subset of $\mathbb{R}^+$, and so for our family of functions we have the following possible cases for $\mu_{p,q}$:

– for $k = 1, 3$:

$$\mu_{p,q}(g, A_k, \Omega_3) = \begin{cases} \left(\frac{A_k(g,\lambda,\theta_p)}{A_k(g,\lambda,\theta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{A_k(g,\lambda,id\cdot\theta_p)}{2\sqrt{p}(A_k(g,\lambda,\theta_p)} - \frac{1}{p}\right), & p = q; \end{cases}$$

– for $k = 2, 4$:

$$\mu_{p,q}(\mathbf{x}, A_k, \Omega_3) = \begin{cases} \left(\frac{A_k(\mathbf{x},\mathbf{p},\theta_p)}{A_k(\mathbf{x},\mathbf{p},\theta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{A_k(\mathbf{x},\mathbf{p},id\cdot\theta_p)}{2\sqrt{p}A_k(\mathbf{x},\mathbf{p},\theta_p)} - \frac{1}{p}\right), & p = q; \end{cases}$$

where g is positive function, and $\overline{g}, \overline{x}, x_i > 0$ $(i = 1, \ldots, n)$.

As before, we conclude that the functions $p \mapsto A_k(g, \lambda, \theta_p)$ (for $k = 1, 3$) and $p \mapsto A_k(\mathbf{x}, \mathbf{p}, \theta_p)$ (for $k = 2, 4$) are exponentially convex, and for $\mu_{p,q}$ we get the monotonicity property.

If $p, q, u, v \in \mathbb{R}^+$ such that $p \leqslant u$, $q \leqslant v$, then we have

$$\mu_{p,q}(g, A_k, \Omega_3) \leqslant \mu_{u,v}(g, A_k, \Omega_3), \text{ for } k = 1, 3,$$
$$\mu_{p,q}(\mathbf{x}, A_k, \Omega_3) \leqslant \mu_{u,v}(\mathbf{x}, A_k, \Omega_3), \text{ for } k = 2, 4.$$

EXAMPLE 6.4. Let

$$\Omega_4 = \{\phi_p : \mathbb{R}^+ \to \mathbb{R}^+ : p \in \mathbb{R}^+\}$$

be family of functions defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{(\log p)^2}, & p \neq 1; \\ \frac{x^2}{2}, & p = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\phi_p(x) = p^{-x} > 0$, ϕ_p is convex function for $p > 0$. From Remark 2.4 it follows that $p \mapsto \frac{d^2}{dx^2}\phi_p(x)$ is exponentially convex function, and from [6] we then also have that $p \mapsto [y_0, y_1, y_2]\phi_p$ is exponentially convex. Our family Ω_4 of functions ϕ_p fulfills the condition given in Corollary 2.1, Corollary 3.1, Corollary 4.1 and Corollary 5.1. We assume again that interval $[\alpha, \beta]$ from these corollaries is a subset of $\mathbb{R}^+$, and so for our family of functions we have the following possible cases for $\mu_{p,q}$:

– for $k = 1,3$:

$$\mu_{p,q}(g, A_k, \Omega_4) = \begin{cases} \left(\frac{A_k(g,\lambda,\phi_p)}{A_k(g,\lambda,\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{A_k(g,\lambda,\phi_p)}{pA_k(g,\lambda,\phi_p)} - \frac{2}{p\log p}\right), & p = q \neq 1; \\ \exp\left(-\frac{1}{3}\frac{A_k(g,\lambda,id\cdot\phi_1)}{A_k(g,\lambda,\phi_1)}\right), & p = q = 1; \end{cases}$$

– for $k = 2,4$:

$$\mu_{p,q}(\mathbf{x}, A_k, \Omega_4) = \begin{cases} \left(\frac{A_k(\mathbf{x},\mathbf{p},\phi_p)}{A_2(\mathbf{x},\mathbf{p},\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{A_k(\mathbf{x},\mathbf{p},\phi_p)}{pA_k(\mathbf{x},\mathbf{p},\phi_p)} - \frac{2}{p\log p}\right), & p = q \neq 1; \\ \exp\left(-\frac{1}{3}\frac{A_k(\mathbf{x},\mathbf{p},id\cdot\phi_1)}{A_k(\mathbf{x},\mathbf{p},\phi_1)}\right), & p = q = 1; \end{cases}$$

where g is positive function, and $\overline{g}, \overline{x}, x_i > 0 \ (i = 1, \ldots, n)$.

As before, we conclude that the functions $p \mapsto A_k(g, \lambda, \phi_p)$ (for $k = 1,3$) and $p \mapsto A_k(\mathbf{x}, \mathbf{p}, \phi_p)$ (for $k = 2,4$) are exponentially convex, and for $\mu_{p,q}$ we have the monotonicity property.

If $p, q, u, v \in \mathbb{R}^+$ such that $p \leqslant u$, $q \leqslant v$, then we have

$$\mu_{p,q}(g, A_k, \Omega_4) \leqslant \mu_{u,v}(g, A_k, \Omega_4), \text{ for } k = 1,3,$$
$$\mu_{p,q}(\mathbf{x}, A_k, \Omega_4) \leqslant \mu_{u,v}(\mathbf{x}, A_k, \Omega_4), \text{ for } k = 2,4.$$

Acknowledgements. The present investigation was supported in part by the Croatian Ministry of Science, Education and Sports under the Research Grant 117-1170889-0888. The research of the first author is funded by Higher Education Commission Pakistan.

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(Received March 6, 2012)

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