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SHARP TWO PARAMETER BOUNDS FOR THE LOGARITHMIC MEAN AND THE ARITHMETIC–GEOMETRIC MEAN OF GAUSS

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Abstract. For fixed $s \geqslant 1$ and $t_1, t_2 \in (0, 1/2)$ we prove that the inequalities $G^s(t_1a+(1-t_1)b, t_1b+(1-t_1)a)A^{1-s}(a,b) > AG(a,b)$ and $G^s(t_2a+(1-t_2)b, t_2b+(1-t_2)a)A^{1-s}(a,b) > L(a,b)$ hold for all $a,b>0$ with $a \neq b$ if and only if $t_1 \geqslant 1/2-\sqrt{2s}/(4s)$ and $t_2 \geqslant 1/2-\sqrt{6s}/(6s)$. Here $G(a,b)$, $L(a,b)$, $A(a,b)$ and $AG(a,b)$ are the geometric, logarithmic, arithmetic and arithmetic-geometric means of a and b, respectively.

1. Introduction

For $a,b>0$ the classical arithmetic-geometric mean $AG(a,b)$ of Gauss is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} &a_0 = a, \qquad b_0 = b, \\ &a_{n+1} = (a_n+b_n)/2 = A(a_n,b_n), \qquad b_{n+1} = \sqrt{a_nb_n} = G(a_n,b_n). \end{aligned} \tag{1.1}$$

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log a - \log b)$ and $A(a,b) = (a+b)/2$ be the classical harmonic, geometric, logarithmic and arithmetic means of two distinct positive real numbers a and b, respectively. Then it is well known that the inequalities $H(a,b) < G(a,b) < L(a,b) < A(a,b)$ hold for all $a,b>0$ with $a \neq b$.

Recently, the harmonic, geometric, logarithmic, arithmetic-geometric and arithmetic means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [5, 6, 10–14, 16–18].

Carlson and Vuorinen [7], and Brackenn [3] proved that

$$L(a,b) < AG(a,b) \tag{1.2}$$

for all $a,b>0$ with $a \neq b$. Vamanamurthy and Vuorinen [19] established that

$$AG(a,b) < \frac{\pi}{2}L(a,b)$$

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for all $a,b>0$ with $a\neq b$.

The following inequalities were proved by Sándor in [17, 18].

$$\sqrt{A(a,b)G(a,b)} < AG(a,b) < \left(\frac{\sqrt{A(a,b)}+\sqrt{G(a,b)}}{2}\right)^2 \tag{1.3}$$

for $a,b>0$ with $a\neq b$.

In order to refine inequality (1.2), Neuman and Sándor [15] proved that

$$L(a,b) < L(a_n,b_n) < AG(a,b), \quad n\geqslant 1$$

for $a,b>0$ with $a\neq b$, where $\{a_n\}$ and $\{b_n\}$ are defined as in (1.1).

For $t_1,t_2,t_3,t_4\in(0,1/2)$, very recently Chu et al. [8, 9] proved that the inequalities

$$G(t_1a+(1-t_1)b,t_1b+(1-t_1)a) > AG(a,b), \tag{1.4}$$

$$H(t_2a+(1-t_2)b,t_2b+(1-t_2)a) > AG(a,b), \tag{1.5}$$

$$G(t_3a+(1-t_3)b,t_3b+(1-t_3)a) > L(a,b) \tag{1.6}$$

and

$$H(t_4a+(1-t_4)b,t_4b+(1-t_4)a) > L(a,b) \tag{1.7}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $t_1\geqslant 1/2-\sqrt{2}/4$, $t_2\geqslant 1/4$, $t_3\geqslant 1/2-\sqrt{6}/6$ and $t_4\geqslant 1/2-\sqrt{3}/6$.

Let $t\in(0,1/2)$, $s\geqslant 1$ and

$$Q_{t,s}(a,b) = G^s(ta+(1-t)b,tb+(1-t)a)A^{1-s}(a,b). \tag{1.8}$$

Then it is not difficult to verify that

$$Q_{t,1}(a,b) = G(ta+(1-t)b,tb+(1-t)a),$$

$$Q_{t,2}(a,b) = H(ta+(1-t)b,tb+(1-t)a)$$

and $Q_{t,s}(a,b)$ is strictly increasing with respect to $t\in(0,1/2)$ for fixed $a,b>0$ with $a\neq b$.

It is natural to ask what are the least values $t_1=t_1(s)$ and $t_2=t_2(s)$ in $(0,1/2)$ such that inequalities $Q_{t_1,s}(a,b)>AG(a,b)$ and $Q_{t_2,s}(a,b)>L(a,b)$ hold for all $a,b>0$ with $a\neq b$ and any $s\geqslant 1$. The aim of this paper is to answer these questions, our main results are the following Theorems 1.1 and 1.2.

THEOREM 1.1. *If $t\in(0,1/2)$ and $s\geqslant 1$, then the inequality*

$$Q_{t,s}(a,b) > AG(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $t\geqslant t_1(s)=1/2-\sqrt{2s}/(4s)$.

THEOREM 1.2. *If* $t \in (0,1/2)$ *and* $s \geqslant 1$*, then the inequality*

$$Q_{t,s}(a,b) > L(a,b)$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $t \geqslant t_2(s) = 1/2 - \sqrt{6s}/(6s)$.

REMARK 1.1. We clearly see that the inequalities (1.4)–(1.7) are the special cases of Theorems 1.1 and 1.2 with $s = 1,2$.

REMARK 1.2. Since $t_1(s) = 1/2 - \sqrt{2s}/(4s) > t_2(s) = 1/2 - \sqrt{6s}/(6s)$ for all $s \geqslant 1$, we clearly see that the results contained in the Theorems 1.1 and 1.2 are not compared to each other. In particular, if $t \geqslant t_1(s)$, then $Q_{t,s}(a,b) > AG(a,b) > L(a,b)$, so the inequality of Theorem 1.1 implies the one from Theorem 1.2.

2. Preliminaries

In order to prove Theorems 1.1 and 1.2 we need some basic knowledge of hypergeometric function and two lemmas, which we present in this section.

For real numbers a,b and c with $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \; |x| < 1. \tag{2.1}$$

Here $(a,0) = 1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$ is the shifted factorial function for $n = 1,2,\cdots$. In connection with the Gaussian hypergeometric function, the well-known complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)(0 < r < 1)$ of the first and second kinds [2, 4] are defined by

$$\begin{cases} \mathscr{K}(r) = \pi F(1/2,1/2;1;r^2)/2 = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathscr{K}(0) = \pi/2, \qquad \mathscr{K}(1) = \infty \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mathscr{E}(r) = \pi F(-1/2,1/2;1;r^2)/2 = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \\ \mathscr{E}(0) = \pi/2, \qquad \mathscr{E}(1) = 1, \end{cases} \tag{2.3}$$

respectively. The following formulas for $\mathscr{K}(r)$ were presented in [1]:

$$\frac{d\mathscr{K}(r)}{dr} = \frac{\mathscr{E}(r) - (1-r^2)\mathscr{K}(r)}{r(1-r^2)}, \tag{2.4}$$

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathscr{K}(r). \tag{2.5}$$

The Gaussian identity [1] shows that

$$AG(1,r)\mathscr{K}(\sqrt{1-r^2}) = \frac{\pi}{2} \tag{2.6}$$

for all $r \in (0,1)$.

LEMMA 2.1. *Let* $u \in [0,1]$, $s \geqslant 1$ *and*

$$f_{u,s}(x) = \frac{s}{2}\log(1-ux^2) - \log\left(\frac{\pi}{2\mathscr{K}(x)}\right). \tag{2.7}$$

Then $f_{u,s} > 0$ *for all* $x \in (0,1)$ *if and only if* $2su \leqslant 1$.

Proof. From (2.4) and (2.7) one has

$$f'_{u,s}(x) = -\frac{usx}{1-ux^2} + \frac{\mathscr{E}(x)-(1-x^2)\mathscr{K}(x)}{x(1-x^2)\mathscr{K}(x)} = \frac{F_{u,s}(x)}{x(1-x^2)(1-ux^2)\mathscr{K}(x)}, \tag{2.8}$$

where

$$F_{u,s}(x) = -sux^2(1-x^2)\mathscr{K}(x) + (1-ux^2)[\mathscr{E}(x)-(1-x^2)\mathscr{K}(x)]. \tag{2.9}$$

It follows from (2.1)–(2.3) and (2.9) together with elaborated computations that

$$\begin{aligned}
&\mathscr{E}(x)-(1-x^2)\mathscr{K}(x)\\
=&\frac{\pi}{2}\left[\sum_{n=0}^{\infty}\frac{(-1/2,n)(1/2,n)}{(n!)^2}x^{2n} - (1-x^2)\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n}\right]\\
=&\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+2},
\end{aligned}$$

$$\begin{aligned}
\frac{2}{\pi}F_{u,s}(x) =& -sux^2(1-x^2)\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n} + (1-ux^2)\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+2}\\
=& -su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+2} + su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+4}\\
&+\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+2} - u\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+4}\\
=& -sux^2 - su\sum_{n=0}^{\infty}\frac{(1/2,n+1)^2}{[(n+1)!]^2}x^{2n+4} + su\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{(n!)^2}x^{2n+4}\\
&+\frac{x^2}{2} + \sum_{n=0}^{\infty}\frac{(1/2,n+1)^2}{2(n+1)!(n+2)!}x^{2n+4} - u\sum_{n=0}^{\infty}\frac{(1/2,n)^2}{2n!(n+1)!}x^{2n+4}\\
=&x^2\left[\frac{1}{2} - su + \sum_{n=0}^{\infty}\frac{(1/2,n)^2A_n}{2(n+1)!(n+2)!}x^{2n+2}\right],
\end{aligned} \tag{2.10}$$

where

$$A_n = su(n+2)(2n+\frac{3}{2}) + (n+\frac{1}{2})^2 - u(n+1)(n+2) > 0. \tag{2.11}$$

We divide the proof into two cases.

Case 1.1. $2su \leqslant 1$. Then (2.8)–(2.11) lead to the conclusion that $f_{u,s}(x)$ is strictly increasing on $(0,1)$. Therefore, $f_{u,s}(x) > f_{u,s}(0^+) = 0$ for all $x \in (0,1)$ follows from (2.2) and (2.7) together with the monotonicity of $f_{u,s}(x)$ on $(0,1)$.

Case 1.2. $2su > 1$. Then (2.8)–(2.10) lead to the conclusion that there exists $\delta_1 \in (0,1)$ such that $f_{u,s}(x)$ is strictly decreasing on $(0,\delta_1)$. Therefore, $f_{u,s}(x) < f_{u,s}(0^+) = 0$ for all $x \in (0,\delta_1)$ follows from (2.2) and (2.7) together with the monotonicity of $f_{u,s}(x)$ on $(0,\delta_1)$. □

LEMMA 2.2. *Let $u \in [0,1]$, $s \geqslant 1$, $arctanh(x) = \log[(1+x)/(1-x)]/2$ be the inverse hyperbolic tangent function, and*

$$g_{u,s}(x) = \frac{s}{2}\log(1-ux^2) + \log\left(\frac{arctanh(x)}{x}\right). \tag{2.12}$$

Then $g_{u,s}(x) > 0$ for all $x \in (0,1)$ if and only if $3su \leqslant 2$.

Proof. From (2.12) one has

$$g'_{u,s}(x) = -\frac{sux}{1-ux^2} + \frac{x-(1-x^2)\mathrm{arctanh}(x)}{x(1-x^2)\mathrm{arctanh}(x)} = \frac{G_{u,s}(x)}{x(1-x^2)(1-ux^2)\mathrm{arctanh}(x)}, \tag{2.13}$$

where

$$G_{u,s}(x) = -sux^2(1-x^2)\mathrm{arctanh}(x) + (1-ux^2)[x-(1-x^2)\mathrm{arctanh}(x)]. \tag{2.14}$$

Making use of series expansion and (2.14) we have

$$\begin{aligned} G_{u,s}(x) =& -sux^2(1-x^2)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} + (1-ux^2)\left[x-(1-x^2)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\right] \\ =& -su\sum_{n=0}^{\infty}\frac{x^{2n+3}}{2n+1} + su\sum_{n=0}^{\infty}\frac{x^{2n+5}}{2n+1} + (1-ux^2)\sum_{n=0}^{\infty}\frac{2x^{2n+3}}{(2n+1)(2n+3)} \\ =& x^3\left[\frac{2}{3} - su + \sum_{n=0}^{\infty}\frac{B_n x^{2n+2}}{(2n+1)(2n+3)(2n+5)}\right], \end{aligned} \tag{2.15}$$

where

$$B_n = 2u(s-1)(2n+5) + 2(2n+1) > 0. \tag{2.16}$$

We divide the proof into two cases.

Case 1.1. $3su \leqslant 2$. Then (2.13)–(2.16) lead to the conclusion that $g_{u,s}(x)$ is strictly increasing on $(0,1)$. Therefore, $g_{u,s}(x) > g_{u,s}(0^+) = 0$ for all $x \in (0,1)$ follows from (2.12) together with the monotonicity of $g_{u,s}(x)$ on $(0,1)$.

Case 1.2. $3su > 2$. Then (2.13)–(2.15) lead to the conclusion that there exists $\delta_2 \in (0,1)$ such that $g_{u,s}(x)$ is strictly decreasing on $(0,\delta_2)$. Therefore, $g_{u,s}(x) < g_{u,s}(0^+) = 0$ for all $x \in (0,\delta_2)$ follows from (2.12) and the monotonicity of $g_{u,s}(x)$ on $(0,\delta_2)$. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Since both $Q_{t,s}(a,b)$ and $AG(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b$. Let $x=(a-b)/(a+b)\in(0,1)$. Then from (2.5) and (2.6) together with $b/a=(1-x)/(1+x)$ we have

$$\begin{aligned}\frac{AG(a,b)}{A(a,b)}=&\frac{AG(1,b/a)}{A(1,b/a)}=\frac{\pi}{\mathscr{K}\sqrt{1-(b/a)^2}(1+b/a)}\\ =&\frac{\pi(1+x)}{2\mathscr{K}(2\sqrt{x}/(1+x))}=\frac{\pi}{2\mathscr{K}(x)}.\end{aligned}\tag{3.1}$$

It follows from (1.8) and (3.1) that

$$\begin{aligned}&\log\left(\frac{Q_{t,s}(a,b)}{AG(a,b)}\right)=\log\left(\frac{Q_{t,s}(a,b)}{A(a,b)}\right)-\log\left(\frac{AG(a,b)}{A(a,b)}\right)\\ =&\frac{s}{2}\log\left[1-(1-2t)^2x^2\right]-\log\left[\frac{\pi}{2\mathscr{K}(x)}\right].\end{aligned}\tag{3.2}$$

Therefore, Theorem 1.1 follows from Lemma 2.1 and (3.2). □

Proof of Theorem 1.2. Since both $Q_{t,s}(a,b)$ and $L(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b$. Let $x=(a-b)/(a+b)\in(0,1)$. Then (1.8) leads to

$$\begin{aligned}&\log\left(\frac{Q_{t,s}(a,b)}{L(a,b)}\right)=\log\left(\frac{Q_{t,s}(a,b)}{A(a,b)}\right)-\log\left(\frac{L(a,b)}{A(a,b)}\right)\\ =&\frac{s}{2}\log\left[1-(1-2t)^2x^2\right]+\log\left(\frac{\operatorname{arctanh}(x)}{x}\right).\end{aligned}\tag{3.3}$$

Therefore, Theorem 1.2 follows from Lemma 2.2 and (3.3). □

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