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A DIMENSIONALITY REDUCTION PRINCIPLE ON THE OPTIMIZATION OF FUNCTION

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Abstract. In this paper, we put out a dimensionality reduction principle on the optimization of function, in other words, we show that $\inf_{a\in\mathbb{R}_+^n}\{f(a)\}=0$ if and only if

$$\inf_{a\in[0,1]^m,k\in K_{m+1}}\left\{f\left(a_1I_{k_1},\ldots,a_mI_{k_m},I_{k_{m+1}},O_{n-k_1-\cdots-k_{m+1}}\right)\right\}=0$$

under the proper hypotheses. As applications, we study the optimal problems of linear inequalities involving function power means. In order to show the significance of our results, we give an example for a discrete case by means of the software Mathematica and another example involving space science.

1. Introduction

We shall use the following some notations and symbols throughout the paper:

$$\begin{aligned}
&a^\theta=(a_1^\theta,\ldots,a_n^\theta);\quad a=a^1;\quad \alpha=(\alpha_1,\ldots,\alpha_m);\quad \lambda=(\lambda_1,\ldots,\lambda_m);\\
&\min\{\alpha\}=\min\{\alpha_1,\ldots,\alpha_m\};\quad \max\{\alpha\}=\max\{\alpha_1,\ldots,\alpha_m\};\\
&O_n=(\underbrace{0,\ldots,0}_{n});\quad I_n=(\underbrace{1,\ldots,1}_{n});\quad I^n=\underbrace{I\times\cdots\times I}_{n};\quad \mathbb{R}=(-\infty,\infty);\\
&\mathbb{R}_+^n=[0,\infty)^n;\quad \mathbb{R}_{++}^n=(0,\infty)^n;\quad \mathbb{N}=\{1,\ldots,n,\ldots\};\\
&K_m=\left\{k\in\mathbb{N}^m:1\leqslant k_1\leqslant\cdots\leqslant k_{m-1},\ \sum_{j=1}^m k_j\leqslant n, m<n\right\};\\
&\Omega_n=\left\{t\in\mathbb{R}_{++}^n:0<t_1\leqslant\ldots\leqslant t_{n-1},\ \sum_{i=1}^n t_i\leqslant 1\right\}.
\end{aligned}$$

If $A=\{a_1,\ldots,a_n\}$ is a finite set, where $a_i\neq a_j$, $1\leqslant i\neq j\leqslant n$, then we define that

$$|A|=n.$$

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Recall the definition of the $t-$th power mean for a sequence $a \in \mathbb{R}^n_{++}$:

$$M_n^{[t]}(a) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^n a_i^t\right)^{1/t}, & \text{if } t \in \mathbb{R}\backslash\{0\} \\ \left(\prod_{i=1}^n a_i\right)^{1/n}, & \text{if } t = 0 \\ \min\{a\}, & \text{if } t = -\infty \\ \max\{a\}, & \text{if } t = \infty \end{cases}.$$

Let $E \subset \mathbb{R}^l$, where $l \in \mathbb{N}$, be a measurable set with the measure (l-dimension volume) $|E| \in (0,\infty)$ and the function $f : E \to (0,\infty)$ be bounded. If the Lebesgue integrals of f^t and $\ln f$ exist, we call the functional

$$M^{[t]}(f) = \begin{cases} \left(\frac{1}{|E|}\int_E f^t\right)^{1/t}, & \text{if } t \in \mathbb{R}\backslash\{0\} \\ \exp\left(\frac{1}{|E|}\int_E \ln f\right), & \text{if } t = 0 \\ \min\{f\}, & \text{if } t = -\infty \\ \max\{f\}, & \text{if } t = \infty \end{cases}$$

the $t-$th power mean of the function f, or function power mean. In particular,

$$\overline{f} = M^{[1]}(f)$$

is the mean of the function f.

As pointed out in [1], the means play many important and vital roles in several different mathematical and scientific specialized areas. The power mean is the most one in all the means. Many inequalities involving the power means or function power mean and the related problems are established (see [2, 3, 4, 5, 6]). For example, the literatures [2, 3] gave the applications of the function power mean in geometry and space science.

Assume an inequality includes some parameters. If we get that these parameters should satisfy necessary and sufficient conditions that the inequality holds, then we call that the inequality is optimized.

In this paper, we put out a dimensionality reduction principle on the optimization of the function. As the applications, we study the optimal problems of linear inequalities involving function power means, as well as we give an example for a discrete case by means of the software Mathematica and another example involving space science, the aim is to show the significance of our results.

2. A dimensionality reduction principle

We first introduce a dimensionality reduction principle on the optimization of function as follows.

THEOREM 2.1. (Dimensionality reduction principle, abbreviated as DRP) *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a homogeneous, symmetrical and continuous function with degree γ, where $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma > 0$, and let f be a differentiable function in $\mathbb{R}^n_{++}$. If there* *exists* $m \in \mathbb{N}:\ m < n$ *such that for any* $p \in \mathbb{N}:\ 1 \leqslant p \leqslant n$*, and any* $\mu \in \mathbb{R}$*, the solution* $(a_1, \ldots, a_p) \in \mathbb{R}^p_{++}$ *of the system of the equations*

$$\frac{\partial}{\partial a_l} f(a_1, \ldots, a_p, O_{n-p}) + \mu = 0, \quad l = 1, \ldots, p \tag{1}$$

satisfies the inequality

$$|\{a_1, \ldots, a_p\}| \leqslant m + 1,$$

then a necessary and sufficient condition such that

$$\inf_{a \in \mathbb{R}^n_+} \{f(a)\} = 0 \tag{2}$$

holds is that

$$\inf_{a \in [0,1]^m, k \in K_{m+1}} \left\{ f\left(a_1 I_{k_1}, \ldots, a_m I_{k_m}, I_{k_{m+1}}, O_{n-k_1-\cdots-k_{m+1}}\right)\right\} = 0 \tag{3}$$

holds.

In (2) and (3), if the infimum inf *is replaced by the supremum* sup*, then the same conclusion also holds.*

Proof. We only prove the case of inf, the similar argument for the case sup is omitted.

Note that the necessary condition is clear, we only need to prove the sufficiency, that is, assume (3) holds, we will prove that (2) holds.

First, we prove that for any $p \in \{1, \ldots, n\}$, and any $(a_1, \ldots, a_p) \in \mathbb{R}^p_+$, the following inequality holds by induction on p:

$$f(a_1, \ldots, a_p, O_{n-p}) \geqslant 0. \tag{4}$$

(A) If $1 \leqslant p \leqslant m$, since

$$f(O_n) = f(0 I_n) = 0^\gamma f(I_n) = 0,$$

we may assume that some a_i among $a_1, \ldots, a_p$ is not equal to zero and

$$\max\{a_1, \ldots, a_p\} = a_p > 0.$$

Set

$$b_j = \frac{a_j}{a_p}, \quad j = 1, \ldots, p-1.$$

Then $(b_1, \ldots, b_{p-1}) \in [0,1]^{p-1}$. From (3) we get

$$f\left(b_1 I_{k_1}, \ldots, b_{p-1} I_{k_{p-1}}, 0, \ldots, 0, I_{k_{m+1}}, O_{n-k_1-\cdots-k_{m+1}}\right) \geqslant 0. \tag{5}$$

Let

$$k_j = 1, \quad j = 1, 2, \ldots, m+1$$

in (5), we get

$$f(b_1,\dots,b_{p-1},1,O_{n-p}) = f(b_1,\dots,b_{p-1},O_{m-p+1},1,O_{n-m-1}) \geqslant 0. \tag{6}$$

By (6), we obtain that

$$f(a_1,\dots,a_p,O_{n-p}) = a_p^{\gamma} f(b_1,\dots,b_{p-1},1,O_{n-p}) \geqslant 0.$$

That is to say, the inequality (4) holds for $p : 1 \leqslant p \leqslant m$.

(B) Assume the inequality (4) holds for $m \leqslant p \leqslant n-1$, we will prove that the inequality

$$f(a_1,\dots,a_{p+1},O_{n-p-1}) \geqslant 0 \tag{7}$$

also holds.

Note that

$$(a_1,\dots,a_{p+1}) \in \mathbb{R}_+^{p+1},$$

we may assume that some a_i among $a_1,\dots,a_{p+1}$ is not equal to zero, then

$$\sum_{q=1}^{p+1} a_q > 0. \tag{8}$$

Set

$$x_j = \frac{(p+1)a_j}{\sum_{q=1}^{p+1} a_q}, \quad j = 1,\dots,p+1,$$

then we get

$$f(a_1,\dots,a_{p+1},O_{n-p-1}) = \left(\frac{\sum_{q=1}^{p+1} a_q}{p+1}\right)^{\gamma} f(x_1,\dots,x_{p+1},O_{n-p-1}),$$

where $(x_1,\cdots,x_{p+1}) \in D_{p+1}$, and

$$D_n := \left\{ x \in \mathbb{R}_+^n : \sum_{j=1}^{n} x_j = n \right\}.$$

Thus, the proof of (7) becomes the proof of the following inequality

$$f(x_1,\dots,x_{p+1},O_{n-p-1}) \geqslant 0 \tag{9}$$

holds when $(x_1,\dots,x_{p+1}) \in D_{p+1}$.

Next, we divide the proof into two steps as follows.

(I) If $(x_1,\dots,x_{p+1}) \in \mathbb{R}_{++}^{p+1}$ is a critical point of $f(x_1,\dots,\, x_{p+1},O_{n-p-1})$ in D_{p+1}, we will prove that (9) holds.

We construct the corresponding Lagrange function:

$$L = f(x_1,\dots,x_{p+1},O_{n-p-1}) + \mu\,(x_1+\cdots+x_{p+1}-p-1),$$

where $\mu \in \mathbb{R}$, then

$$\frac{\partial L}{\partial x_j} = \frac{\partial}{\partial x_j} f(x_1,\ldots,x_{p+1},O_{n-p-1}) + \mu = 0, \quad j = 1,\cdots,p+1.$$

Since $m+1 \leqslant p+1 \leqslant n$ and the hypothesis of Theorem 2.1, we get

$$|\{x_1,\ldots,x_{p+1}\}| \leqslant m+1.$$

By symmetry, just as well, we may assume that

$$\{x_1,\ldots,x_{p+1}\} = \{y_1,\ldots,y_{m+1}\},$$

and

$$\max\{y_1,\ldots,y_{m+1}\} = y_{m+1} > 0.$$

By symmetry, we confirm that there is a $k = (k_1,\ldots,k_{m+1}) \in K_{m+1}$, where

$$k_1 + \cdots + k_{m+1} = p+1,$$

such that

$$f(x_1,\ldots,x_{p+1},O_{n-p-1}) = f\left(y_1 I_{k_1},\ldots,y_{m+1} I_{k_{m+1}},O_{n-p-1}\right).$$

Set

$$z_j = \frac{y_j}{y_{m+1}}, \quad j = 1,\cdots,m.$$

Since

$$(x_1,\ldots,x_{p+1}) \in \mathbb{R}^{p+1}_{++},$$

then

$$(z_1,\ldots,z_m) \in (0,1]^m,$$

and

$$\begin{aligned} & f(x_1,\ldots,x_{p+1},O_{n-p-1}) \\ & = y^{\gamma}_{m+1} f\left(z_1 I_{k_1},\ldots,z_m I_{k_m},I_{k_{m+1}},O_{n-k_1-\ldots-k_{m+1}}\right). \end{aligned} \tag{10}$$

Since

$$(z_1,\ldots,z_m) \in (0,1]^m \subset [0,1]^m \text{ and } (k_1,\ldots,k_{m+1}) \in K_{m+1},$$

the $(a_1,\ldots,a_m)$ in (3) may be replaced by $(z_1,\ldots,z_m)$, that is,

$$f\left(z_1 I_{k_1},\ldots,z_m I_{k_m},I_{k_{m+1}},O_{n-k_1-\ldots-k_{m+1}}\right) \geqslant 0,$$

which implies the inequality (9) holds by (10).

(II) If $(x_1,\ldots,x_{p+1})$ is a boundary point of D_{p+1}, we will prove that (9) also holds as follows.

We know that some component of $(x_1,\ldots,x_{p+1})$ must be zero. Without loss of generality, we may assume that $x_{p+1}=0$, then the inequality (9) becomes

$$f(x_1,\ldots,x_p,O_{n-p})\geqslant 0. \tag{11}$$

Based on $(x_1,\ldots,x_p)\in\mathbb{R}_+^p$, $m\leqslant p\leqslant n-1$, so the inequality (11) can be deduced from induction, the inequality (9) holds.

Therefore, the inequality (4) holds for any $p\in\{1,\ldots,n\}$, and any $(a_1,\ldots,a_p)\in\mathbb{R}_+^p$.

Finally, set $p=n$ in (4), we can obtain the inequality

$$f(a)\geqslant 0,\forall a\in\mathbb{R}_+^n. \tag{12}$$

In other words, we show that (2) holds if and only if (3) holds. The proof of Theorem 2.1 is completed. □

3. Applications of DRP

3.1. Main results

As applications of DRP, we obtain the main results for the optimizations of linear inequalities involving function power means as follows.

THEOREM 3.1. *Let E be a closed and bounded region in $\mathbb{R}^l$, $l\in\mathbb{N}$, the function $f:E\to(0,\infty)$ be bounded, the Lebesgue integral $\int_E f^t$ of f^t exist for any real number $t\in(0,\infty)$, and let*

$$\lambda\in\mathbb{R}_{++}^m,\quad \sum_{j=1}^m\lambda_j=1,\quad \alpha\in\mathbb{R}_{++}^m,\quad \alpha_i\neq\alpha_j,\quad 1\leqslant i\neq j\leqslant m,\quad m\geqslant 2,\quad \gamma,\ \theta\in(0,\infty). \tag{13}$$

Then the necessary and sufficient condition that the inequality

$$\sum_{j=1}^m\lambda_j\left\{M^{[\alpha_j]}(f)\right\}^\gamma\leqslant\left\{M^{[\theta]}(f)\right\}^\gamma \tag{14}$$

holds is that

$$\theta\geqslant\max\{\alpha\}. \tag{15}$$

The sign of the equality holding in (14) if $f\equiv Constant$. That is to say, the equation

$$\sup_{f>0}\left\{\sum_{j=1}^m\lambda_j\left\{M^{[\alpha_j]}(f)\right\}^\gamma-\left\{M^{[\theta]}(f)\right\}^\gamma\right\}=0 \tag{16}$$

holds if and only if the inequality (15) holds.

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, the necessary and sufficient condition that the inequality*

$$\sum_{j=1}^{m} \lambda_j \left\{M^{[\alpha_j]}(f)\right\}^{\gamma} \geqslant \left\{M^{[\theta]}(f)\right\}^{\gamma} \tag{17}$$

holds is that

$$0 < \theta \leqslant \left(\sum_{j=1}^{m} \frac{\lambda_j}{\alpha_j}\right)^{-1}. \tag{18}$$

The sign of the equality holding in (17) if $f \equiv Constant$. That is to say, the equation

$$\inf_{f>0} \left\{\sum_{j=1}^{m} \lambda_j \left\{M^{[\alpha_j]}(f)\right\}^{\gamma} - \left\{M^{[\theta]}(f)\right\}^{\gamma}\right\} = 0 \tag{19}$$

holds if and only if the inequality (18) holds.

3.2. Related lemmas

In order to prove Theorem 3.1 and Theorem 3.2, we need four lemmas as follows.

LEMMA 3.1. (see Lemma 3.2 in [6]) *Let $m \in \mathbb{N}$, $b_j \in \mathbb{R}\backslash\{0\}$, $\gamma_j \in \mathbb{R}$, $j = 0,1,\ldots,m$, $\gamma_i \neq \gamma_j$, $0 \leqslant i \neq j \leqslant m$. Then the number of the zeroes of the function*

$$u : (0,\infty) \to \mathbb{R}, \quad u(t) = \sum_{j=0}^{m} b_j t^{\gamma_j} \tag{20}$$

is not greater than m, i.e.,

$$|U_m| \leqslant m,$$

where

$$U_m = \{t \in (0,+\infty) : u(t) = 0\}.$$

LEMMA 3.2. *Let (13) hold. Then a necessary and sufficient condition that the inequality*

$$\sum_{j=1}^{m} \lambda_j \left[M_n^{[\alpha_j]}(a)\right]^{\gamma} \leqslant \left[M_n^{[\theta]}(a)\right]^{\gamma} \tag{21}$$

holds for any $a \in \mathbb{R}_+^n$, $n > m$ is that the inequality (21) holds under the following conditions:

$$(a_1,\ldots,a_{m-1}) \in [0,1]^{m-1}, \quad (k_1,\ldots,k_m) \in K_m \tag{22}$$

and

$$a = \left(a_1 I_{k_1}, \cdots, a_{m-1} I_{k_{m-1}}, I_{k_m}, O_{n-k_1-\cdots-k_m}\right). \tag{23}$$

If the inequality (21) is reversed, the same conclusion also holds.

Proof. Firstly, we prove that the especial case $\theta = 1$. Consider the auxiliary function:

$$f(a) := \left[\sum_{j=1}^{m} \lambda_j \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\right]^{\frac{1}{\gamma}} - \frac{1}{n}\sum_{i=1}^{n} a_i.$$

Clearly, the inequality (21) is equivalent to

$$f(a) \leqslant 0, \quad \forall a \in \mathbb{R}_+^n. \tag{24}$$

Note that the $f : \mathbb{R}_+^n \to \mathbb{R}$ is a homogeneous, symmetrical and continuous function with degree $1 > 0$, f is a differentiable function in $\mathbb{R}_{++}^n$. Combining the fact with Theorem 2.1, we will only prove that for any $p \in \mathbb{N} : 1 \leqslant p \leqslant n$, and any $\mu \in \mathbb{R}$, the solution $(a_1, \ldots, a_p) \in \mathbb{R}_{++}^p$ of the system of equations (1) satisfies that

$$|\{a_1, \ldots, a_p\}| \leqslant m.$$

Set

$$a^* := (a_1, \ldots, a_p, O_{n-p}),$$

and

$$M_{m,n}^{[\gamma]}(a^*; \alpha; \lambda) := \left[\sum_{j=1}^{m} \lambda_j \left(\frac{1}{n}\sum_{i=1}^{p} a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\right]^{\frac{1}{\gamma}}.$$

Thus the system (1) simplifies into

$$\frac{\partial}{\partial a_l} f(a^*) + \mu = 0, \;\; l = 1, \ldots, p. \tag{25}$$

Note that

$$f(a^*) = \left[\sum_{j=1}^{m} \lambda_j \left(\frac{1}{n}\sum_{i=1}^{p} a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\right]^{\frac{1}{\gamma}} - \frac{1}{n}\sum_{i=1}^{p} a_i,$$

$$\frac{\partial f(a^*)}{\partial a_l} = \frac{1}{n}\left[M_{m,n}^{[\gamma]}(a^*; \alpha; \lambda)\right]^{1-\gamma} \sum_{j=1}^{m} \left[\lambda_j \left(\frac{1}{n}\sum_{i=1}^{p} a_i^{\alpha_j}\right)^{\frac{\gamma-\alpha_j}{\alpha_j}} a_l^{\alpha_j - 1}\right] - \frac{1}{n}.$$

Define the function (20) as follows:

$$\gamma_0 := 0, \quad \gamma_j := \alpha_j - 1, \;\; j = 1, \cdots, m;$$

$$b_0 := -\frac{1}{n} + \mu, \quad b_j := \frac{\lambda_j}{n}\left[M_{m,n}^{[\gamma]}(a^*; \alpha; \lambda)\right]^{1-\gamma} \left(\frac{1}{n}\sum_{i=1}^{p} a_i^{\alpha_j}\right)^{\frac{\gamma-\alpha_j}{\alpha_j}}, \;\; j = 1, \cdots, m.$$

Thus (25) can be rewritten as

$$u(a_l) = 0, \ \ l = 1, \dots, p. \tag{26}$$

(A) Let $b_0 \neq 0$. Then

$$b_j \in \mathbb{R} \backslash \{0\}, \ \ j = 0, \dots, m.$$

If $\gamma_j \neq 0, j = 1, \dots, m$. From

$$\alpha \in \mathbb{R}^m_{++}, \quad \alpha_i \neq \alpha_j, \quad 1 \leqslant i \neq j \leqslant m,$$

we get

$$\gamma_j \in \mathbb{R}, \quad j = 0, \dots, m, \quad \gamma_i \neq \gamma_j, \quad 0 \leqslant i \neq j \leqslant m.$$

From the fact

$$\{a_1, \dots, a_p\} \subset U_m = \{t \in (0, +\infty) : u(t) = 0\}$$

and Lemma 3.1, we have

$$\left| \{a_1, \dots, a_p\} \right| \leqslant |U_m| \leqslant m.$$

If some $\gamma_j = 0, 1 \leqslant j \leqslant m$, for example, $\gamma_m = 0$, then

$$u : (0, \infty) \to \mathbb{R}, \quad u(t) = (b_0 + b_m) + \sum_{j=1}^{m-1} b_j t^{\gamma_j}.$$

Based on the above analysis, we get

$$\left| \{a_1, \dots, a_p\} \right| \leqslant |U_m| \leqslant m - 1 < m.$$

(B) If $b_0 = 0$, then we have that

$$\left| \{a_1, \dots, a_p\} \right| \leqslant |U_m| \leqslant m - 1 < m$$

based on the above analysis. This completes the proof of above assertion.

Secondly, we consider the case of $0 < \theta \neq 1$. Using the transformation:

$$a^* = a^\theta, \quad \alpha^* = \frac{\alpha}{\theta}, \quad \gamma^* = \frac{\gamma}{\theta},$$

we can rewrite the inequality (21) in the form

$$\sum_{j=1}^{m} \lambda_j \left[M_n^{[\alpha_j^*]}(a^*) \right]^{\gamma^*} \leqslant \left[M_n^{[1]}(a^*) \right]^{\gamma^*}. \tag{27}$$

It follows from (27) that the necessary and sufficient condition that the inequality (21) holds has been transformed into the above case. Hence we obtain the conditional expression shown as Lemma 3.2.

This completes our proof. □

REMARK 3.1. In Lemma 3.2, if some $\lambda_j < 0$ among $\lambda_1, \ldots, \lambda_m$, then Lemma 3.2 still holds by Lemma 3.1 and the proof of Lemma 3.2.

In fact, in the proof of Lemma 3.2, if we define a new auxiliary function as follows:

$$f(a) = \begin{cases} [g(a)]^{\frac{1}{\gamma}} - \frac{1}{n}\sum_{i=1}^{n} a_i, & \text{if } g(a) \geqslant 0 \\ -[-g(a)]^{\frac{1}{\gamma}} - \frac{1}{n}\sum_{i=1}^{n} a_i, & \text{if } g(a) < 0 \end{cases},$$

where

$$g(a) = \sum_{j=1}^{m} \lambda_j \left(\frac{1}{n} \sum_{i=1}^{n} a_i^{\alpha_j} \right)^{\frac{\gamma}{\alpha_j}},$$

then the proof of Lemma 3.2 is still correct.

Let E be a bounded and closed region in $\mathbb{R}^l$, $l \in \mathbb{N}$, and $E \subset [c,d]^l$. Write the interval

$$J_k := \left[c + (k-1)\frac{d-c}{m}, c + k\frac{d-c}{m} \right], \quad m \in \mathbb{N}, \quad k \in \{1, \ldots, m\}.$$

Define the partition P_E of E as follows:

$$P_E : E = \left(\bigcup_{i=1}^{n} E_i \right) \bigcup \left(\bigcup_{i=1}^{n'} E_i' \right), \quad n, n' \in \mathbb{N},$$

where there exists $J_{j_1} \times \cdots \times J_{j_l} \subset E, j_1, \cdots, j_l \in \{1, \ldots, m\}$ such that

$$E_i = J_{j_1} \times \cdots \times J_{j_l}, \quad i = 1, \ldots, n,$$

and there exists $J_{j_1} \times \cdots \times J_{j_l} \not\subset E, j_1, \cdots, j_l \in \{1, \ldots, m\}$ such that

$$E_i' = \left(J_{j_1} \times \cdots \times J_{j_l} \right) \bigcap E \neq \Phi, \quad i = 1, \ldots, n',$$

and the Φ is the empty set.

Here, we need to note that:

(i) If the partition P_E exists, then the number of the elements of the finite set $J := \left\{ J_{j_1} \times \cdots \times J_{j_l} : j_1, \cdots, j_l \in \{1, \ldots, m\} \right\}$ satisfies that

$$|J| = m^l \geqslant n + n';$$

(ii) There exist $m \in \mathbb{N}$ and $m^l \geqslant n + n'$ such that the partition P_E exists, and

$$n \to \infty \Leftrightarrow m \to \infty;$$

(iii) If the partition P_E exists, the measure $\left| \bigcup_{i=1}^{n'} E_i' \right|$ may be zero.

For the above partition P_E, we have the following Lemma 3.3.

LEMMA 3.3. *Let E be a bounded and closed region in $\mathbb{R}^l$, $l \in \mathbb{N}$, and $E \subset [c,d]^l$, and the Lebesgue integral $\int_E f$ of the bounded function $f : E \to \mathbb{R}$ exist. Then for the above partition P_E, we have that*

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f(x)\}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sup_{x\in E_i}\{f(x)\}\right) = \frac{1}{|E|}\int_E f. \tag{28}$$

Proof. Note that the expression (28) has meaning, because E is a bounded and closed region in $\mathbb{R}^l$, thus

$$0 < |E| < \infty.$$

Consider the above partition P_E, where $m^l \geqslant n + n'$. Since the Lebesgue integral $\int_E f$ of $f : E \to \mathbb{R}$ exists, then for any given $\varepsilon \in (0, |E|/2)$, there exists a $n_1 \in \mathbb{N}$, when $n > n_1$ we have

$$\sum_{i=1}^{n}\left(\sup_{x\in E_i}\{f(x)\}\right)|E_i| + \sum_{i=1}^{n'}\left(\sup_{x\in E_i'}\{f(x)\}\right)|E_i'| < \int_E f + \varepsilon, \tag{29}$$

and

$$\sum_{i=1}^{n}\left(\inf_{x\in E_i}\{f(x)\}\right)|E_i| + \sum_{i=1}^{n'}\left(\inf_{x\in E_i'}\{f(x)\}\right)|E_i'| > \int_E f - \varepsilon. \tag{30}$$

For the above partition P_E, it is easy to know that

$$\lim_{n\to\infty}\sum_{i=1}^{n}|E_i| = |E|, \quad \lim_{n\to\infty}\sum_{i=1}^{n'}|E_i'| = 0, \tag{31}$$

and

$$\bigcup_{i=1}^{n}E_i \subset E, \quad \bigcup_{i=1}^{n'}E_i' \subset E, \quad \sum_{i=1}^{n}|E_i| + \sum_{i=1}^{n'}|E_i'| = |E|. \tag{32}$$

Thus, there exists a $n_2 \in N$, when $n > n_2$, we have

$$|E| - \varepsilon < \sum_{i=1}^{n}|E_i| \leqslant |E|, \tag{33}$$

and

$$0 \leqslant \sum_{i=1}^{n'}|E_i'| < \varepsilon. \tag{34}$$

From $|E_1| = \cdots = |E_n|$, we can rewrite the inequality (33) in the form:

$$\frac{|E| - \varepsilon}{n} < |E_i| \leqslant \frac{|E|}{n}, \ \ i = 1, \ldots, n. \tag{35}$$

Since the function $f : E \to \mathbb{R}$ is a bounded function, there exists a constant $M > 0$ such that

$$-M \leqslant f(x) \leqslant M, \ \ \forall x \in E. \tag{36}$$

Then

$$-M \leqslant \inf_{x\in E_i'}\{f(x)\} \leqslant M, \ \ i=1,\ldots,n, \tag{37}$$

and

$$-M \leqslant \sup_{x\in E_i'}\{f(x)\} \leqslant M, \ \ i=1,\ldots,n'. \tag{38}$$

Thus, when $n>\max\{n_1,n_2\}$, from (29), (34), (35) and (38) we have that

$$\begin{aligned}
\int_E f+\varepsilon &> \sum_{i=1}^{n}\left(\sup_{x\in E_i}\{f(x)\}\right)|E_i| + \sum_{i=1}^{n'}\left(\sup_{x\in E_i'}\{f(x)\}\right)|E_i'| \\
&\geqslant \frac{|E|-\varepsilon}{n}\sum_{i=1}^{n}\sup_{x\in E_i}\{f(x)\} + \sum_{i=1}^{n'}\left(\sup_{x\in E_i'}\{f(x)\}\right)|E_i'| \\
&\geqslant \frac{|E|-\varepsilon}{n}\sum_{i=1}^{n}\sup_{x\in E_i}\{f(x)\} - M\sum_{i=1}^{n'}|E_i'| \\
&> \frac{|E|-\varepsilon}{n}\sum_{i=1}^{n}\sup_{x\in E_i}\{f(x)\} - \varepsilon M,
\end{aligned} \tag{39}$$

and from (30), (34), (35) and (37), we have that

$$\begin{aligned}
\int_E f-\varepsilon &< \sum_{i=1}^{n}\left(\inf_{x\in E_i}\{f(x)\}\right)|E_i| + \sum_{i=1}^{n'}\left(\inf_{x\in E_i'}\{f(x)\}\right)|E_i'| \\
&\leqslant \frac{|E|}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f(x)\} + \sum_{i=1}^{n'}\left(\inf_{x\in E_i'}\{f(x)\}\right)|E_i'| \\
&\leqslant \frac{|E|}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f(x)\} + M\sum_{i=1}^{n'}|E_i'| \\
&< \frac{|E|}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f(x)\} + \varepsilon M.
\end{aligned} \tag{40}$$

From (39–40), we get

$$\begin{aligned}
\frac{1}{|E|}\int_E f - \frac{(1+M)\varepsilon}{|E|} &< \frac{1}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f(x)\} \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}\sup_{x\in E_i}\{f(x)\} \\
&< \frac{1}{|E|-\varepsilon}\int_E f + \frac{(1+M)\varepsilon}{|E|-\varepsilon} \\
&= \frac{1}{|E|}\int_E f + \frac{\varepsilon}{|E|(|E|-\varepsilon)}\int_E f + \frac{(1+M)\varepsilon}{|E|-\varepsilon} \\
&\leqslant \frac{1}{|E|}\int_E f + \frac{M\varepsilon}{|E|-\varepsilon} + \frac{(1+M)\varepsilon}{|E|-\varepsilon}
\end{aligned}$$

$$= \frac{1}{|E|}\int_E f + \frac{(1+2M)\varepsilon}{|E|-\varepsilon}$$
$$< \frac{1}{|E|}\int_E f + \frac{2(1+2M)\varepsilon}{|E|}.$$

For the above inequalities, we get the equalities (28) in Lemma 3.3. The proof of Lemma 3.3 is completed. □

REMARK 3.2. In Lemma 3.3, if E is a curve segment in $\mathbb{R}^l$ (see [8, 9]), and the partition $E = \bigcup_{i=1}^n E_i$ of E satisfies that

$$|E_1| = \cdots = |E_n| = \frac{1}{n}|E|,$$

then the equalities (28) still holds by the proof of Lemma 3.3, where the integral in (28) is the curve integral.

LEMMA 3.4. *Under the hypotheses of Theorem 3.1, a necessary and sufficient condition that the inequality (14) holds is that the inequality*

$$\sum_{j=1}^m \lambda_j \left(t_m + \sum_{i=1}^{m-1} t_i a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}} \leqslant \left(t_m + \sum_{i=1}^{m-1} t_i a_i^{\theta}\right)^{\frac{\gamma}{\theta}} \tag{41}$$

holds for any $(a_1,\ldots,a_{m-1}) \in [0,1]^{m-1}$, *and any* $(t_1,\ldots,t_m) \in \Omega_m$.

If both the inequalities (14) and (41) are reversed, then the same conclusion also holds.

Proof. We only consider the case of $\leqslant$. Since E is a bounded and closed region in $\mathbb{R}^l$, there exists $[c,d]^l \subset \mathbb{R}^l$ such that $E \subset [c,d]^l$. For any positive integral number $n > m$, consider the above partition P_E of E. Since $f(x) > 0$ for any $x \in E$, the $\inf_{x\in E_i}\{f(x)\}$ exists for any $i \in \{1,\ldots,n\}$. Let

$$a = (a_1,\ldots,a_n) = \left(\inf_{x\in E_1}\{f(x)\},\ldots,\inf_{x\in E_n}\{f(x)\}\right). \tag{42}$$

Then $a \in \mathbb{R}_+^n$. By Lemma 3.3, the inequality (14) can be rewritten in the form

$$\lim_{n\to\infty} \sum_{j=1}^m \lambda_j \left[M_n^{[\alpha_j]}(a)\right]^\gamma \leqslant \lim_{n\to\infty}\left[M_n^{[\theta]}(a)\right]^\gamma. \tag{43}$$

Sufficiency. Assume for any $a = (a_1,\ldots,a_{m-1}) \in [0,1]^{m-1}$, and any $(t_1,\ldots,t_m) \in \Omega_m$, the inequality (41) holds, we prove the inequality (43) also holds as follows.

In the inequality (41), set

$$(t_1,\ldots,t_m) = \left(\frac{k_1}{n},\ldots,\frac{k_m}{n}\right) \in \Omega_m,\ \ n \in \mathbb{N},\ \ n > m, \tag{44}$$

where $(k_1,\ldots,k_m)\in\mathbb{N}^m$, then $(k_1,\ldots,k_m)\in K_m$ and the inequality (21) holds under the condition (23) by the inequality (41).

According to Theorem 2.1, the inequality (21) holds for any $a\in\mathbb{R}^n_+$. In the inequality (21), Taking the limit as $n\to\infty$, we find the inequality (43), therefore the inequality (14) holds by Lemma 3.3. This ends the proof of the sufficiency.

Necessity. Suppose now that the inequality (43) holds, we will prove that the inequality (41) also holds as follows.

According to the inequality (43) and the basic properties of the limit , there exists $n_0\in\mathbb{N}, n_0\geqslant m$, when $n>n_0$ the inequality (21) holds under the condition (42). According to the arbitrariness of the function $f:E\to(0,\infty)$ and Lemma 3.2, the inequality (21) holds under the conditions (22) and (23).

Since an arbitrary real number can be approached by a sequence of rational numbers, we can assume that (44) hold in (41), then $(t_1,\ldots,t_m)\in\Omega_m$ in (41). Since (41) is equivalent with (21), the inequality (41) holds from (21). This ends the proof of the necessity.

So far, we have proven Lemma 3.4. □

3.3. Proof of Theorem 3.1

Proof. Sufficiency. Assume that (15) holds, then

$$0<\alpha_j\leqslant\theta,\ \ j=1,\ldots,m,$$

and

$$\left(\frac{1}{|E|}\int_E f^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\leqslant\left(\frac{1}{|E|}\int_E f^{\theta}\right)^{\frac{\gamma}{\theta}},\ \ j=1,\ldots,m,$$

by the famous power mean inequality. Hence

$$\sum_{j=1}^m\lambda_j\left(\frac{1}{|E|}\int_E f^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\leqslant\sum_{j=1}^m\lambda_j\left(\frac{1}{|E|}\int_E f^{\theta}\right)^{\frac{\gamma}{\theta}}=\left(\frac{1}{|E|}\int_E f^{\theta}\right)^{\frac{\gamma}{\theta}}.$$

That is to say, the inequality (14) holds. This ends the proof of the sufficiency.

Necessity. Suppose now that (14) holds, we will prove that (15) holds as follows.

Without loss of generality, we can assume that $\max\{\alpha\}=\alpha_m$.

Assume $0<\theta<\max\{\alpha\}=\alpha_m$. By Lemma 3.4, the inequality (41) holds. Let $t_m>0$. Setting $t_1=\cdots=t_{m-1}=0$ in (41), and dividing from both sides of (41) by $(t_m)^{\gamma/\alpha_m}$, we get

$$\sum_{j=1}^{m-1}\lambda_j(t_m)^{\frac{\gamma}{\alpha_j}-\frac{\gamma}{\alpha_m}}+\lambda_m\leqslant(t_m)^{\frac{\gamma}{\theta}-\frac{\gamma}{\alpha_m}}.$$

In the above inequality, taking $t_m\to0^+$, from

$$\frac{\gamma}{\alpha_j}-\frac{\gamma}{\alpha_m}>0,\ \ j=1,\ldots,m-1\quad\text{and}\quad\frac{\gamma}{\theta}-\frac{\gamma}{\alpha_m}>0$$

we find that $\lambda_m \leqslant 0$, which is a contradiction with $\lambda \in \mathbb{R}^m_{++}$. This ends the proof of the necessity.

Therefore, this completes the proof of Theorem 3.1. $\square$

3.4. Proof of Theorem 3.2

Proof. Sufficiency. Suppose that (18) holds, we will prove the inequality (17) holds as follows.

Recall the well-known Hölder inequality (see [1] and Lemma 2.2 in [7]): Let $\left(x_1^{(j)},\ldots,x_n^{(j)}\right) \in \mathbb{R}^n_+$, $j=1,\ldots,m$, and $(p_1,\ldots,p_m) \in \mathbb{R}^m_{++}$. If

$$\sum_{j=1}^{m} \frac{1}{p_j} \leqslant 1, \tag{45}$$

then we have that

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{m} x_i^{(j)} \leqslant \prod_{j=1}^{m}\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{(j)}\right)^{p_j}\right]^{\frac{1}{p_j}}. \tag{46}$$

Set

$$\frac{1}{p_j} = \frac{\lambda_j \theta}{\alpha_j}, \quad j=1,\ldots,m,$$

then $(p_1,p_2,\ldots,p_m) \in \mathbb{R}^m_{++}$, and (45) holds by (18). Then for any $(x_1^{(j)},\ldots,x_n^{(j)}) \in \mathbb{R}^n_{++}$, $j=1,\ldots,m$, the inequality (46) holds.

Note that for any $y \in \mathbb{R}^m_+$ we have the following AM-GM inequality:

$$\sum_{j=1}^{m} \lambda_j \cdot y_j \geqslant \prod_{j=1}^{m} y_j^{\lambda_j}. \tag{47}$$

Hence for any $a \in \mathbb{R}^n_+$ we have that

$$\begin{aligned}
\sum_{j=1}^{m} \lambda_j \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}} &\geqslant \prod_{j=1}^{m}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\alpha_j}\right)^{\frac{\gamma\lambda_j}{\alpha_j}} \\
&= \left\{\prod_{j=1}^{m}\left[\frac{1}{n}\sum_{i=1}^{n}\left(a_i^{\lambda_j\theta}\right)^{p_j}\right]^{\frac{1}{p_j}}\right\}^{\frac{\gamma}{\theta}} \\
&\geqslant \left[\frac{1}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{m} a_i^{\lambda_j\theta}\right)\right]^{\frac{\gamma}{\theta}} \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\sum_{j=1}^{m}\lambda_j\theta}\right)^{\frac{\gamma}{\theta}} \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{\theta}\right)^{\frac{\gamma}{\theta}}.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{m}\lambda_j\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\geqslant\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\theta}\right)^{\frac{\gamma}{\theta}}. \tag{48}$$

Consider the above partition P_E of E. Since $f(x)>0$ for any $x\in E$, the $\inf_{x\in E_i}\{f(x)\}$ exists for any $i\in\{1,\dots,n\}$. Let (42) holds in (48). Then

$$a_i^t=\inf_{x\in E_i}\{f^t(x)\}\geqslant 0,\ \ i=1,\dots,n$$

for any $t\in(0,\infty)$, and

$$\lim_{n\to\infty}\sum_{j=1}^{m}\lambda_j\left(\frac{1}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f^{\alpha_j}(x)\}\right)^{\frac{\gamma}{\alpha_j}}\geqslant\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\inf_{x\in E_i}\{f^{\theta}(x)\}\right)^{\frac{\gamma}{\theta}}. \tag{49}$$

According to Lemma 3.3, (49) and the Lebesgue integral $\int_E f^t$ of f^t exists for any real number $t\in(0,\infty)$, we get the inequality (17). This ends the proof of the sufficiency.

Necessity. Assume that the inequality (17) holds, we will prove the inequality (18) holds as follows.

By means of Lemma 3.4, for any $(a_1,\dots,a_{m-1})\in[0,1]^{m-1}$, and any $(t_1,\dots,t_m)\in\Omega_m$, the inequality

$$\sum_{j=1}^{m}\lambda_j\left(t_m+\sum_{i=1}^{m-1}t_ia_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\geqslant\left(t_m+\sum_{i=1}^{m-1}t_ia_i^{\theta}\right)^{\frac{\gamma}{\theta}} \tag{50}$$

holds. Setting $a_j=0,\ j=1,\dots,m-1,\ 0<t_m<1$ in (50), we get

$$\frac{\sum_{j=1}^{m}\lambda_j(t_m)^{\frac{\gamma}{\alpha_j}}-(t_m)^{\frac{\gamma}{\theta}}}{1-t_m}\geqslant 0. \tag{51}$$

In (51), take the limit as $t_m\to 1^-$. The limit has the $0/0$ form, so by L'Hôpital's Rule, we have the inequality (18). This ends the proof of the necessity.

Therefore, this completes the proof of Theorem 3.2. $\square$

3.5. Two effective examples

Under the hypotheses of Theorem 3.1, if $\min\{\alpha\}<\theta<\max\{\alpha\}$, then the inequality (14) does not hold by Theorem 3.1. However, for any $a\in\mathbb{R}_+^n$, $n>m$, the following inequality may hold under the conditions (13):

$$\sum_{j=1}^{m}\lambda_j\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\alpha_j}\right)^{\frac{\gamma}{\alpha_j}}\leqslant\left(\frac{1}{n}\sum_{i=1}^{n}a_i^{\theta}\right)^{\frac{\gamma}{\theta}}, \tag{52}$$

where

$$\min\{\alpha\} < \theta < \max\{\alpha\}.$$

EXAMPLE 3.1. Let $a \in \mathbb{R}_+^5$ and $\lambda \in (0,1)$. Then the inequality

$$\frac{1-\lambda}{2}\left[\left(\frac{1}{5}\sum_{i=1}^{5} a_i^3\right)^{7/3} + \left(\frac{1}{5}\sum_{i=1}^{5} a_i^5\right)^{7/5}\right] + \lambda\left(\frac{1}{5}\sum_{i=1}^{5} a_i^9\right)^{7/9} \leqslant \left(\frac{1}{5}\sum_{i=1}^{5} a_i^6\right)^{7/6}, \tag{53}$$

holds if and only if

$$0 < \lambda \leqslant 0.3517133118615961\ldots. \tag{54}$$

Proof. By Lemma 3.2, the inequality (53) is equivalent to an inequality of two variables x, y as follows.

$$\frac{1-\lambda}{2}\left[\left(\frac{ix^3+jy^3+k}{5}\right)^{7/3} + \left(\frac{ix^5+jy^5+k}{5}\right)^{7/5}\right] + \lambda\left(\frac{ix^9+jy^9+k}{5}\right)^{7/9} \leqslant \left(\frac{ix^6+jy^6+k}{5}\right)^{7/6}. \tag{55}$$

The condition that this inequality holds is that

$$\lambda \leqslant \inf\left\{f(x,y;i,j,k) : (x,y) \in [0,1]^2, 1 \leqslant i \leqslant j, k \geqslant 1, i+j+k \leqslant 5\right\}, \tag{56}$$

where

$$f(x,y;i,j,k) := \frac{2\left(\frac{ix^6+jy^6+k}{5}\right)^{7/6} - \left(\frac{ix^3+jy^3+k}{5}\right)^{7/3} - \left(\frac{ix^5+jy^5+k}{5}\right)^{7/5}}{2\left(\frac{ix^9+jy^9+k}{5}\right)^{7/9} - \left(\frac{ix^3+jy^3+k}{5}\right)^{7/3} - \left(\frac{ix^5+jy^5+k}{5}\right)^{7/5}}.$$

By means of the software Mathematica, we obtain that

$\inf\{f(x,y;1,1,1)\} = f(0.508003...,0.508003;1,1,1) = 0.388872\ldots;$
$\inf\{f(x,y;1,1,2)\} = f(0.634119...,0.634119;1,1,2) = 0.464969\ldots;$
$\inf\{f(x,y;1,1,3)\} = f(1,1;1,1,3) = 0.4;$
$\inf\{f(x,y;1,2,1)\} = f(0.553056...,0.553084;1,2,1) = 0.378517\ldots;$
$\inf\{f(x,y;1,2,2)\} = f(0.901875...,0.901873;1,2,2) = 0.395445\ldots;$
$\inf\{f(x,y;1,3,1)\} = f(0.6988211...,0.6988204;1,3,1) = 0.3517133118624215\ldots;$
$\inf\{f(x,y;2,2,1)\} = f(0.698822...,0.698822;2,2,1) = 0.3517133118615961\ldots.$

Therefore,

$$\inf\left\{f(x,y;i,j,k):(x,y)\in[0,1]^2, 1\leqslant i\leqslant j, k\geqslant 1, i+j+k\leqslant 5\right\}$$
$$=0.3517133118615961\ldots,$$

that is to say, the inequality (53) holds if and only if the inequality (54) holds.

This completes the proof of Example 3.1. □

Next, we give an application of Theorem 3.2 in space science as follows.

EXAMPLE 3.2. Let the image $\Gamma=\gamma([a,b])$ of the vector function $\gamma:[a,b]\to\mathbb{R}^3$ be a smooth and closed curve (see [2, 3]), $P\in\mathbb{R}^3$ be a fixed point, and $A\in\Gamma$ be a moving point, and let (13) hold. If the inequality (18) holds, then we have that

$$\sum_{j=1}^{m}\lambda_j\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\|A-P\|^{-\alpha_j}ds\right)^{\frac{\gamma}{\alpha_j}}\geqslant\left(\frac{1}{|\Gamma|}\oint_{\Gamma}\|A-P\|^{-\theta}ds\right)^{\frac{\gamma}{\theta}} \tag{57}$$

by Remark 3.2 and Theorem 3.2, where $\oint_{\Gamma}\bullet ds$ is curve integral of the function $\bullet$, and $\|A-P\|$ is the distance between the point A and the point P.

In Example 3.2, we may regard the P as the earth with the mass M, A as a satellite with the mass m, and the Γ as the trajectory of the satellite. According to the law of universal gravity, the norm $\|\mathbf{F}(A,P)\|$ of gravity $\mathbf{F}(A,P)$ between the satellite A and the earth P is that

$$\|\mathbf{F}(A,P)\|=\frac{GmM}{\|A-P\|^2},$$

where G is the gravitational constant of solar system. Without loss of generality, we can assume that $GmM=1$. When the satellite A traverse one cycle along its orbit Γ, the average of the norm $\|\mathbf{F}(\Gamma,P)\|$ of the gravity $\mathbf{F}(A,P)$ between the satellite A and the earth P is that

$$\overline{\|\mathbf{F}(\Gamma,P)\|}=\frac{1}{|\Gamma|}\oint_{\Gamma}\|A-P\|^{-2}ds. \tag{58}$$

If we define that

$$\overline{\|\mathbf{F}_{\theta}(\Gamma,P)\|}:=\frac{1}{|\Gamma|}\oint_{\Gamma}\|A-P\|^{-\theta}ds \tag{59}$$

is θ-average gravity norm between A and P, where $\theta\in(0,\infty)$, then the inequality (57) can be rewritten as

$$\sum_{j=1}^{m}\lambda_j\overline{\|\mathbf{F}_{\alpha_j}(A,P)\|}^{\frac{\gamma}{\alpha_j}}\geqslant\overline{\|\mathbf{F}_{\theta}(\Gamma,P)\|}^{\frac{\gamma}{\theta}}. \tag{60}$$

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