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SHARP INEQUALITIES INVOLVING NEUMAN–SÁNDOR AND LOGARITHMIC MEANS

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Abstract. Sharp bounds for the Neuman-Sándor mean and for the logarithmic mean are established. The bounding quantities are the one-parameter bivariate means called the p-means. In this paper best values of the parameters of the bounding means are obtained.

1. Introduction

In recent years a significant progress has been made in developing new inequalities for bivariate means. In particular, means such as the logarithmic mean, two Seiffert means, and recently introduced mean by E. Neuman and J. Sándor (see [11]) have attracted attention of many researchers. All these means belong to a larger family of means called the Schwab-Borchardt means (see, e.g., [1], [2], [11], [12], [7]). The latter family of means, however, is not utilized in this paper. The Neuman-Sándor mean has been studied extensively in [8], [9], [5], [13], and in [14].

In this paper we present results which complement those reported in [10]. In Section 2 we give definitions and some properties of all bivariate means used in this paper. Main results of the present work are established in Section 3. They involve sharp bounds for the Neuman-Sándor mean and the logarithmic mean. The bounding means belong to the one-parameter family of means which in [10] are called the p-means.

2. Defnitions and preliminaries

In this section we give definitions of several bivariate means that are used in the sequel.

Let $a,b>0$. In order to avoid trivialities we will always assume that $a\neq b$. The unweighted arithmetic mean of a and b is defined as

$$A=\frac{a+b}{2}.$$

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The bivariate means discussed in this paper include the Neuman-Sándor mean M and the logarithmic mean L. Recall that

$$M = A\frac{v}{\sinh^{-1} v}, \quad L = A\frac{v}{\tanh^{-1} v}, \tag{1}$$

where

$$v = \frac{a-b}{a+b}. \tag{2}$$

(see [11], [1], [2]). Clearly $0 < |v| < 1$.

Other unweighted bivariate means used in this paper are the harmonic mean H, geometric mean G, root-square mean Q and the contra-harmonic mean C which are defined in usual way

$$H = \frac{2ab}{a+b}, \quad G = \sqrt{ab}, \quad Q = \sqrt{\frac{a^2+b^2}{2}}, \quad C = \frac{a^2+b^2}{a+b}. \tag{3}$$

One can easily verify that the means defined in (3) all can be expressed in terms of A and v. We have

$$\begin{aligned} H &= A(1-v^2), \quad G = A\sqrt{1-v^2}, \\ Q &= A\sqrt{1+v^2}, \quad C = A(1+v^2). \end{aligned} \tag{4}$$

All the means mentioned above are comparable. It is known that

$$H < G < L < A < M < Q < C \tag{5}$$

(see, e.g., [11]).

Following [10] we introduce a family of bivariate means which depend on the parameter p which satisfies $|p| \leqslant 1$. First we define two nonnegative numbers w_1 and w_2:

$$w_1 = \frac{1+p}{2}, \quad w_2 = \frac{1-p}{2}. \tag{6}$$

Clearly $w_1 + w_2 = 1$. We associate with the pair (a,b) a pair of positive numbers (x,y), where

$$x = w_1 a + w_2 b, \quad y = w_1 b + w_2 a. \tag{7}$$

Thus x and y are the convex combinations of a and b.

For the sake of presentation let N stand for a bivariate symmetric mean. We define a mean $N_p(a,b) \equiv N_p$ as follows

$$N_p(a,b) = N(x,y). \tag{8}$$

We will call the mean N_p the p-mean or the p-mean generated by N.

We will present now some elementary properties of the p-mans. Using (8), (6), and (7) we see that

$$N_{-p}(a,b) = N(y,x) = N(x,y) = N_p(a,b).$$

Thus the function $p \to N_p$ is an even function. To this end we will assume that $0 \leqslant p \leqslant 1$. It follows from (6) and (7) that

$$N_0 = A, \quad N_1 = N. \tag{9}$$

Moreover, the function $p \to N_p$ is strictly decreasing if $N < A$, i.e.,

$$N_1 \leqslant N_p \leqslant N_0 \tag{10}$$

or is strictly increasing if $N > A$, i.e.,

$$N_0 \leqslant N_p \leqslant N_1. \tag{11}$$

We now present formulas for the p-means mentioned in this section. Let us begin with the case when $N = A$. We have

$$A_p = A_p(a,b) = A(x,y) = A.$$

Thus we shall always write A instead of A_p when no confusion would arise. To obtain the p-versions of the means listed in (1) and (3) let us introduce a quantity u, where

$$u = \frac{x-y}{x+y}. \tag{12}$$

Using (12) and (2) we obtain

$$u = pv. \tag{13}$$

Since $0 < |v| < 1$, $0 < |u| < p \leqslant 1$

It is easy to verify that the formulas for the p-means derived from those listed in (3) read as follows

$$\begin{aligned} H_p &= A(1-u^2), \quad G_p = A\sqrt{1-u^2}, \\ Q_p &= A\sqrt{1+u^2}, \quad C_p = A(1+u^2). \end{aligned} \tag{14}$$

3. Main results

The goal of this section is to establish sharp bounds for means M and L means defined in (1). Bounding quantities are pairs of the p-means listed in (14). For similar results involving two Seiffert means, the interested reader is referred to [10], [3], and [4].

For the later use we define $\mu = \sinh^{-1}(1) = \ln(1+\sqrt{2})$.

In the first problem discussed here we deal with the question: For which numbers p and q, where $0 < p,q < 1$, the two-sided inequality

$$Q_p < M < Q_q \tag{15}$$

is satisfied? Answer to the last one is contained in the following

THEOREM 1. *In order for the inequalities (15) to be satisfied it is necessary and sufficient that*

$$p \leqslant \sqrt{\frac{1}{\mu^2}-1} = 0.536... \quad \textit{and} \quad q \geqslant \sqrt{\frac{1}{3}} = 0.577.... \tag{16}$$

Proof. Making use of (1), (14), and (13) we see that (15) is the same as

$$A\sqrt{1+p^2v^2} < A\frac{v}{\sinh^{-1}v} < A\sqrt{1+q^2v^2}.$$

Letting $v = \sinh t \ \ (0 \leqslant t \leqslant \mu)$ we can write the last two-sided inequality as

$$p^2 < \varphi(t) < q^2, \tag{17}$$

where

$$\varphi(t) = \frac{1}{t^2} - \frac{1}{\sinh^2 t}. \tag{18}$$

Differentiation yields

$$\varphi'(t) = 2\Big(\frac{\cosh t}{\sinh^3 t} - \frac{1}{t^3}\Big) = \frac{2}{\sinh^3 t}\Big(\cosh t - \Big(\frac{\sinh t}{t}\Big)^3\Big).$$

Application of Lazarević inequality (see, e.g., [6])

$$\cosh t < \Big(\frac{\sinh t}{t}\Big)^3 \tag{19}$$

$(t \neq 0)$ gives $\varphi'(t) < 0$, where the last inequality is valid for all $0 \leqslant t \leqslant \mu$. Thus the function $\varphi(t)$ is strictly decreasing on the stated domain. Moreover, $\lim_{t\to 0^+}\varphi(t) = 1/3$ and $\lim_{t\to\mu^-}\varphi(t) = 1/\mu^2 - 1$. This in conjunction with (17) implies that $p^2 \leqslant 1/\mu^2 - 1$ and $q^2 \geqslant 1/3$. Hence the asserted result (16) follows. □

Another pair of bounds for the Neuman-Sándor mean is obtained in the following

THEOREM 2. *The following inequalities*

$$C_p < M < C_q \tag{20}$$

are satisfied if and only if

$$p \leqslant \sqrt{\frac{1}{\mu}-1} = 0.366... \quad \textit{and} \quad q \geqslant \sqrt{\frac{1}{6}} = 0.408.... \tag{21}$$

Proof. We follow the initial lines in the proof of the previous theorem to write (20) in the equivalent form

$$p^2 < \frac{1}{v^2}\Big(\frac{v}{\sinh^{-1}v} - 1\Big) < q^2.$$

Again, we let $v = \sinh t \ \ (0 \leqslant t \leqslant \mu)$, to obtain

$$p^2 < \varphi(t) < q^2, \tag{22}$$

where now

$$\varphi(t) = \frac{1}{t \sinh t} - \frac{1}{\sinh^2 t}.$$

Differentiating function φ we obtain

$$\varphi'(t) = -\frac{1}{\sinh^3 t} f(t), \tag{23}$$

where

$$f(t) = -2\cosh t + \left(\frac{\sinh t}{t}\right)^2 + \frac{\sinh t}{t} \cdot \cosh t.$$

Use of Lazarević inequality (19) yields

$$\left(\frac{\sinh t}{t}\right)^2 > (\cosh t)^{2/3}$$

and

$$\frac{\sinh t}{t} \cosh t > (\cosh t)^{4/3}.$$

These inequalities are now utilized to obtain

$$f(t) > -2\cosh t + (\cosh t)^{2/3} + (\cosh t)^{4/3}.$$

Letting $(\cosh t)^{1/3} = c$ we can write the last inequality in the form $f(t) > -2c^3 + c^2 + c^4$. A factorization of the last expression yields

$$f(t) > [c(c-1)]^2 > 0.$$

This and (3.9) imply that $\varphi'(t) < 0$. Easy computations also give $\lim_{t \to 0^+} \varphi(t) = 1/6$ and $\lim_{t \to \mu^-} \varphi(t) = 1/\mu - 1$. The last two limits together with (22) give the desired result (23). □

We will establish now sharp lower and upper bounds for the logarithmic mean L. Those bounds involve either pairs of the p-harmonic means or the p-geometric means. We have the following

THEOREM 3. *The following two-sided inequality*

$$H_p < L < H_q \tag{24}$$

holds true if and only if

$$p = 1, \quad \textit{and} \quad q \leqslant \sqrt{\frac{1}{3}} = 0.577.... \tag{25}$$

Proof. It follows from (1), (14), and (13) that the two-sided inequality (24) is equivalent to

$$A(1-p^2v^2) < A\frac{v}{\tanh^{-1}v} < A(1-q^2v^2).$$

We rewrite the last inequalities as follows

$$q^2 < \frac{1}{v^2} - \frac{1}{v\tanh^{-1}v} < p^2.$$

Letting above $v = \tanh v$ $(t \geqslant 0)$ we obtain

$$q^2 < \varphi(t) < p^2, \tag{26}$$

where

$$\varphi(t) = \frac{1}{\tanh^2 t} - \frac{1}{t\tanh t}.$$

We shall demonstrate now that the function $\varphi(t)$ is strictly increasing on the nonnegative semi-axis. Differentiation gives

$$\varphi'(t) = \frac{1}{\sinh^3 t} f(t), \tag{27}$$

where

$$f(t) = \frac{\sinh t}{t} + \left(\frac{\sinh t}{t}\right)^2 \cosh t - 2\cosh t.$$

Two applications of Lazarević inequality (19) yield

$$f(t) > (\cosh t)^{1/3} + (\cosh t)^{5/3} - 2\cosh t.$$

With $c = (\cosh t)^{1/3}$ $(t \neq 0)$ the last inequality can be written as $f(t) > c(c^2-1)^2$. Since $c > 1$, $f(t) > 0$. This in conjunction with (27) gives $\varphi'(t) > 0$. Thus the function $\varphi(t)$ is strictly increasing on its domain. This, (26), and the fact that $\lim_{t\to 0^+}\varphi(t) = 1/3$ and $\lim_{t\to\infty}\varphi(t) = 1$ give the assertion (25). □

We close this section with the following

THEOREM 4. *In order for the simultaneous inequalities*

$$G_p < L < G_q \tag{28}$$

to be satisfied it is necessary and sufficient that

$$p = 1, \quad \textit{and} \quad q \leqslant \sqrt{\frac{2}{3}} = 0.816.... \tag{29}$$

Proof. We follow the lines used in the proofs of the previous theorems of this section. Making use of (1), (14), and (13) we see that the inequality (28) can be written as

$$A\sqrt{1-p^2v^2} < A\frac{v}{\tanh^{-1}v} < A\sqrt{1-q^2v^2}.$$

Substituting $v = \tanh t \ \ (t > 0)$ we obtain

$$q^2 < \varphi(t) < p^2, \tag{30}$$

where now

$$\varphi(t) = \frac{1}{\tanh^2 t} - \frac{1}{t^2}. \tag{31}$$

Differentiation yields

$$\varphi'(t) = \frac{2}{\sinh^3 t}\left(\left(\frac{\sinh t}{t}\right)^3 - \cosh t\right).$$

Again we use Lazarević inequality (19) to conclude that the function $\varphi(t)$ is strictly increasing on its domain. Easy computations yield $\lim_{t\to 0^+} \varphi(t) = 2/3$ and $\lim_{t\to\infty} \varphi(t) = 1$. This in conjunction with the monotonicity property of the function $\varphi(t)$ gives the desired result. The proof is complete. □

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