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ON CERTAIN INEQUALITIES FOR HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS

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Abstract. Starting from certain inequalities for hyperbolic and trigonometric functions, we obtain some general inequalities for functions and their inverses. As applications, refinements and new inequalities for hyperbolic and trigonometric functions are pointed out.

1. Introduction

During the last years there has been a great interest in trigonometric and hyperbolic inequalities. For papers on the famous Jordan's, Cusa-Huygens, Wilker's or Huygens', etc. inequalities we quote e.g. [4], [3], [2], [1], [5], [6], [7], [8], [9], [10] and the references therein.

In the recent interesting paper [1], the authors have considered some hyperbolic, trigonometric or hyperbolic-trigonometric inequalities. Among many inequalities, the following have been proved: For all $x \in (0,1)$ one has

$$\begin{aligned} &(i) \quad x/\arcsin x \leqslant \sin x/x;\\ &(ii) \quad x/\operatorname{arcsinh} x \leqslant \sinh x/x;\\ &(iii) \quad x/\arctan x \leqslant \tan x/x;\\ &(iv) \quad x/\operatorname{arctanh} x \leqslant \tanh x/x. \end{aligned} \tag{1.1}$$

The aim of this note is to offer extensions of these inequalities for arbitrary functions satisfying certain conditions. As particular cases, relations (1.1) will be reobtained, in more precise forms. Reverse type inequalities will be pointed out, too.

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2. Inequalities connecting functions with their inverses

In this section we will consider a general result on functions and their inverses.

THEOREM 2.1. *Let $f : I \to J$ be a bijective function, where I, J are nonempty subsets of $(0,+\infty)$. Suppose that the function $g(x) = \frac{f(x)}{x}$, $x \in I$ is strictly increasing. Then for any $x \in I$, $y \in J$ such that $f(x) \geqslant y$ one has*

$$f(x)f^{-1}(y) \geqslant xy, \tag{2.1}$$

where $f^{-1} : J \to I$ denotes the inverse function of f.
Under the same conditions, if $f(x) \leqslant y$ one has the reverse inequality

$$f(x)f^{-1}(y) \leqslant xy. \tag{2.2}$$

Proof. First remark that f must be strictly increasing, too. Indeed, if x_1, $x_2 \in I$ and $x_1 < x_2$, then $\frac{f(x_1)}{x_1} < \frac{f(x_2)}{x_2}$, so $f(x_1) < \frac{x_1}{x_2} f(x_2) < f(x_2)$. Thus f^{-1} is also strictly increasing. Put $t = f^{-1}(y)$. Since f^{-1} is strictly increasing, we can write $t \leqslant x$, so $\frac{f(t)}{t} \leqslant \frac{f(x)}{x}$ so $yx \leqslant f(x)f^{-1}(y)$, i.e. inequality (2.1) holds true.

When $y \geqslant f(x)$ we can write similarly $\frac{f(x)}{x} \leqslant \frac{f(t)}{t}$, where $t = f^{-1}(y) \geqslant x$, and (2.2) follows. □

Clearly one has equality in (2.1) or (2.2) only when $y = f(x)$.
The following result will be obtained with the aid of (2.1).

THEOREM 2.2.
(1) For any $x \in (0,1)$ and $y \in \left(0, \frac{\pi}{2}\right)$ such that $y < \arcsin x$ one has

$$\arcsin x \cdot \sin y > xy. \tag{2.3}$$

(2) For any $x > 0$, $y > 0$ such that $\sinh x > y$ one has

$$\sinh x \cdot \operatorname{arcsinh} y > xy. \tag{2.4}$$

(3) For any $x \in \left(0, \frac{\pi}{2}\right)$ and $y \in (0,\infty)$ such that $\tan x > y$ one has

$$\tan x \cdot \arctan y > xy. \tag{2.5}$$

(4) For any $x \in (0,1)$ and $y \in (0,\infty)$ such that $\operatorname{arctanh} x > y$ one has

$$\operatorname{arctanh} x \cdot \tanh y > xy. \tag{2.6}$$

Proof. (1) Let $I = (0,1)$, $J = \left(0, \frac{\pi}{2}\right)$ and $f(x) = \arcsin x$. Then

$$g(x) = \frac{\arcsin x}{x}$$

is strictly increasing, as

$$g'(x) = \frac{x - \sqrt{1-x^2}\arcsin x}{x^2\sqrt{1-x^2}} > 0$$

by $\arcsin x < \dfrac{x}{\sqrt{1-x^2}}$. Indeed, by letting $x = \sin p$, this becomes

$$p < \frac{\sin p}{\cos p} = \tan p,$$

which is well-known.

(2) Let $I = J = (0,\infty)$ and $f(x) = \sinh x$. Then

$$\left(\frac{\sinh x}{x}\right)' = \frac{\cosh x \cdot x - \sinh x}{x^2} > 0$$

by $\tanh x < x$, which is known.

(3) For $I = \left(0, \dfrac{\pi}{2}\right)$, $J = (0,\infty)$ and $f(x) = \tan x$ one has

$$\left(\frac{f(x)}{x}\right)' = \frac{x - \sin x \cos x}{x^2 \cos^2 x} > 0$$

as $x > \sin x$ and $1 > \cos x$.

(4) $I = (0,1)$, $J = (0,\infty)$, $f(x) = \operatorname{arctanh} x$. As

$$\left(\frac{\operatorname{arctanh} x}{x}\right)' = \left(\frac{x}{1-x^2} - \operatorname{arctanh} x\right)\Big/ x^2 > 0$$

by $\operatorname{arctanh} x > \dfrac{x}{1-x^2}$, which is equivalent, by letting $x = \tanh p$ by $p < \sinh p \cdot \cosh p$. This is true, as $\sinh p > 0$ and $\cosh p > 1$ for $p > 0$. $\square$

As a corollary, the above theorem gives:

COROLLARY 2.3.

$$\begin{aligned}
&(i) \quad \frac{x}{\arcsin x} < \frac{\sin x}{x} \qquad \text{for } x \in (0,1);\\
&(ii) \quad \frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x} \qquad \text{for } x > 0;\\
&(iii) \quad \frac{x}{\arctan x} < \frac{\tan x}{x} \qquad \text{for } x \in \left(0, \frac{\pi}{2}\right);\\
&(iv) \quad \frac{x}{\operatorname{arctanh} x} < \frac{\tanh x}{x} \qquad \text{for } x \in (0,1).
\end{aligned} \tag{2.7}$$

Proof. (i) by (2.3) from $\arcsin x > x$ for $x \in (0,1)$.

(ii) by (2.4) from $\sinh x > x$ for $x > 0$.
(iii) by (2.5) from $\tan x > x$ for $x \in \left(0, \frac{\pi}{2}\right)$.
(iv) by (2.6) from $\operatorname{arctanh} x > x$ for $x \in (0,1)$. □

REMARK. Clearly, the results hold true in symmetric intervals, too. E.g. (i) for $x \in (-1,1)$, etc.

These exact results may be compared with relations (1.1), proved in [1].

The following result will be an application of (2.2):

THEOREM 2.3.

$$(i) \quad \frac{x}{\arcsin x} > \frac{\sin\left(\frac{\pi}{2}x\right)}{\frac{\pi}{2}x}, \qquad \text{for } x \in (0,1);$$

$$(ii) \quad \frac{\sinh x}{x} < \frac{x}{a \cdot \operatorname{arcsinh}(x/a)}, \text{ for } x \in (0,k), \text{ where } k > 0 \text{ and } a = \frac{k}{\sinh k};$$

$$(iii) \quad \frac{\tan x}{x} < \frac{bx}{\arctan(bx)}, \qquad \text{for } x \in (0,k), \text{ where } 0 < k < \pi/2 \text{ and } b = \frac{\tan k}{k};$$

$$(iv) \quad \frac{x}{\operatorname{arctanh} x} > \frac{\tanh(cx)}{cx}, \qquad \text{for } x \in (0,k), \text{ where } k \in (0,1) \text{ and } c = \frac{k}{\operatorname{arctanh} k}.$$

Proof. (i) Since $\frac{\arcsin x}{x}$ is a strictly increasing function of x,

$$\frac{\arcsin x}{x} < \frac{\arcsin(\pi/2)}{\pi/2},$$

so $\frac{2}{\pi}\arcsin x = f(x) < x$. Now, f is bijective, having the inverse

$$f^{-1}(x) = \sin\left(\frac{\pi}{2} \cdot x\right).$$

Thus, relation (2.2) of Theorem 2.1, applied to $y = x$ implies (i).

(ii) Put $f(x) = a \cdot \sinh x < x$, and apply the same method.

(iii) Let $f(x) = \frac{1}{b} \cdot \tan x$ and use the monotonicity of $\frac{\tan x}{x}$.

(iv) Let $f(x) = c \cdot \operatorname{arctanh} x$ and use the monotonicity of Corollary 2.3 (iv). □

REMARKS.

1) When $k = 1$ and $a = \frac{1}{\sinh 1}$ in Theorem 2.3 (ii), we get

$$\frac{\sinh x}{x} < (\sinh 1) \cdot \frac{x}{\arcsin((\sinh 1)x)} \quad \text{for} \quad x \in (0,1).$$

2) When $k = \pi/4$ in Theorem 2.3 (iii) , we get

$$\frac{\tan x}{x} < \frac{\left(\frac{4}{\pi}x\right)}{\arctan\left(\frac{4}{\pi}x\right)} \quad \text{for} \quad x \in \left(0, \frac{\pi}{4}\right).$$

3) When $k = \frac{1}{2}$ in Theorem 2.3 (iv) , we get

$$\frac{x}{\operatorname{arctanh} x} > \frac{\tanh(x/\ln 3)}{x/\ln 3} \quad \text{for} \quad x \in \left(0, \frac{1}{2}\right).$$

4) The results hold in symmetric intervals, too.

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