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DIFFERENCE OF COMPOSITION OPERATORS ON HARDY SPACE

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Abstract. Suppose φ is an analytic self-map of open unit disk $\mathbb{D}$ and w is an analytic function on $\mathbb{D}$. Then a weighted composition operator induced by φ with weight w is given by $(W_{w,\varphi}f)(z) = w(z)f(\varphi(z))$, for $z \in \mathbb{D}$ and f analytic on $\mathbb{D}$. We find a sufficient condition under which two composition operators lie in the same path component of $\mathscr{C}(H^2)$, and we find a sufficient condition for the difference of such operators to be compact on $H^2(\mathbb{D})$. Then we provide another example that answers a question raised by Shapiro and Sundberg [18] negatively. Moreover, we characterize the Hilbert-Schmidt difference of two composition operators on $H^2(\mathbb{D})$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The space $H^\infty(\mathbb{D})$ is the set of bounded analytic functions on $\mathbb{D}$, with

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of functions f analytic on $\mathbb{D}$ that satisfy

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where σ is the normalized Lebesgue measure on the boundary of the unit disk. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of those functions f analytic on $\mathbb{D}$ that satisfy

$$\|f\|_{A^p_\alpha}^p = \int_{\mathbb{D}} |f(z)|^p d\lambda_\alpha(z) < \infty,$$

where $d\lambda_\alpha(z) = \frac{(1+\alpha)}{\pi}\left(1-|z|^2\right)^\alpha dA(z)$ is a weighted area measure. It is well-known that when $p = 2$, $H^2(\mathbb{D})$ and $A^2_\alpha(\mathbb{D})$ are Hilbert spaces. Observe that $H^p(\mathbb{D})$ is contained in $H^q(\mathbb{D})$ whenever $0 < q \leqslant p \leqslant \infty$, with $\|f\|_{H^q} \leqslant \|f\|_{H^p}$. If f belongs to any space $H^p(\mathbb{D})$, then the radial limit

$$f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$$

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exists for almost all ζ on $\partial\mathbb{D}$ ([17], p. 25). Moreover

$$\|f\|_{H^p}^p = \int_{\partial\mathbb{D}} |f^*(\zeta)|^p d\sigma(\zeta),$$

for all finite values of p ([16], Section 5.6). This gives us an alternative representation for the norm $\|f\|_{H^p}$. Because of this representation, we have to consider the radial limit function of $f\circ\varphi$, that is $(f\circ\varphi)^*$, where f belongs to $H^p(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. In particular, we have to relate $(f\circ\varphi)^*$ to the composition of radial limit functions $f^*\circ\varphi^*$. From Proposition 2.25 in [6], we know that these two functions agree almost everywhere on the boundary of the unit disk. More interestingly, it is true whenever the operator C_φ takes $H^p(\mathbb{B}_n)$ into itself for all $0<p<\infty$ ([11], Lemma 1.6).

Suppose φ is an analytic function mapping $\mathbb{D}$ into itself and w is an analytic function on $\mathbb{D}$, the weighted composition operator $W_{w,\varphi}$ is defined on the space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ by

$$(W_{w,\varphi}f)(z) = w(z)C_\varphi f(z) = w(z)f(\varphi(z)),$$

for all $f\in H(\mathbb{D})$ and $z\in\mathbb{D}$. The composition operator C_φ is a weighted composition operator with the weight function w identically equal to 1. It is clear that C_φ preserves $H(\mathbb{D})$. Littlewood [10] proved that C_φ also preserves $H^2(\mathbb{D})$. In addition for $1\leqslant p<\infty$, the Littlewood subordination theorem shows that an analytic $\varphi:\mathbb{D}\to\mathbb{D}$ induces a bounded operator $C_\varphi: H^p(\mathbb{D})\to H^p(\mathbb{D})$ ([6], Corollary 3.7). A necessary and sufficient condition for the boundedness of the composition operator C_φ on $H^p(\mathbb{B}_n)$ is known ([11], Theorem 1.1). As a corollary, MacCluer [11] proved that if C_φ is bounded on $H^p(\mathbb{B}_n)$ for some finite value of p then it is bounded for all p, $0<p<\infty$ ([11], Corollary 1.2).

In this paper, we will consider the problem of finding conditions on φ and ψ such that $C_\varphi - C_\psi$ is either compact or Hilbert-Schmidt. The first result on this problem was proved by Berkson [2]. Berkson proved that if $\varphi:\mathbb{D}\to\mathbb{D}$ is an analytic self-map whose radial limit function satisfies $|\varphi^*(\zeta)|=1$ for $\zeta\in E\subset\partial\mathbb{D}$, then for any analytic self-map of the unit disk $\psi\neq\varphi$,

$$\|C_\varphi - C_\psi\|_{H^2} \geqslant \sqrt{\frac{\sigma(E)}{2}},$$

where σ denotes the normalized Lebesgue measure on the unit circle. In other words, if the extreme set E of φ has a positive measure then C_φ is isolated in $\mathscr{C}(H^2)$, the collection of all composition operators on H^2 endowed with the metric induced by the operator norm. Berkson's isolation theorem raised the problem of describing the connected component in $\mathscr{C}(H^2)$ containing a given C_φ. Shapiro and Sundberg [18] improved the lower bound of Berkson's estimate and showed that

$$\|C_\varphi - C_\psi\|_{H^2} \geqslant \sqrt{\frac{\sigma(E)+\sigma(F)}{2}},$$

where E denotes the extreme set of φ and F the extreme set of ψ. Their work revealed a connection between the isolation problem and the problem of when the difference of two composition operators is compact. Moreover, Shapiro and Sundberg conjectured that if two composition operators belong to the same component of $\mathscr{C}(H^2)$, then their difference is compact. At about the same time MacCluer [12] proved that if the composition operators C_φ and C_ψ belong to the same component of $\mathscr{C}(D_\alpha)$ then φ and ψ have the same first-order boundary data, where for $\alpha > 0$ the weighted Dirichlet space D_α is the set of analytic functions on the unit disk $\mathbb{D}$ with derivatives in A^2_α. Also, MacCluer gave a necessary condition for two composition operators on D_α to have a compact difference. Bourdon [3] showed that MacCluer's condition is necessary and sufficient for C_φ and C_ψ to belong to the same component of $\mathscr{C}(H^2)$ when φ and ψ are linear-fractional transformations. Moreover for self-maps φ and ψ of the unit disk, Bourdon proved that φ and ψ must have the same second-order boundary data for $C_\varphi - C_\psi$ to be compact. Moorhouse and Toews [15] used Carleson measure techniques to give a sufficient condition for the difference of two composition operators in A^2_α to be compact. Using the pseudo-hyperbolic distance, Moorhouse [14] characterized the compact difference of two composition operators in A^2_α. Recently, using Aleksandro-Clark measures, Gallardo-Gutiérrez et al. [8] showed that there exist non-compact composition operators in the connected component of the compact ones on the Hardy space $H^2(\mathbb{D})$. Their main theorem states that "For $0 \leqslant t \leqslant 1$ there are analytic maps $\varphi_t : \mathbb{D} \to \mathbb{D}$ such that C_{φ_0} is compact and C_{φ_1} is non-compact on H^p and $t \mapsto C_{\varphi_t}$ is continuous from $[0,1]$ into $\mathscr{C}(H^p)$, where $1 \leqslant p < \infty$". Their result answered Shapiro and Sundberg question negatively.

In Section 2, by using Carleson-type measure techniques we find a sufficient condition for two composition operators C_φ and C_ψ to lie in the same path component of $\mathscr{C}(H^2)$ and we find a sufficient condition for $C_\varphi - C_\psi$ to be compact from $H^2(\mathbb{D})$ into $H^2(\mathbb{D})$. Moreover, we provide an example of two composition operators that lie in the same component and fail to have a compact difference. In particular, this example answers the question raised by Shapiro and Sundberg [18] negatively.

In Section 3, we show that the pseudo-hyperbolic distance is a good measure for characterizing the Hilbert-Schmidt difference of two composition operators on $H^2(\mathbb{D})$. Moreover, we use a boundary-data argument to find a necessary condition for the difference of two composition operators to be Hilbert-Schmidt.

2. Compact difference of composition operators

A linear operator on a Banach space is said to be compact if the image of the unit ball under the operator is relatively compact. Further information on compactness of (weighted) composition operators comes from Carleson measure techniques. For a point ζ in the boundary of the unit disk and $\delta > 0$ we define a Carleson set $S(\zeta,\delta) = \{z \in \mathbb{D} : |z-\zeta| < \delta\}$. Given a positive, finite Borel measure μ on the open unit disk $\mathbb{D}$, we say μ is an $\alpha-$Carleson measure if and only if

$$\|\mu\|_\alpha = \sup_{S(\zeta,\delta)} \frac{\mu\left(S(\zeta,\delta)\right)}{\delta^{\alpha+2}} < \infty,$$

where the supremum is taken over all $\zeta \in \partial\mathbb{D}$ and $\delta > 0$. If in addition,

$$\lim_{\delta\to 0}\sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\zeta,\delta))}{\delta^{\alpha+2}} = 0,$$

then we say μ is a compact $\alpha-$Carleson measure. This notion has many applications in the study of composition operators. This was first observed by Carleson for the Hardy space and was later extended to a variety of spaces by several authors. For references and historical development consult Section 2.2 in [6] or Section 8.2 in [20].

To apply Carleson measure characterizations to the (weighted) composition operators we define the following Borel measure. Let φ be an analytic self-map of $\mathbb{D}$ and w be a bounded analytic function on $\mathbb{D}$. We define a positive Borel measure $|w|^2\sigma\varphi^{-1}$ on the closed unit disk $\overline{\mathbb{D}}$ that assigns to each Borel set E the value:

$$|w|^2\sigma\varphi^{-1}(E) = \int_{\varphi^{*-1}(E)} |w^*(\zeta)|^2\, d\sigma(\zeta),$$

where φ^* and w^* denote the radial limit functions of φ and w. To make this paper self-contained we need the following two lemmas. The first lemma is a special case of (Proposition 2, [15]) and the second lemma is a special case of Theorem 1 and Corollary 1 in [15].

LEMMA 2.1. *Let φ be an analytic self-map of the unit disk $\mathbb{D}$ and suppose w is a bounded analytic function not identically zero on $\mathbb{D}$. Then:*

1. $W_{w,\varphi} : A^2_\alpha \to H^2$ *is bounded if and only if φ has radial limits of modulus strictly less than 1 almost everywhere and*

$$\|\,|w|^2\sigma\varphi^{-1}\|_{A^2_\alpha} = \sup_{S(\zeta,\delta)} \frac{|w|^2\sigma\varphi^{-1}(S(\zeta,\delta))}{\delta^{\alpha+2}} < \infty.$$

2. $W_{w,\varphi} : H^2 \to H^2$ *is bounded if and only if*

$$\|\,|w|^2\sigma\varphi^{-1}\|_{H^2} = \sup_{S(\zeta,\delta)} \frac{|w|^2\sigma\varphi^{-1}(S(\zeta,\delta))}{\delta} < \infty.$$

In all cases, the supremum is comparable to the norm of $W_{w,\varphi}$ acting on the appropriate spaces, and if the displayed quotient goes to 0 uniformly in ζ as $\delta \to 0$, then $W_{w,\varphi}$ is compact on these spaces.

LEMMA 2.2. *Let φ and ψ be analytic self-maps of the unit disk, and define $\varphi_s(z) = s\varphi(z) + (1-s)\psi(z)$ for $0 \leqslant s \leqslant 1$. Let $w(z)$ denote the bounded analytic function $\varphi(z) - \psi(z)$. If the weighted composition operators $W_{w,\varphi_s} : A^2_1 \to H^2$ are uniformly norm bounded in s, then C_{φ_s} is an arc of composition operators in $\mathscr{C}(H^2)$. Moreover, if the weighted composition operators W_{w,φ_s} are compact for each s, then $C_\varphi - C_\psi$ is compact from H^2 into H^2.*

Note that $\varphi_s(z)$ lies on a straight line path between $\varphi(z)$ and $\psi(z)$, thus

$$\min\{1-|\varphi(z)|, 1-|\psi(z)|\} \leqslant 1-|\varphi_s(z)|, \tag{1}$$

hence,

$$\frac{1}{1-|\varphi_s(z)|} \leqslant \frac{1}{1-|\varphi(z)|} + \frac{1}{1-|\psi(z)|}. \tag{2}$$

If φ is an analytic self-map of the unit disk $\mathbb{D}$, then we have the following estimate ([6], Corollary 3.7) for the composition operator C_φ on $H^p(\mathbb{D})$ when $p \geqslant 1$,

$$\frac{1}{1-|\varphi(0)|^2} \leqslant \|C_\varphi\|_{H^p}^p \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}. \tag{3}$$

In the following Theorem 2.3, we give a sufficient condition for the composition operators C_φ and C_ψ to be in the same path component of $\mathscr{C}(H^2)$.

THEOREM 2.3. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. If there exists a positive constant A such that*

$$\frac{|\varphi(\zeta)-\psi(\zeta)|^2}{(1-|\varphi(\zeta)|)(1-|\psi(\zeta)|)} < A$$

for almost all $\zeta \in \partial\mathbb{D}$, then C_φ and C_ψ lie in the same path component of $\mathscr{C}(H^2)$.

Proof. Assume that the hypothesis is satisfied, then

$$\begin{aligned}
& |\varphi-\psi|^2 < A(1-|\varphi|)(1-|\psi|) \\
\Rightarrow\quad & ||\varphi|-|\psi||^2 < |\varphi-\psi|^2 < A(1-|\varphi|)(1-|\psi|) \\
\Rightarrow\quad & \left|\frac{1-|\varphi|}{1-|\psi|}-1\right|^2 (1-|\psi|)^2 < |\varphi-\psi|^2 < A\left(\frac{1-|\varphi|}{1-|\psi|}\right)(1-|\psi|)^2.
\end{aligned}$$

Now, we claim that $\dfrac{1-|\varphi|}{1-|\psi|} \leqslant A+2$. To prove this, note from the previous inequality we have

$$\left|\frac{1-|\varphi|}{1-|\psi|}-1\right|^2 < A\left(\frac{1-|\varphi|}{1-|\psi|}\right).$$

Let $x = \dfrac{1-|\varphi|}{1-|\psi|}$, then we get:

$$\begin{aligned}
& |x-1|^2 \leqslant Ax \\
\Rightarrow\quad & (|x|-1)^2 \leqslant |x-1|^2 \leqslant Ax \\
\Rightarrow\quad & |x|^2 \leqslant (A+2)|x|-1 \\
\Rightarrow\quad & |x| \leqslant A+2,
\end{aligned}$$

which proves the claim. By a similar argument, we get

$$\frac{1-|\psi|}{1-|\varphi|} \leqslant A+2.$$

Now, since

$$|\varphi-\psi|^2 < A\left(\frac{1-|\varphi|}{1-|\psi|}\right)(1-|\psi|)^2,$$

and

$$|\varphi-\psi|^2 < A\left(\frac{1-|\psi|}{1-|\varphi|}\right)(1-|\varphi|)^2,$$

we get, by using Equation 1, for almost all $\zeta \in \partial\mathbb{D}$

$$\begin{aligned}|\varphi(\zeta)-\psi(\zeta)|^2 &< A\,(A+2)\min\{(1-|\varphi(\zeta)|)^2,(1-|\psi(\zeta)|)^2\}\\ &< A\,(A+2)\,(1-|\varphi_s(\zeta)|)^2.\end{aligned}$$

Now for $s \in [0,1]$ we have,

$$\begin{aligned}\|\,|\varphi-\psi|^2\,\sigma\varphi_s^{-1}\|_{A_1^2} &= \sup_{S(\zeta,\delta)} \frac{|\varphi-\psi|^2\,\sigma\varphi_s^{-1}(S(\zeta,\delta))}{\delta^3}\\ &= \sup_{S(\zeta,\delta)}\left(\frac{1}{\delta^3}\int_{\varphi_s^{*-1}(S(\zeta,\delta))}|\varphi-\psi|^2\,d\sigma\right)\\ &\leqslant \frac{A(A+2)}{\delta^3}\int_{\varphi_s^{*-1}(S(\zeta,\delta))}(1-|\varphi_s|)^2\,d\sigma\\ &< \frac{A(A+2)}{\delta^3}\delta^2\int_{\varphi_s^{*-1}(S(\zeta,\delta))}d\sigma\\ &= A(A+2)\frac{\sigma\varphi_s^{-1}(S(\zeta,\delta))}{\delta}.\end{aligned} \tag{4}$$

The last inequality can be seen as, if $z \in \varphi_s^{*-1}(S(\zeta,\delta))$ then $\varphi_s(z) \in S(\zeta,\delta)$, that is $|\varphi_s-\zeta| < \delta$. Thus, $1-|\varphi_s| < |\varphi_s-\zeta| < \delta$. Now for any $f \in H^2$, by using the change of variables formula ([9], p. 163) we have

$$\begin{aligned}\|C_{\varphi_s}f\|_{H^2}^2 &= \int_{\partial\mathbb{D}}|(f\circ\varphi_s)^*|^2\,d\sigma\\ &= \int_{\partial\mathbb{D}}|f^*\circ\varphi_s^*|^2\,d\sigma\\ &= \int_{\overline{\mathbb{D}}}|f^*|^2\,d(\sigma\varphi_s^{-1})\\ &= \|f^*\|_{L^2(\sigma\varphi_s^{-1})}^2.\end{aligned}$$

Thus, we get

$$\int_{\overline{\mathbb{D}}}|f^*|^2\,d(\sigma\varphi_s^{-1}) \leqslant \|C_{\varphi_s}\|_{H^2}^2\,\|f\|_{H^2}^2.$$

Now let $f(z) = K_{(1-\delta)\zeta}(z)$, the reproducing kernel function, where $0 < \delta < 1$ and $\zeta \in \partial\mathbb{D}$. For $z \in S(\zeta,\delta)$ it is clear that $\left|K_{(1-\delta)\zeta}(z)\right|^2 \geqslant \frac{1}{4\delta^2}$ since

$$\begin{aligned}\left|K_{(1-\delta)\zeta}(z)\right| &= \frac{1}{\left|1-(1-\delta)\overline{\zeta}z\right|}\\ &\geqslant \frac{1}{|\zeta - z| + \delta|z|}\\ &\geqslant \frac{1}{2\delta},\end{aligned}$$

where the last inequality can be verified as, $|z-\zeta| < \delta$ whenever $z \in S(\zeta,\delta)$. Hence

$$\begin{aligned}\frac{\sigma\varphi_s^{-1}(S(\zeta,\delta))}{4\delta^2} &= \frac{1}{4\delta^2}\int_{S(\zeta,\delta)} d(\sigma\varphi_s^{-1})\\ &\leqslant \int_{S(\zeta,\delta)} \left|K_{(1-\delta)\zeta}\right|^2 d(\sigma\varphi_s^{-1})\\ &\leqslant \int_{\mathbb{D}} \left|K_{(1-\delta)\zeta}\right|^2 d(\sigma\varphi_s^{-1})\\ &\leqslant \|C_{\varphi_s}\|_{H^2}^2 \|K_{(1-\delta)\zeta}\|_{H^2}^2\\ &= \|C_{\varphi_s}\|_{H^2}^2 \frac{1}{2\delta - \delta^2}\\ &\leqslant \frac{\|C_{\varphi_s}\|_{H^2}^2}{\delta}.\end{aligned}$$

Therefore, from the above argument we get

$$\sigma\varphi_s^{-1}(S(\zeta,\delta)) \leqslant 4\delta\|C_{\varphi_s}\|_{H^2}^2. \tag{5}$$

Hence, from (4) and (5) we have

$$\begin{aligned}\|\,|\varphi-\psi|^2\,\sigma\varphi_s^{-1}\|_{A_1^2} &< A(A+2)\frac{\sigma\varphi_s^{-1}(S(\zeta,\delta))}{\delta}\\ &\leqslant \frac{A(A+2)}{\delta}4\delta\|C_\varphi\|_{H^2}^2\\ &= 4A(A+2)\|C_\varphi\|_{H^2}^2.\end{aligned} \tag{6}$$

But by using (1) and (3) we have

$$\begin{aligned}\|C_{\varphi_s}\|_{H^2}^2 &\leqslant \frac{1+|\varphi_s(0)|}{1-|\varphi_s(0)|}\\ &\leqslant \frac{2}{1-|\varphi_s(0)|}\\ &\leqslant \frac{2}{\min\{1-|\varphi(0)|,1-|\psi(0)|\}}.\end{aligned} \tag{7}$$

Hence, by (6) and (7) we get

$$\| |\varphi-\psi|^2 \sigma \varphi_s^{-1} \|_{A_1^2} < \frac{8A(A+2)}{\min\{1-|\varphi(0)|, 1-|\psi(0)|\}}.$$

Therefore by Lemma 2.1, we get $W_{w,\varphi_s} : A_1^2 \to H^2$ is uniformly norm bounded in s. Hence by Lemma 2.2, we get C_{φ_s} is an arc of composition operators in $\mathscr{C}(H^2)$, that is $s \mapsto C_{\varphi_s}$ forms a path between C_φ and C_ψ. Thus C_φ and C_ψ lie in the same path component of $\mathscr{C}(H^2)$. $\square$

By using argument similar to Theorem 2.3, in the following Theorem 2.4 we present a sufficient condition for the compactness of the difference of two composition operators.

THEOREM 2.4. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. If there are positive constants A and $0 < a < 2$ such that*

$$\frac{|\varphi(\zeta)-\psi(\zeta)|^a}{(1-|\varphi(\zeta)|)(1-|\psi(\zeta)|)} < A$$

for almost all $\zeta \in \partial\mathbb{D}$, then $C_\varphi - C_\psi$ is compact from H^2 into H^2.

Proof. First, we are going to prove that for $\zeta \in \partial\mathbb{D}$ and $0 < a < 2$:

$$|\varphi(\zeta)-\psi(\zeta)|^a < A(A+2)\,(1-|\varphi_s(\zeta)|)^2.$$

To do this there are two cases: If $1 \leqslant |\varphi(\zeta)-\psi(\zeta)| < 2$, then $|\varphi(\zeta)-\psi(\zeta)|^a \leqslant |\varphi(\zeta)-\psi(\zeta)|^2$. But from Theorem 2.3, we have

$$|\varphi(\zeta)-\psi(\zeta)|^2 < A(A+2)\,(1-|\varphi_s(\zeta)|)^2,$$

hence,

$$|\varphi(\zeta)-\psi(\zeta)|^a < A(A+2)\,(1-|\varphi_s(\zeta)|)^2.$$

If $0 < |\varphi(\zeta)-\psi(\zeta)| \leqslant 1$, then $|\varphi(\zeta)-\psi(\zeta)|^2 \leqslant |\varphi(\zeta)-\psi(\zeta)|^a$. By using the hypothesis we get

$$\begin{aligned} ||\varphi(\zeta)|-|\psi(\zeta)||^2 \leqslant |\varphi(\zeta)-\psi(\zeta)|^2 &\leqslant |\varphi(\zeta)-\psi(\zeta)|^a \\ &< A(1-|\varphi(\zeta)|)(1-|\psi(\zeta)|) \qquad (8) \end{aligned}$$

that is

$$||\varphi(\zeta)|-|\psi(\zeta)||^2 < A(1-|\varphi(\zeta)|)(1-|\psi(\zeta)|).$$

Now, by following the same argument as that in the proof of Theorem 2.3 we get

$$\frac{1-|\varphi|}{1-|\psi|} \leqslant A+2 \qquad \text{and} \qquad \frac{1-|\psi|}{1-|\varphi|} \leqslant A+2.$$

Now, from (8) we get

$$|\varphi-\psi|^a < A\left(\frac{1-|\varphi|}{1-|\psi|}\right)(1-|\psi|)^2,$$

and

$$|\varphi-\psi|^a < A\left(\frac{1-|\psi|}{1-|\varphi|}\right)(1-|\varphi|)^2,$$

By using Equation 1, we get

$$\begin{aligned}|\varphi-\psi|^a &< A(A+2)\min\{(1-|\varphi|)^2,(1-|\psi|)^2\}\\ &< A(A+2)(1-|\varphi_s|)^2.\end{aligned}$$

Hence, for all $0<a<2$ we have

$$|\varphi(\zeta)-\psi(\zeta)|^a < A(A+2)(1-|\varphi_s(\zeta)|)^2.$$

Now by similar argument in the proof of Theorem 2.3, we have for $s\in[0,1]$

$$\begin{aligned}\|\,|\varphi-\psi|^2\sigma\varphi_s^{-1}\|_{A_1^2} &= \sup_{S(\zeta,\delta)}\frac{|\varphi-\psi|^2\sigma\varphi_s^{-1}(S(\zeta,\delta))}{\delta^3}\\ &= \sup_{S(\zeta,\delta)}\left(\frac{1}{\delta^3}\int_{\varphi_s^{*-1}(S(\zeta,\delta))}|\varphi-\psi|^2\,d\sigma\right)\\ &\leqslant \frac{(A(A+2))^{2/a}}{\delta^3}\int_{\varphi_s^{*-1}(S(\zeta,\delta))}(1-|\varphi_s|)^{4/a}\,d\sigma\\ &< \frac{(A(A+2))^{2/a}}{\delta^3}\delta^{4/a}\int_{\varphi_s^{*-1}(S(\zeta,\delta))}d\sigma\\ &= (A(A+2))^{2/a}\delta^{(4/a)-3}\sigma\varphi_s^{-1}(S(\zeta,\delta)). \end{aligned} \tag{9}$$

Hence, from (5) and (9) we get

$$\begin{aligned}\|\,|\varphi-\psi|^2\sigma\varphi_s^{-1}\|_{A_1^2} &\leqslant (A(A+2))^{2/a}\delta^{(4/a)-3}\sigma\varphi_s^{-1}(S(\zeta,\delta))\\ &\leqslant 4(A(A+2))^{2/a}\delta^{(4/a)-3}\delta\|C_{\varphi_s}\|_{H^2}^2\\ &= 4(A(A+2))^{2/a}\delta^{(4/a)-2}\|C_{\varphi_s}\|_{H^2}^2.\end{aligned}$$

But we know that,

$$\begin{aligned}\|C_{\varphi_s}\|_{H^2}^2 &\leqslant \frac{1+|\varphi_s(0)|}{1-|\varphi_s(0)|}\\ &\leqslant \frac{2}{1-|\varphi_s(0)|}\\ &\leqslant \frac{2}{\min\{1-|\varphi(0)|,1-|\psi(0)|\}}.\end{aligned}$$

Hence,

$$\| |\varphi - \psi|^2 \sigma \varphi_s^{-1} \|_{A_1^2} \leqslant \frac{8\,(A(A+2))^{2/a}\ \ \delta^{(4/a)-2}}{\min\{1-|\varphi(0)|\,,1-|\psi(0)|\}}.$$

Since $\left(\frac{4}{a}-2\right) > 0$, by applying Lemma 2.1 with $w = \varphi - \psi$ we get $W_{w,\varphi_s} : A_1 \to H^2$ is uniformly bounded in s. When $\delta \to 0$, $\| |\varphi - \psi|^2 \sigma \varphi_s^{-1} \|_{A_1^2} \to 0$. Thus using Lemma 2.1 again, we get W_{w,φ_s} is compact. Hence by Lemma 2.2, $C_\varphi - C_\psi$ is compact from H^2 into itself, which completes the proof. $\square$

By combining Theorem 2.3 and Theorem 2.4 we get the next corollary.

COROLLARY 2.5. *Let φ and ψ be analytic self-maps of $\mathbb{D}$. If there are positive constants A and $0 < a \leqslant 2$ such that*

$$\frac{|\varphi(\zeta) - \psi(\zeta)|^a}{(1-|\varphi(\zeta)|)(1-|\psi(\zeta)|)} < A$$

for almost all $\zeta \in \partial\mathbb{D}$, then

1. *C_φ and C_ψ lie in the same path component of $\mathscr{C}(H^2)$, when $a = 2$.*
2. *$C_\varphi - C_\psi$ is compact from H^2 into H^2, when $0 < a < 2$.*

The analytic maps φ and ψ in the next example are from Cowen and MacCluer [6] (p. 337 and Exercise 9.3.3) and they also appeared in ([15], p. 211). By using Corollary 2.5 we show in this example that the composition operators C_φ and C_ψ lie in the same component $\mathscr{C}(H^2)$ yet their difference is not compact. This example answers Shapiro and Sundberg's conjecture negatively, when $b = 2$.

EXAMPLE 2.6. Let $\varphi(z) = \dfrac{z+1}{2}$ and $\psi(z) = \varphi(z) + t\,(z-1)^b$. If $t > 0$ is sufficiently small, then

1. C_φ and C_ψ lie in the same path component of $\mathscr{C}(H^2)$, when $b \geqslant 2$.
2. $C_\varphi - C_\psi$ is compact from H^2 into H^2, when $b > 2$.
3. $C_\varphi - C_\psi$ is not compact, when $b = 2$.

Proof. We know that for every z in the closed disk, $\varphi(z)$ lies in the internally tangent disk $\{z : |1-z|^2 < 1-|z|^2\}$ with center at $1/2$ and radius $1/2$. Therefore for $z \in \overline{\mathbb{D}}$ we have

$$|1-\varphi(z)|^2 < 1-|\varphi(z)|^2.$$

Thus,

$$\begin{aligned} 1-|\varphi(z)| &= \frac{1-|\varphi(z)|^2}{1+|\varphi(z)|} \\ &\geqslant \frac{1-|\varphi(z)|^2}{2} \\ &\geqslant \frac{|1-\varphi(z)|^2}{2} \end{aligned}$$

$$= \frac{|z-1|^2}{8}. \tag{10}$$

While for $b \geqslant 2$ and t sufficiently small, there is a positive constant c_1 such that

$$\begin{aligned} 1 - |\psi(z)| &\geqslant 1 - |\varphi(z)| - |t|\,|z-1|^b \\ &\geqslant \frac{|z-1|^2}{8} - \frac{2^b}{4}\,|t|\,|z-1|^2 \\ &\geqslant c_1\,|z-1|^2. \end{aligned} \tag{11}$$

From next-to-last inequality in (11) it follows that $|\psi(z)| < 1$ for $z \in \mathbb{D}$ and t sufficiently small.

Using the above inequalities (10) and (11) we see that for $\zeta \in \partial\mathbb{D}$, there is a constant c_2 such that

$$\begin{aligned} \frac{|\varphi(\zeta) - \psi(\zeta)|^a}{(1 - |\varphi(\zeta)|)(1 - |\psi(\zeta)|)} &\leqslant c_2 \frac{|t|^a\,|\zeta - 1|^{ab}}{|\zeta - 1|^4} \\ &= c_2\,|t|^a\,|\zeta - 1|^{ab-4}, \end{aligned}$$

which is bounded over $\partial\mathbb{D}$ whenever $ab - 4 \geqslant 0$. When $a = 2$ we can find $b \geqslant 2$ such that $ab - 4 \geqslant 0$. Similarly when $0 < a < 2$ we can find $b > 2$ such that $ab - 4 \geqslant 0$. In both cases we get the conclusions (1) and (2) by applying Corollary 2.5.

The conclusion (3) can be found in ([6], Exercise 9.3.3). $\square$

3. Hilbert-Schmidt difference of composition operators

An operator $T \in B(H,K)$, Banach space of bounded operators acting between Hilbert spaces H and K, is said to be Hilbert-Schmidt if for any orthonormal basis $\{e_n\}$ of H the sum $\sum_{n=0}^{\infty} \|Te_n\|_K^2$ is finite. It is well-known that any Hilbert-Schmidt operator is compact, but there are compact operators that are not Hilbert-Schmidt. Moreover, there are compact composition operators not in any Schatten classes (see for example, [5]). In particular, a bounded operator T on $H^2(\mathbb{D})$ is a Hilbert-Schmidt operator if and only if

$$\|T\|_{HS}^2 = \sum_{n=0}^{\infty} \|T(z^n)\|_{H^2}^2 < \infty.$$

Now, using the orthonormal basis $\{z^n\}$ for $H^2(\mathbb{D})$ we get: C_φ is Hilbert-Schmidt on $H^2(\mathbb{D})$ if and only if $\|C_\varphi\|_{HS}^2 = \int_{\partial\mathbb{D}} \frac{1}{1-|\varphi|^2} d\sigma < \infty$ (see, Theorem 3.1 in [19]).

The pseudo-hyperbolic distance in the unit disk $\mathbb{D}$ is defined as

$$\rho(z,w) = |\varphi_w(z)| \qquad \text{where} \qquad \varphi_w(z) = \frac{w-z}{1-\overline{w}z},$$

for $z, w \in \mathbb{D}$. This represents a metric. Moreover the triangle inequality takes a stronger form

$$\rho(z,w) \leqslant \frac{\rho(z,\alpha) + \rho(\alpha,w)}{1 + \rho(z,\alpha)\rho(\alpha,w)}.$$

From Schwarz-Pick theorem ([6], Theorem 2.39), this metric is Möbius-invariant in the sense that

$$\rho(\varphi_\beta(z), \varphi_\beta(w)) = \rho(z, w),$$

for any $\beta \in \mathbb{D}$. For further details on pseudo-hyperbolic distance see [7].

Moorhouse [14] has shown that the pseudo-hyperbolic distance is a good measure for characterizing compact difference of two composition operators on the Bergman space $A^2_\alpha(\mathbb{D})$ (see, Theorem 4 in [14]). Thus, we proceed with the intuition that the pseudo-hyperbolic distance is also a good measure for characterizing Hilbert-Schmidt difference of two composition operators on the Hardy space $H^2(\mathbb{D})$. For the rest of this section, we will denote $\rho(z) = \left|\dfrac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}\right|$, which gives the pseudo-hyperbolic distance between the image values $\varphi(z)$ and $\psi(z)$.

THEOREM 3.1. *Let φ and ψ be analytic self-maps of $\mathbb{D}$ such that $|\varphi|$ and $|\psi|$ are almost everywhere less than 1 on $\partial\mathbb{D}$. If*

$$\frac{\rho(z)}{(1-|\varphi|)(1-|\psi|)} \in L^1(\partial\mathbb{D}, \sigma),$$

where σ is the normalized Lebesgue measure on $\partial\mathbb{D}$, then $C_\varphi - C_\psi$ is Hilbert-Schmidt.

Proof. Using the definition, we have

$$\begin{aligned}
\|C_\varphi - C_\psi\|^2_{HS} &= \sum_{n=0}^{\infty} \|\left(C_\varphi - C_\psi\right)(z^n)\|^2_{H^2} \\
&= \sum_{n=0}^{\infty} \|\varphi^n - \psi^n\|^2_{H^2} \\
&= \sum_{n=0}^{\infty} \int_{\partial\mathbb{D}} |\varphi^n - \psi^n|^2 \, d\sigma \\
&= \int_{\partial\mathbb{D}} \sum_{n=0}^{\infty} |\varphi^n - \psi^n|^2 \, d\sigma \\
&= \int_{\partial\mathbb{D}} \sum_{n=0}^{\infty} \left(|\varphi^n|^2 + |\psi^n|^2 - (\overline{\varphi}\psi)^n - (\overline{\psi}\varphi)^n\right) d\sigma \\
&= \int_{\partial\mathbb{D}} \left(\frac{1}{1-|\varphi|^2} - \frac{1}{1-\overline{\varphi}\psi} + \frac{1}{1-|\psi|^2} - \frac{1}{1-\overline{\psi}\varphi}\right) d\sigma \\
&= \int_{\partial\mathbb{D}} \left(\frac{\overline{\varphi}(\varphi-\psi)}{\left(1-|\varphi|^2\right)(1-\overline{\varphi}\psi)} + \frac{\overline{\psi}(\psi-\varphi)}{\left(1-|\psi|^2\right)(1-\varphi\overline{\psi})}\right) d\sigma
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\partial\mathbb{D}} \left(\frac{|\overline{\varphi}|\,|\varphi-\psi|}{(1-|\varphi|)\,(1+|\varphi|)\,|1-\overline{\varphi}\psi|} + \frac{|\overline{\psi}|\,|\psi-\varphi|}{(1-|\psi|)\,(1+|\psi|)\,|1-\varphi\overline{\psi}|} \right) d\sigma \\
&\leqslant \int_{\partial\mathbb{D}} \frac{|\varphi-\psi|}{|1-\overline{\varphi}\psi|} \left(\frac{|\varphi|}{1-|\varphi|} + \frac{|\psi|}{1-|\psi|} \right) d\sigma \\
&< \int_{\partial\mathbb{D}} \rho(z) \frac{|\varphi|+|\psi|}{(1-|\varphi|)\,(1-|\psi|)} d\sigma \\
&\leqslant 2 \int_{\partial\mathbb{D}} \frac{\rho(z)}{(1-|\varphi|)\,(1-|\psi|)} d\sigma < \infty.
\end{aligned}
$$

This completes the proof. □

In the next theorem, we give a useful sufficient condition for the Hilbert-Schmidt difference, which is similar to the compactness difference theorem ([18], Theorem 3.2).

THEOREM 3.2. *Let φ and ψ be analytic self-maps of $\mathbb{D}$ such that $|\varphi|$ and $|\psi|$ are almost everywhere less than 1 on $\partial\mathbb{D}$. If*

$$
\frac{|\varphi-\psi|}{(\min\{1-|\varphi|\,,1-|\psi|\})^2} \in L^1(\partial\mathbb{D},\sigma),
$$

then $C_\varphi - C_\psi$ is Hilbert-Schmidt.

Proof. From the proof of Theorem 3.1 we have the first inequality,

$$
\begin{aligned}
\|C_\varphi - C_\psi\|_{HS}^2 &\leqslant \int_{\partial\mathbb{D}} \left(\frac{|\overline{\varphi}|\,|\varphi-\psi|}{(1-|\varphi|)\,(1+|\varphi|)\,|1-\overline{\varphi}\psi|} + \frac{|\overline{\psi}|\,|\psi-\varphi|}{(1-|\psi|)\,(1+|\psi|)\,|1-\varphi\overline{\psi}|} \right) d\sigma \\
&\leqslant \int_{\partial\mathbb{D}} \frac{|\varphi-\psi|}{|1-\overline{\varphi}\psi|} \left(\frac{1}{1-|\varphi|} + \frac{1}{1-|\psi|} \right) d\sigma \\
&\leqslant \int_{\partial\mathbb{D}} \frac{|\varphi-\psi|}{|1-\overline{\varphi}\psi|} \; \frac{2}{\min\{1-|\varphi|\,,1-|\psi|\}} d\sigma \\
&\leqslant 2 \int_{\partial\mathbb{D}} \frac{|\varphi-\psi|}{(\min\{1-|\varphi|\,,1-|\psi|\})^2} d\sigma.
\end{aligned}
$$

The last inequality can be verified as $1-|\psi| \leqslant 1-|\overline{\varphi}|\,|\psi| \leqslant |1-\overline{\varphi}\psi|$, and $1-|\varphi| \leqslant 1-|\overline{\varphi}|\,|\psi| \leqslant |1-\overline{\varphi}\psi|$.
Then, $\min\{1-|\varphi|\,,1-|\psi|\} \leqslant |1-\overline{\varphi}\psi|$. This completes the proof. □

In the following, we characterize the Hilbert-Schmidt difference of two composition operators in terms of the pseudo-hyperbolic distance between the image values $\varphi(z)$ and $\psi(z)$.

THEOREM 3.3. *Let φ and ψ be analytic self-maps of $\mathbb{D}$ such that $|\varphi|$ and $|\psi|$ are almost everywhere less than 1 on $\partial\mathbb{D}$. $C_\varphi - C_\psi$ is Hilbert-Schmidt if and only if*

$$
\frac{\rho^2(z)}{1-\max\{|\varphi|^2,|\psi|^2\}} \in L^1(\partial\mathbb{D},\sigma).
$$

Proof. By using ([6], Lemma 9.12) we have,

$$\|C_\varphi - C_\psi\|_{HS}^2 = \sum_{n=0}^{\infty} \|(C_\varphi - C_\psi)(z^n)\|^2$$
$$= \int_{\partial\mathbb{D}} \frac{|\varphi(z)-\psi(z)|^2}{\left|1-\overline{\varphi(z)}\psi(z)\right|^2}\left(\frac{1}{1-|\varphi(z)|^2}+\frac{1}{1-|\psi(z)|^2}-1\right)d\sigma(z).$$

On the one hand,

$$\frac{1}{1-|\varphi|^2}+\frac{1}{1-|\psi|^2}-1 \leqslant \frac{1}{1-|\varphi|^2}+\frac{1}{1-|\psi|^2}$$
$$\leqslant \frac{2}{1-\max\{|\varphi|^2,|\psi|^2\}}.$$

On the other hand, if $\max\{|\varphi|,|\psi|\} \leqslant |\varphi|$ then

$$\frac{1}{1-|\varphi|^2}+\frac{1}{1-|\psi|^2}-1 = \frac{1-|\varphi|^2|\psi|^2}{\left(1-|\varphi|^2\right)\left(1-|\psi|^2\right)}$$
$$\geqslant \frac{1-|\psi|^2}{\left(1-|\varphi|^2\right)\left(1-|\psi|^2\right)}$$
$$= \frac{1}{1-|\varphi|^2}$$
$$\geqslant \frac{1}{1-\max\{|\varphi|^2,|\psi|^2\}}.$$

We arrive at the same conclusion if $\max\{|\varphi|,|\psi|\} \leqslant |\psi|$. Hence, we get

$$\int_{\partial\mathbb{D}} \frac{\rho^2}{1-\max\{|\varphi|^2,|\psi|^2\}}d\sigma \leqslant \|C_\varphi - C_\psi\|_{HS}^2 \leqslant \int_{\partial\mathbb{D}} \frac{2\rho^2}{1-\max\{|\varphi|^2,|\psi|^2\}}d\sigma,$$

which completes the proof. □

The following Example 3.4 shows the difference $C_\varphi - C_\psi$ is Hilbert-Schmidt if and only if $b \geqslant 5/2+\varepsilon$, where φ and ψ are given in Example 2.6. In Example 2.6, we showed that the difference is compact whenever $b > 2$.

EXAMPLE 3.4. Let $\varphi(z) = \dfrac{z+1}{2}$ and $\psi(z) = \varphi(z) + t\,(z-1)^b$. If $t > 0$ is sufficiently small, then $C_\varphi - C_\psi$ is Hilbert-Schmidt if and only if $b \geqslant 5/2+\varepsilon$.

Proof. From Example 2.6 we know that for $z \in \overline{\mathbb{D}}$

$$1-|\varphi(z)| \geqslant \frac{|z-1|^2}{8},$$

and for $z \in \mathbb{D}$ and $b \geqslant 2$ there is a constant $c_1 > 0$ such that

$$1 - |\psi(z)| \geqslant c_1 |z-1|^2 .$$

By an argument similar to Example 2.6, we get

$$|1 - \overline{\varphi\psi}| \geqslant c_2 |z-1|^2 .$$

Now,

$$\begin{aligned}\int_{\partial\mathbb{D}} \frac{|\varphi-\psi|^2}{|1-\overline{\varphi}\psi|^2} \frac{1}{1-|\psi|^2} d\sigma &\leqslant c_3 \int_{\partial\mathbb{D}} \frac{|t|^2 |\zeta-1|^{2b}}{|\zeta-1|^4} \frac{1}{|\zeta-1|^2} d\sigma(\zeta) \\ &= c_3 \int_{\partial\mathbb{D}} |t|^2 |\zeta-1|^{2b-6} d\sigma(\zeta),\end{aligned}$$

which is finite if and only if $b \geqslant 5/2 + \varepsilon$. Hence from Theorem 3.3 we get the desired result. □

We say the angular derivative of φ exists at a point $\zeta \in \partial\mathbb{D}$ if there exists $\eta \in \partial\mathbb{D}$ such that the difference quotient $\dfrac{\varphi(z)-\eta}{z-\zeta}$ has a (finite) limit as z tends non-tangentially to ζ in $\mathbb{D}$ and this limit is denoted $\varphi'(\zeta)$ (see, [13]).

The goal of the rest of this paper is to show that a boundary data argument up to the third-order derivative along with smoothness assumptions placed on φ and ψ is sufficient for $C_\varphi - C_\psi$ to be Hilbert-Schmidt. MacCluer ([12], Theorem 2.2) showed that the necessary condition for compactness of $C_\varphi - C_\psi$ is that φ and ψ have the same first-order boundary data. Bourdon ([3], Theorem 4.3) proved a necessary condition for the compact difference $C_\varphi - C_\psi$ is that φ and ψ have the same second-order boundary data, where extra smoothness assumptions are placed on φ and ψ. Moreover, Bourdon et.al. ([4], Theorem 7.5) proved in the presence of more smoothness, along with boundary-contact restriction Bourdon's condition became sufficient for the compact difference. We follow the terminology of [4].

DEFINITION 3.5. Let n be a positive integer, let $\zeta \in \partial\mathbb{D}$, and let $0 \leqslant \varepsilon < 1$. We say that the self-map φ of $\mathbb{D}$ belongs to $C^{n+\varepsilon}(\zeta)$ provided that φ is differentiable at ζ up to order n (viewed as a function with domain $\mathbb{D} \cup \zeta$) and, for $z \in \mathbb{D}$, has the expansion

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(\zeta)}{k!} (z-\zeta)^k + \gamma(z),$$

where $\gamma(z) = o\left(|z-\zeta|^{n+\varepsilon}\right)$ as $z \to \zeta$ from within $\mathbb{D}$.

DEFINITION 3.6. We say φ and ψ have the same third-order boundary data at $\zeta \in \partial\mathbb{D}$ provided that both functions belong to $C^3(\zeta)$ and

1. $\varphi(\zeta) = \psi(\zeta)$,

2. φ and ψ have the same (finite) angular derivative at ζ,

3. $\varphi''(\zeta) = \psi''(\zeta)$, and

4. $\varphi'''(\zeta) = \psi'''(\zeta)$.

For a self-map φ and ψ of $\mathbb{D}$ and for a point $\eta \in \partial\mathbb{D}$, let $\varphi^{-1}(\{\eta\}) = \{\zeta \in \partial\mathbb{D} :$ η belongs to the cluster set of φ *at* $\zeta\}$. Thus $\zeta \in \varphi^{-1}(\{\eta\})$ if and only if there is a sequence $\{z_n\}$ in $\mathbb{D}$ with limit ζ such that $\{\varphi(z_n)\}$ has limit η. By using argument similar to ([4], Theorem 7.5) we get the next Theorem 3.7, where we show that the sufficient condition for $C_\varphi - C_\psi$ to be Hilbert-Schmidt is that φ and ψ have the same third-order boundary data along with smoothness assumptions that are placed on φ and ψ.

THEOREM 3.7. *Suppose that φ and ψ are analytic self-maps of $\mathbb{D}$ such that:*

1. *φ and ψ each takes $\mathbb{D}$ into a proper subdisk of $\mathbb{D}$ that is internally tangent to the unit circle at η;*

2. *$\varphi^{-1}(\{\eta\}) = \psi^{-1}(\{\eta\}) = \{\zeta\}$ where $\zeta \in \partial\mathbb{D}$;*

3. *φ and ψ each belongs to $C^{3+\varepsilon}(\zeta)$ for some $0 < \varepsilon < 1$; and*

4. *φ and ψ have the same third-order boundary data at ζ.*

Then $C_\varphi - C_\psi$ is Hilbert-Schmidt.

Proof. Let E be a set of points at which both φ and ψ have a radial limit. Then from the radial limit theorem ([17], p. 25), $E = \partial\mathbb{D}$ almost everywhere. By using Definition 3.5 and Definition 3.6 together with hypotheses (3) and (4), there exists a boundary analytic function γ on $\mathbb{D}$ such that $\gamma(z) = o\left(|z-\zeta|^{3+\varepsilon}\right)$ *as* $z \to \zeta$, and for every $z \in \mathbb{D} \cup E$ there is a constant c_1 such that

$$|\varphi(z) - \psi(z)| \leqslant c_1 |\zeta - z|^{3+\varepsilon}.$$

Since φ and ψ have the same (finite) angular derivative at ζ, by Julia-Carathéodory Theorem ([6], Theorem 2.44) together with hypothesis (2) we get

$$\begin{aligned}\varphi'(\zeta) &= d(\zeta)\overline{\zeta}\varphi(\zeta)\\ &= d(\zeta)\overline{\zeta}\psi(\zeta)\\ &= \psi'(\zeta),\end{aligned}$$

that is, $\varphi'(\zeta) = \psi'(\zeta)$ is non-zero. This together with hypothesis (2), there is a constant c_2 such that for every $z \in \mathbb{D} \cup E$

$$\frac{|\zeta - z|}{|\eta - \varphi(z)|} \leqslant c_2 \qquad \text{and} \qquad \frac{|\zeta - z|}{|\eta - \psi(z)|} \leqslant c_2.$$

Finally, by the hypothesis (1) we get $\varphi(z)$ and $\psi(z)$ lie in the internally tangent disk $\left\{z : |\eta - z|^2 < 1 - |z|^2\right\}$ for every $z \in \mathbb{D} \cup E$. Hence, there is a constant c_3 such that for every $z \in \mathbb{D} \cup E$,

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|} \leqslant c_3 \qquad \text{and} \qquad \frac{|\eta - \psi(z)|^2}{1 - |\psi(z)|} \leqslant c_3.$$

Now for $\lambda \in E$, if $|\varphi(\lambda)| \leqslant |\psi(\lambda)|$ then

$$\begin{aligned}
&\int_E \frac{|\varphi(\lambda) - \psi(\lambda)|}{(\min\{1 - |\varphi(\lambda)|, 1 - |\psi(\lambda)|\})^2} d\sigma(\lambda) \\
=& \int_E \frac{|\varphi(\lambda) - \psi(\lambda)|}{(1 - |\psi(\lambda)|)^2} d\sigma(\lambda) \\
\leqslant& \int_E c_1 \frac{|\zeta - \lambda|^{3+\varepsilon}}{(1 - |\psi(\lambda)|)^2} d\sigma(\lambda) \\
\leqslant& \int_E c_1 (c_2)^{3+\varepsilon} \frac{|\eta - \psi(\lambda)|^{3+\varepsilon}}{(1 - |\psi(\lambda)|)^2} d\sigma(\lambda) \\
\leqslant& c_1 (c_2)^{3+\varepsilon} (c_3)^2 \int_E \frac{1}{|\eta - \psi(\lambda)|^{1-\varepsilon}} d\sigma(\lambda) \\
\leqslant& c_1 (c_2)^4 (c_3)^2 \int_E \frac{1}{|\zeta - \lambda|^{1-\varepsilon}} d\sigma(\lambda) < \infty.
\end{aligned}$$

The same is true if $|\psi(\lambda)| \leqslant |\varphi(\lambda)|$. Since $E = \partial\mathbb{D}$ almost everywhere,

$$\frac{|\varphi - \psi|}{(\min\{1 - |\varphi|, 1 - |\psi|\})^2} \in L^1(\partial\mathbb{D}, \sigma).$$

Hence using Theorem 3.2 we complete the proof. □

LEMMA 3.8. *([19], Theorem 2.1) If φ has an angular derivative at a point $\zeta \in \partial\mathbb{D}$, then C_φ is not compact on $H^2(\mathbb{D})$.*

As an application of the above lemma we see that whenever the hypotheses of Theorem 3.7 are satisfied, C_φ and C_ψ are not Hilbert-Schmidt while their difference is Hilbert-Schmidt.

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