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BOUNDS FOR THE ZEROS OF A CLASS OF LACUNARY–TYPE POLYNOMIALS

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Abstract. In this paper, we present certain results concerning the location of the zeros of lacunary-tpye polynomials which generalize and refine some known Cauchy type bounds for the zeros of polynomials.

1. Introduction

The following classical result is due to Cauchy [1](see also [6, p. 123].

THEOREM A. *If*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

is a polynomial of degree n *and*

$$Q = \left\{ \max_{0 \leqslant j \leqslant n-1} \left| \frac{a_j}{a_n} \right| \right\}^{1/n},$$

then all the zeros of $P(z)$ *lie in circle*

$$|z| < 1 + Q^n.$$

In literature [6, 8, 9], there exist a variety of results giving bounds which are valid for all the zeros or for p of the zeros, $p \leqslant n$, of the polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

In either case the bounds were expressed as the functions of all the coefficients $a_0, a_1, \ldots,$ a_n of $P(z)$.

An important class of polynomials are those of the lacunary type

$$P(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + a_{n_3} z^{n_3} + \cdots + a_{n_k} z^{n_k},$$

$$0 < n_0 = p < n_1 < n_2 < \cdots < n_k, \quad a_0 a_p a_{n_1} a_{n_2} \cdots a_{n_k} \neq 0.$$

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Here the coefficients $a_j,\ 0 \leqslant j \leqslant p$, are fixed; $a_{n_j},\ j = 1,2,\cdots,k$, are arbitrary and remaining coefficients are zero. Landau [3, 4] initiated the study of polynomials of this form in 1906-7 and proved that every trinomial

$$a_0 + a_1 z + a_n z^n,\ \ a_1 a_n \neq 0,\ \ n \geqslant 2,$$

has at least one zero in the circle $|z| \leqslant 2|a_0/a_1|$ and every quadrinomial

$$a_0 + a_1 z + a_m z^m + a_n z^n,\ \ a_1 a_m a_n \neq 0,\ \ 2 \leqslant m < n,$$

has at least one zero in the circle $|z| \leqslant (17/3)|a_0/a_1|$.

About sixty years ago, Simeon Reich proposed and among others, O. P. Lossers [5] proved the following:

THEOREM B. *If*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-2} z^{n-2} + a_n z^n$$

is a polynomial of degree n with

$$Q = \left\{ \max_{0 \leqslant j \leqslant n-2} \left| \frac{a_j}{a_n} \right| \right\}^{1/n} \quad \text{and} \quad Q \geqslant 1,$$

then all the zeros of $P(z)$ lie in the circle

$$|z| \leqslant Q + Q^2 + \cdots + Q^{n-1}. \tag{1}$$

2. Main results

Here we first present the following generalization of Theorem B to lacunary type polynomials which among other things considerably improves the bound (1) for $r = n-2$ and further shows that the assertion (1) remains valid even if we do not assume that $Q > 1$.

THEOREM 1. *Let*

$$P(z) = a_0 + a_1 z + \cdots + a_r z^r + a_n z^n,\ \ a_r \neq 0,\ \ 0 \leqslant r \leqslant n-1,$$

be a polynomial of degree n. If

$$Q = \left\{ \max_{0 \leqslant j \leqslant r} \left| \frac{a_j}{a_n} \right| \right\}^{1/n},$$

then all the zeros of $P(z)$ lie in circle

$$|z| \leqslant \left\{ Q^n + Q^{n-1} + \cdots + Q^{n-r} \right\}^{1/n-r}. \tag{2}$$

COROLLARY 1. *All the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-2} z^{n-2} + a_n z^n$$

of degree n lie in circle

$$|z| \leqslant \left\{Q^n + Q^{n-1} + \cdots + Q^2\right\}^{1/2} \tag{3}$$

where

$$Q = \left\{\max_{0 \leqslant j \leqslant n-2} \left|\frac{a_j}{a_n}\right|\right\}^{1/n}.$$

REMARK 1. Since it can be easily verified with the help of mathematical induction that

$$(Q^2 + Q^3 + \cdots + Q^n)^{1/2} \leqslant Q + Q^2 + \cdots + Q^{n-1}$$

for $n \geqslant 2$, it follows that the bound (3) of Corollary 1 is sharper than the bound (1) of the Theorem B.

The following result was proved by Mohammad [7,Theorem 1].

THEOREM C. *All the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

of degree n lie in circle

$$|z| \leqslant \max\left\{L_p, L_p^{1/n}\right\}$$

where

$$L_p = n^{1/q} \left\{\sum_{j=0}^{n-1} \left|\frac{a_j}{a_n}\right|^p\right\}^{1/p},$$

$p > 1,\ q > 1$ *with* $p^{-1} + q^{-1} = 1$.

Here we next generalize this result to lacunary polynomials and prove the following:

THEOREM 2. *For any given positive number t, all the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_r z^r + a_n z^n,\ \ a_r \neq 0,\ \ 0 \leqslant r \leqslant n-1$$

of degree n lie in the circle

$$|z| \leqslant t \max\left\{L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n}\right\} \tag{4}$$

where

$$L_{p,t} = (r+1)^{1/q} \left\{\sum_{j=0}^{r} \left|a_j / a_n t^{n-j}\right|^p\right\}^{1/p},$$

$p > 1,\ q > 1$ *with* $p^{-1} + q^{-1} = 1$. *The bound is sharp.*

REMARK 2. The limit in Theorem 2 is attained by

$$P(z) = t^n + t^{n-1}z + \cdots + t^{n-r}z^r - (r+1)z^n \ , t > 0.$$

To see this, we have

$$\sum_{j=0}^{r} \left| \frac{a_j}{a_n t^{n-j}} \right|^p = \sum_{j=0}^{r} \left\{ \frac{t^{n-j}}{(r+1)t^{n-j}} \right\}^p = \sum_{j=0}^{r} \frac{1}{(r+1)^p} = (r+1)^{1-p},$$

which gives

$$L_{p,t} = (r+1)^{1/q} \left\{ \sum_{j=0}^{r} \left| a_j / a_n t^{n-j} \right|^p \right\}^{1/p} = 1,$$

so that

$$t \max \left\{ L_{p,t}^{1/(n-r)},\ L_{p,t}^{1/n} \right\} = t$$

and $z = t$ is a zero of $P(z)$.

REMARK 3. If we take $t = 1$ and $r = n-1$ in Theorem 2, we get Theorem C.

COROLLARY 2. *For any given positive number t, all the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$$

of degree n lie in the circle

$$|z| \leqslant t \max \left\{ N_{p,t},\ N_{p,t}^{1/n} \right\} \tag{5}$$

where

$$N_{p,t} = n^{1/q} \left\{ \sum_{j=0}^{n-1} \left| a_j / a_n t^{n-j} \right|^p \right\}^{1/p},$$

$p > 1$, $q > 1$ *with* $p^{-1} + q^{-1} = 1$. *The bound is sharp.*

Finally in this paper we present the following result.

THEOREM 3. *For every positive number t, all the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$$

of degree n lie in circle

$$|z| \leqslant (n+1)^{1/q} \left\{ \sum_{j=0}^{n} \left| \frac{t a_j - a_{j-1}}{a_n t^{n-j}} \right|^p \right\}^{1/p} \tag{6}$$

where $p > 1$, $q > 1$ *with* $p^{-1} + q^{-1} = 1$.

The following result is the limiting case of the Theorem 3 when $q \to \infty$ so that $p \to 1$.

COROLLARY 3. *For any given positive number t, all the zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

of degree n lie in the circle

$$|z| \leqslant \sum_{j=0}^{n} \left| \frac{t a_j - a_{j-1}}{t^{n-j} a_n} \right|.$$

REMARK 4. If a_j is real and $t a_j - a_{j-1} \geqslant 0,\ j = 1, 2, \cdots, n$, then it follows from Corollary 3 that for every $t > 0$, all the zeros of polynoimial

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

of degree n lie in the circle

$$|z| \leqslant \frac{t^n a_n - a_0 + |a_0|}{t^{n-1} |a_n|}.$$

For $t = 1$, we get a result due to Joyal, Labelle and Rahman [2]. Further, for $t = 1$ and $a_0 > 0$, it reduces to the Eneström-Kakeya Theorem (see [6, p. 136]).

3. Proofs of the main results

Proof of Theorem 1. We observe that

$$\left\{Q^n + Q^{n-1} + \cdots + Q^{n-r}\right\}^{1/n-r} \geqslant Q$$

for $0 \leqslant r \leqslant n-1$. Also by hypothesis, $|a_j/a_n| \leqslant Q^n$ for $j = 0, 1, \cdots, r.$ Therefore, we have

$$\begin{aligned} |P(z)| &= |a_n z^n + a_r z^r + \cdots a_1 z + a_0| \\ &\geqslant |a_n||z^n| \left\{ 1 - \sum_{j=0}^{r} \left| \frac{a_j}{a_n} \right| \frac{1}{|z|^{n-j}} \right\} \\ &\geqslant |a_n||z^n| \left\{ 1 - \sum_{j=0}^{r} \frac{Q^n}{|z|^{n-j}} \right\}. \end{aligned} \tag{7}$$

Let $|z| > Q$, then $(Q/|z|) < 1$ and therefore,

$$(Q/|z|)^{n-j} \leqslant (Q/|z|)^{n-r} \ \text{ for } \ j = 0, 1, \cdots, r.$$

This gives

$$\frac{1}{|z|^{n-j}} \leqslant \frac{1}{|z|^{n-r} Q^{r-j}} \quad \text{for } \ j = 0, 1, \cdots, r.$$

Hence if $|z| > Q$, we get from (7),

$$
\begin{aligned}
|P(z)| &\geqslant |a_n||z^n| \left\{ 1 - \sum_{j=0}^{r} \frac{Q^n}{|z|^{n-r} Q^{r-j}} \right\} \\
&= |a_n||z^n| \left\{ 1 - \frac{1}{|z|^{n-r}} \sum_{j=0}^{r} Q^{n-r+j} \right\}.
\end{aligned}
$$

Thus $|P(z)| > 0$ if $|z|^{n-r} > \sum_{j=0}^{r} Q^{n-r+j}$, that is, if

$$
|z| > \left\{ Q^n + Q^{n-1} + \cdots + Q^{n-r} \right\}^{1/n-r} (\geqslant Q).
$$

Hence all the zeros of $P(z)$ whose modulus is greater than Q lie in circle defined by (2). But those zeros of $P(z)$ whose modulus is less or equal to Q already satisfy (2), the Theorem 1 is proved. □

Proof of Theorem 2. We have

$$
\begin{aligned}
|P(z)| &= |a_n z^n + a_r z^r + \cdots a_1 z + a_0| \\
&\geqslant |a_n||z|^n \left\{ 1 - \sum_{j=0}^{r} \left| \frac{a_j}{a_n} \right| \frac{1}{|z|^{n-j}} \right\} \\
&= |a_n||z|^n \left\{ 1 - \sum_{j=0}^{r} \left(\left| \frac{a_j}{a_n} \right| \frac{1}{t^{n-j}} \right) \left(\frac{t}{|z|} \right)^{n-j} \right\},
\end{aligned}
$$

which implies with the help of Holder's inequality that

$$
\begin{aligned}
|P(z)| &\geqslant |a_n||z|^n \left[1 - \left\{ \sum_{j=0}^{r} \left(\left| \frac{a_j}{a_n} \right| \frac{1}{t^{n-j}} \right)^p \right\}^{1/p} \left\{ \sum_{j=0}^{r} \left(\frac{t}{|z|} \right)^{q(n-j)} \right\}^{1/q} \right] \\
&= |a_n||z|^n \left[1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{ \sum_{j=0}^{r} \left(\frac{t}{|z|} \right)^{q(n-j)} \right\}^{1/q} \right]. \qquad (8)
\end{aligned}
$$

Now if $L_{p,t} \geqslant 1$, then $\max\left\{ L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n} \right\} = L_{p,t}^{1/(n-r)}$. Let $|z| \geqslant t$, then $(t/|z|)^{n-j} \leqslant (t/|z|)^{n-r}$ for $j = 0, 1, \cdots, r$. Hence (8) implies that if $|z| > t L_{p,t}^{1/(n-r)}$, then

$$
\begin{aligned}
|P(z)| &\geqslant |a_n||z|^n \left[1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{ \sum_{j=0}^{r} \left(\frac{t}{|z|} \right)^{q(n-r)} \right\}^{1/q} \right] \\
&= |a_n||z|^n \left\{ 1 - L_{p,t} \left(\frac{t}{|z|} \right)^{n-r} \right\} > 0. \qquad (9)
\end{aligned}
$$

Again if $L_{p,t} \leqslant 1$, then $\max\left\{L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n}\right\} = L_{p,t}^{1/n}$. Let $|z| \leqslant t$, then $(t/|z|)^{n-j} \leqslant (t/|z|)^n$ for $j = 0, 1, \cdots, r$. Hence from (8), we infer that if $|z| > tL_{p,t}^{1/n}$, then

$$\begin{aligned} |P(z)| &\geqslant |a_n||z|^n \left[1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{\sum_{j=0}^{r} \left(\frac{t}{|z|}\right)^{nq}\right\}^{1/q}\right] \\ &= |a_n||z|^n \left\{1 - L_{p,t}\left(\frac{t}{|z|}\right)^n\right\} > 0. \end{aligned} \tag{10}$$

From (9) and (10), it follows that $P(z)$ does not vanish for

$$|z| > \max\left\{tL_{p,t}^{1/(n-r)}, tL_{p,t}^{1/n}\right\}.$$

Cosequently all the zeros of $P(z)$ lie in region defined by (4). This completes the proof of Theorem 2. □

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned} F(z) &= (t-z)P(z) \\ &= ta_0 + (ta_1 - a_0)z + (ta_2 - a_1)z^2 + \cdots + (ta_n - a_{n-1})z^n - a_n z^{n+1} \end{aligned}$$

Applying Corollary 2 to the polynomial $F(z)$, it follows that every $t > 0$, all the zeros of $F(z)$ lie in

$$|z| \leqslant t \max\left\{N_{p,t}, N_{p,t}^{1/n}\right\}$$

where

$$N_{p,t} = (n+1)^{1/q} \left\{\sum_{j=0}^{n} \left|\frac{ta_j - a_{j-1}}{a_n t^{n-j+1}}\right|^p\right\}^{1/p}.$$

But

$$\begin{aligned} 1 &= \frac{\left|(t^{n+1}a_n - t^n a_{n-1}) + (t^n a_{n-1} - t^{n-1}a_{n-2}) + \cdots + (t^2 a_1 - ta_0) + ta_0\right|}{t^{n+1}|a_n|} \\ &\leqslant \sum_{j=0}^{n} \frac{|t^{j+1}a_j - t^j a_{j-1}|}{t^{n+1}|a_n|} = \sum_{j=0}^{n} \left|\frac{ta_j - a_{j-1}}{t^{n-j+1}a_n}\right| \\ &\leqslant (n+1)^{1/q} \left\{\sum_{j=0}^{n} \left|\frac{ta_j - a_{j-1}}{t^{n-j+1}a_n}\right|^p\right\}^{1/p} = N_{p,t}, \end{aligned}$$

by Holder's inequality. Therefore, we conclude that the zeros of $F(z)$ and hence that of $P(z)$ lie in the circle

$$|z| \leqslant (n+1)^{1/q} \left\{\sum_{j=0}^{n} \left|\frac{ta_j - a_{j-1}}{t^{n-j}a_n}\right|^p\right\}^{1/p}.$$

This completes the proof of Theorem 3. □

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REFERENCES

[1] A. L. CAUCHY, *Exercises de mathématiques*, IV Annee de Bure Freses, 1829.
[2] A. JOYAL, G. LABELLE AND Q. I. RAHMAN, *On the location of the zeros of polynomials*, Bull. Canad. Math., **10** (1967), 53–63.
[3] E. LANDAU, *Ueber den Picardschen Satz*, Vierteljahrsschrift Naturforsch. Gesellschaft Zürich, **51** (1906), 252–318.
[4] E. LANDAU, *Sur quelques généralisations du théoréme de M. Picard*, Ann. Ecole Norm. **24**, 3 (1907), 179–201.
[5] O. P. LOSSERS, *Advanced problem 5737*, Amer. Math. Monthly, **78** (1971), 681–683.
[6] M. MARDEN, *Geometry of Polynomials*, Math. Surveys No. 3, Amer. Math. Soc., providence R.I. (1967).
[7] Q. G. MOHAMMAD, *Location of the zeros of polynomials*, Amer. Math. Monthly, **74** (1967), 290–292.
[8] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ AND TH. M. RASSIAS, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World scientific Publishing Co., Singapore,(1994).
[9] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Oxford University Press Inc., New York (2002).

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