BOUNDS FOR THE ZEROS OF A CLASS OF LACUNARY–TYPE POLYNOMIALS

A. AZIZ AND N. A. RATHER

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Abstract. In this paper, we present certain results concerning the location of the zeros of lacunary-type polynomials which generalize and refine some known Cauchy type bounds for the zeros of polynomials.

1. Introduction

The following classical result is due to Cauchy [1](see also [6, p. 123].

THEOREM A. If

\[ P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \]

is a polynomial of degree \( n \) and

\[ Q = \left\{ \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right| \right\}^{1/n}, \]

then all the zeros of \( P(z) \) lie in circle

\[ |z| < 1 + Q^n. \]

In literature [6, 8, 9], there exist a variety of results giving bounds which are valid for all the zeros or for \( p \) of the zeros, \( p \leq n \), of the polynomial

\[ P(z) = a_0 + a_1 z + \cdots + a_n z^n. \]

In either case the bounds were expressed as the functions of all the coefficients \( a_0, a_1, \ldots, a_n \) of \( P(z) \).

An important class of polynomials are those of the lacunary type

\[ P(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + a_{n_3} z^{n_3} + \cdots + a_{n_k} z^{n_k}, \]

\[ 0 < n_0 = p < n_1 < n_2 < \cdots < n_k, \quad a_0 a_1 a_2 \cdots a_n \neq 0. \]


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Here the coefficients $a_j$, $0 \leq j \leq p$, are fixed; $a_n$, $j = 1, 2, \ldots, k$, are arbitrary and remaining coefficients are zero. Landau [3, 4] initiated the study of polynomials of this form in 1906-7 and proved that every trinomial

$$a_0 + a_1z + a_nz^n, \quad a_1a_n \neq 0, \quad n \geq 2,$$

has at least one zero in the circle $|z| \leq 2|a_0/a_1|$ and every quadrinomial

$$a_0 + a_1z + a_mz^m + a_nz^n, \quad a_1a_mn \neq 0, \quad 2 \leq m < n,$$

has at least one zero in the circle $|z| \leq (17/3)|a_0/a_1|$.

About sixty years ago, Simeon Reich proposed and among others, O. P. Lossers [5] proved the following:

**Theorem B.** If

$$P(z) = a_0 + a_1z + \cdots + a_{n-2}z^{n-2} + a_nz^n$$

is a polynomial of degree $n$ with

$$Q = \left\{ \max_{0 \leq j \leq n-2} \left| \frac{a_j}{a_n} \right| \right\}^{1/n} \quad \text{and} \quad Q \geq 1,$$

then all the zeros of $P(z)$ lie in the circle

$$|z| \leq Q + Q^2 + \cdots + Q^{n-1}. \quad (1)$$

**2. Main results**

Here we first present the following generalization of Theorem B to lacunary type polynomials which among other things considerably improves the bound (1) for $r = n - 2$ and further shows that the assertion (1) remains valid even if we do not assume that $Q > 1$.

**Theorem 1.** Let

$$P(z) = a_0 + a_1z + \cdots + a_rz^r + a_nz^n, \quad a_r \neq 0, \quad 0 \leq r \leq n - 1,$$

be a polynomial of degree $n$. If

$$Q = \left\{ \max_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{1/n},$$

then all the zeros of $P(z)$ lie in circle

$$|z| \leq \left\{ Q^n + Q^{n-1} + \cdots + Q^{n-r} \right\}^{1/n-r}. \quad (2)$$
COROLLARY 1. All the zeros of the polynomial
\[ P(z) = a_0 + a_1z + \cdots + a_{n-2}z^{n-2} + a_nz^n \]
of degree \( n \) lie in circle
\[ |z| \leq \left\{ Q^n + Q^{n-1} + \cdots + Q^2 \right\}^{1/2} \tag{3} \]
where
\[ Q = \left\{ \max_{0 \leq j \leq n-2} \left| \frac{a_j}{a_n} \right| \right\}^{1/n}. \]

REMARK 1. Since it can be easily verified with the help of mathematical induction that
\[ (Q^2 + Q^3 + \cdots + Q^n)^{1/2} \leq Q + Q^2 + \cdots + Q^{n-1} \]
for \( n \geq 2 \), it follows that the bound (3) of Corollary 1 is sharper than the bound (1) of the Theorem B.

The following result was proved by Mohammad [7, Theorem 1].

THEOREM C. All the zeros of the polynomial
\[ P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \]
of degree \( n \) lie in circle
\[ |z| \leq \max \left\{ L_p, L_p^{1/n} \right\} \]
where
\[ L_p = n^{1/q} \left( \sum_{j=0}^{n-1} \left| \frac{a_j}{a_n} \right|^p \right)^{1/p}, \]
\[ p > 1, \quad q > 1 \quad \text{with} \quad p^{-1} + q^{-1} = 1. \]

Here we next generalize this result to lacunary polynomials and prove the following:

THEOREM 2. For any given positive number \( t \), all the zeros of the polynomial
\[ P(z) = a_0 + a_1z + \cdots + a_rz^r + a_nz^n, \quad a_r \neq 0, \quad 0 \leq r \leq n - 1 \]
of degree \( n \) lie in the circle
\[ |z| \leq t \max \left\{ L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n} \right\} \tag{4} \]
where
\[ L_{p,t} = (r+1)^{1/q} \left( \sum_{j=0}^{r} \left| \frac{a_j}{a_n}t^{n-j} \right|^p \right)^{1/p}, \]
\[ p > 1, \quad q > 1 \quad \text{with} \quad p^{-1} + q^{-1} = 1. \] The bound is sharp.
Remark 2. The limit in Theorem 2 is attained by

\[ P(z) = t^n + t^{n-1}z + \cdots + t^{n-r}z^r - (r+1)z^n, \quad t > 0. \]

To see this, we have

\[ \sum_{j=0}^{r} \left| \frac{a_j}{a_nt^{n-j}} \right|^p = \sum_{j=0}^{r} \left( \frac{t^{n-j}}{(r+1)t^{n-j}} \right)^p = \sum_{j=0}^{r} \frac{1}{(r+1)^p} = (r+1)^{1-p}, \]

which gives

\[ L_{p,t} = (r+1)^{1/q} \left\{ \sum_{j=0}^{r} \frac{a_j/a_nt^{n-j}}{a_{n-1}} \right\}^{1/p} = 1, \]

so that

\[ t \max \left\{ L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n} \right\} = t \]

and \( z = t \) is a zero of \( P(z) \).

Remark 3. If we take \( t = 1 \) and \( r = n - 1 \) in Theorem 2, we get Theorem C.

Corollary 2. For any given positive number \( t \), all the zeros of the polynomial

\[ P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \]

of degree \( n \) lie in the circle

\[ |z| \leq t \max \left\{ N_{p,t}, N_{p,t}^{1/n} \right\} \tag{5} \]

where

\[ N_{p,t} = n^{1/q} \left\{ \sum_{j=0}^{n-1} \left| \frac{a_j}{a_nt^{n-j}} \right|^p \right\}^{1/p}, \]

\( p > 1, \ q > 1 \) with \( p^{-1} + q^{-1} = 1 \). The bound is sharp.

Finally in this paper we present the following result.

Theorem 3. For every positive number \( t \), all the zeros of the polynomial

\[ P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \]

of degree \( n \) lie in circle

\[ |z| \leq (n + 1)^{1/q} \left\{ \sum_{j=0}^{n} \left| \frac{ta_j - a_{j-1}}{a_nt^{n-j}} \right|^p \right\}^{1/p} \tag{6} \]

where \( p > 1, \ q > 1 \) with \( p^{-1} + q^{-1} = 1 \).
The following result is the limiting case of the Theorem 3 when \( q \to \infty \) so that \( p \to 1 \).

**Corollary 3.** For any given positive number \( t \), all the zeros of the polynomial

\[
P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n
\]

of degree \( n \) lie in the circle

\[
|z| \leq \sum_{j=0}^{n} \left| \frac{t a_j - a_{j-1}}{t^{n-j} a_n} \right|.
\]

**Remark 4.** If \( a_j \) is real and \( t a_j - a_{j-1} \geq 0 \), \( j = 1, 2, \ldots, n \), then it follows from Corollary 3 that for every \( t > 0 \), all the zeros of polynomial

\[
P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n
\]

of degree \( n \) lie in the circle

\[
|z| \leq \frac{t^n a_n - a_0 + |a_0|}{t^{n-1}|a_n|}.
\]

For \( t = 1 \), we get a result due to Joyal, Labelle and Rahman [2]. Further, for \( t = 1 \) and \( a_0 > 0 \), it reduces to the Eneström-Kakeya Theorem (see [6, p. 136]).

### 3. Proofs of the main results

**Proof of Theorem 1.** We observe that

\[
\left\{ Q^n + Q^{n-1} + \cdots + Q^{n-r} \right\}^{1/n-r} \geq Q
\]

for \( 0 \leq r \leq n-1 \). Also by hypothesis, \( |a_j/a_n| \leq Q^n \) for \( j = 0, 1, \ldots, r \). Therefore, we have

\[
|P(z)| = |a_n z^n + a_r z^r + \cdots + a_1 z + a_0| \\
\geq |a_n||z^n| \left\{ 1 - \sum_{j=0}^{r} \left| \frac{a_j}{a_n} \right| \frac{1}{|z|^{n-j}} \right\} \\
\geq |a_n||z^n| \left\{ 1 - \sum_{j=0}^{r} \frac{Q^n}{|z|^{n-j}} \right\}.
\]

(7)

Let \( |z| > Q \), then \( (Q/|z|) < 1 \) and therefore,

\[
(Q/|z|)^{n-j} \leq (Q/|z|)^{n-r} \quad \text{for} \quad j = 0, 1, \ldots, r.
\]

This gives

\[
\frac{1}{|z|^{n-j}} \leq \frac{1}{|z|^{n-r}Q^{r-j}} \quad \text{for} \quad j = 0, 1, \ldots, r.
\]
Hence if $|z| > Q$, we get from (7),

$$|P(z)| \geq |a_n||z^n| \left( 1 - \sum_{j=0}^{r} \frac{Q^n}{|z|^{n-r} Q^{r-j}} \right) = |a_n||z^n| \left( 1 - \frac{1}{|z|^{n-r}} \sum_{j=0}^{r} Q^{n-r+j} \right).$$

Thus $|P(z)| > 0$ if $|z|^{n-r} > \sum_{j=0}^{r} Q^{n-r+j}$, that is, if

$$|z| > \left\{ Q^n + Q^{n-1} + \cdots + Q^{n-r} \right\}^{1/n-r} (\geq Q).$$

Hence all the zeros of $P(z)$ whose modulus is greater than $Q$ lie in circle defined by (2). But those zeros of $P(z)$ whose modulus is less or equal to $Q$ already satisfy (2), the Theorem 1 is proved. □

**Proof of Theorem 2.** We have

$$|P(z)| = |a_n z^n + a_r z^r + \cdots + a_1 z + a_0| \geq |a_n||z^n| \left( 1 - \sum_{j=0}^{r} \left( \frac{a_j}{a_n} \right) \left( \frac{1}{|z|^{n-j}} \right)^p \right)^{1/q} \left( \sum_{j=0}^{r} \left( \frac{t}{|z|} \right)^{q(n-j)} \right)^{1/q}$$

which implies with the help of Holder’s inequality that

$$|P(z)| \geq |a_n||z^n| \left[ 1 - \left\{ \sum_{j=0}^{r} \left( \frac{a_j}{a_n} \right) \left( \frac{1}{|z|^{n-j}} \right)^p \right\}^{1/p} \left\{ \sum_{j=0}^{r} \left( \frac{t}{|z|} \right)^{q(n-j)} \right\}^{1/q} \right]$$

$$= |a_n||z^n| \left[ 1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{ \sum_{j=0}^{r} \left( \frac{t}{|z|} \right)^{q(n-j)} \right\}^{1/q} \right]. \tag{8}$$

Now if $L_{p,t} \geq 1$, then $\max \left\{ L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n} \right\} = L_{p,t}^{1/(n-r)}$. Let $|z| \geq t$, then $(t/|z|)^{n-j} \leq (t/|z|)^{n-r}$ for $j = 0, 1, \cdots, r$. Hence (8) implies that if $|z| > t L_{p,t}^{1/(n-r)}$, then

$$|P(z)| \geq |a_n||z^n| \left[ 1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{ \sum_{j=0}^{r} \left( \frac{t}{|z|} \right)^{q(n-j)} \right\}^{1/q} \right]$$

$$= |a_n||z^n| \left[ 1 - L_{p,t} \left( \frac{t}{|z|} \right)^{n-r} \right] > 0. \tag{9}$$
Again if \( L_{p,t} \leq 1 \), then \( \max \left\{ L_{p,t}^{1/(n-r)}, L_{p,t}^{1/n} \right\} = L_{p,t}^{1/n} \). Let \( |z| \leq t \), then \( (t/|z|)^{n-j} \leq (t/|z|)^n \) for \( j = 0, 1, \ldots, r \). Hence from (8), we infer that if \( |z| > t L_{p,t}^{1/n} \), then

\[
|P(z)| \geq |a_n| |z|^n \left[ 1 - \frac{L_{p,t}}{(r+1)^{1/q}} \left\{ \sum_{j=0}^{r} \left( \frac{t}{|z|} \right)^{nq} \right\}^{1/q} \right] 
\]

\[
= |a_n| |z|^n \left\{ 1 - L_{p,t} \left( \frac{t}{|z|} \right)^n \right\} > 0. \tag{10}
\]

From (9) and (10), it follows that \( P(z) \) does not vanish for \( |z| > \max \left\{ tL_{p,t}^{1/(n-r)}, tL_{p,t}^{1/n} \right\} \). Consequently all the zeros of \( P(z) \) lie in region defined by (4). This completes the proof of Theorem 2. \( \square \)

**Proof of Theorem 3.** Consider the polynomial

\[
F(z) = (t - z)P(z) = ta_0 + (ta_1 - a_0)z + (ta_2 - a_1)z^2 + \cdots + (ta_n - a_{n-1})z^n - a_nz^{n+1}
\]

Applying Corollary 2 to the polynomial \( F(z) \), it follows that every \( t > 0 \), all the zeros of \( F(z) \) lie in

\[
|z| \leq t \max \left\{ N_{p,t}, N_{p,t}^{1/n} \right\}
\]

where

\[
N_{p,t} = (n+1)^{1/q} \left\{ \sum_{j=0}^{n} \left| \frac{ta_j - a_{j-1}}{a_n t^{n-j+1}} \right|^p \right\}^{1/p}
\]

But

\[
1 = \frac{|(t^{n+1}a_n - t^n a_{n-1}) + (t^n a_{n-1} - t^{n-1} a_{n-2}) + \cdots + (t a_1 - a_0) + a_0|}{t^{n+1}|a_n|}
\]

\[
\leq \sum_{j=0}^{n} \frac{|t^{j+1}a_j - t^j a_{j-1}|}{t^{n+1}|a_n|} = \sum_{j=0}^{n} \frac{|ta_j - a_{j-1}|}{t^{n-j+1}|a_n|}
\]

\[
\leq (n+1)^{1/q} \left\{ \sum_{j=0}^{n} \left| \frac{ta_j - a_{j-1}}{t^{n-j+1}a_n} \right|^p \right\}^{1/p} = N_{p,t},
\]

by Holder’s inequality. Therefore, we conclude that the zeros of \( F(z) \) and hence that of \( P(z) \) lie in the circle

\[
|z| \leq (n+1)^{1/q} \left\{ \sum_{j=0}^{n} \left| \frac{ta_j - a_{j-1}}{t^{n-j}a_n} \right|^p \right\}^{1/p} .
\]
This completes the proof of Theorem 3. □

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REFERENCES


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