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DIFFERENTIAL INEQUALITIES FOR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, some basic fractional differential inequalities for a finite system of an IVP of hybrid fractional differential equations with linear perturbations of second type are proved. An existence and a comparison theorem for the considered hybrid fractional differential have also been established.

1. Introduction

Given a closed and bounded interval $J=[t_0,t_0+a]$ in $\mathbb{R}$, $\mathbb{R}$ being the real line, let

$$ {}_{t_0}D_t^{-n}f(t)=\frac{1}{\Gamma(n)}\int_{t_0}^{t}(t-t_n)^{n-1}f(t_n)\,dt_n,\ t\in J, $$

for any real number n which is called the fractional integral of order n for the integrable function $f:J\to\mathbb{R}$.

Let $m-p=q$, where m is the least integer greater than q and $0<p\leqslant 1$. Then the derivative for an arbitrary order q denoted by ${}_{t_0}D_t^q$, we have

$$ {}_{t_0}D_t^q x(t)={}_{t_0}D_t^{m-p}x(t)=\frac{d^m}{dt^m}\frac{1}{\Gamma(p)}\int_{t_0}^{t}(t-s)^{p-1}x(s)\,ds, $$

where we take advantage of the fact ${}_{t_0}D_t^m$ is the ordinary m^{th} derivative $\frac{d^m}{dt^m}$. We have that $D^{m-p}=D^mD^{-p}$. From now on we delete t_0,t in the notation ${}_{t_0}D_t^q$. So, if $0<q<1$, then the above equation reduces to

$$ D^q x(t)=\frac{1}{\Gamma(p)}\frac{d}{dt}\int_{t_0}^{t}(t-s)^{p-1}x(s)\,ds $$

which is the fractional derivative of order $0<q<1$.

Now, given the Euclidean space $\mathbb{R}^n$, consider the finite system of perturbed fractional differential equations (in short FDE)

$$ \left.\begin{aligned} &D^q[x(t)-f(t,x(t))]=g(t,x(t)),\ t\in J\\ &[x(t)-f(t,x(t))](t-t_0)^{q-1}\Big|_{t=t_0}=X^o, \end{aligned}\right\} \tag{1.1} $$

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where D^q is the fractional derivative of non-integer order q, $0<q<1$ and $f,g: J\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous.

Let $0<q<1$ and $p=1-q$. Denote by $C_p(J,\mathbb{R}^n)$, the function space

$$C_p(J,\mathbb{R}^n)=\big\{u\in C(J,\mathbb{R}^n)\mid (t-t_0)^p u(t)\in C(J,\mathbb{R}^n)\big\}.$$

By a *solution* of the FDE (1.1) we mean a function $x\in C_p(J,\mathbb{R}^n)$ satisfying

(i) the map $t\mapsto x-f(t,x)$ is continuous for each $x\in\mathbb{R}^n$, and

(ii) $D^q[x(t)-f(t,x(t))]$ exists and satisfies (1.1) on J.

The FDE (1.1) is a hybrid non-integer order fractional differential equation with a linear perturbation of second type and include the following system of FDE,

$$\left.\begin{aligned} &D^q x(t)=g(t,x(t)),\ t\in J\\ &x(t)(t-t_0)^{q-1}\Big|_{t=t_0}=x^o,\end{aligned}\right\}\tag{1.2}$$

as a special case. A systematic account of different types of perturbed differential equations is given in Dhage [2]. The FDE (1.2) has been studied for different aspects of the solution by several authors in the literature. The details of fractional differential equations and their applications are given in Kilbas *et. al* [6] and Podlubny [7]. In this paper, we discuss some of the basic differential inequalities for the hybrid FDE (1.1) on J under suitable conditions.

2. Strict and nonstrict inequalities

We need the following definitions in what follows.

DEFINITION 2.1. A function $f(t,x)$ is said to be quasi-monotone increasing in $x\in\mathbb{R}^n$ if $x,y\in\mathbb{R}^n$ with $x<y$, then $f_i(t,x)<f_i(t,y)$ for each $i=1,2,\dots,n$ and for each $t\in J$, where $x<y$ if and only if $x_i<y_i$ for each $i=1,2,\dots,n$.

DEFINITION 2.2. A function $f(t,x)$ is said to be quasi-monotone nondecreasing in $x\in\mathbb{R}^n$ if $x,y\in\mathbb{R}^n$ with $x\leqslant y$, then $f_i(t,x)\leqslant f_i(t,y)$ for each $i=1,2,\dots,n$ and for each $t\in J$, where $x\leqslant y \iff x_i\leqslant y_i$ for each $i=1,2,\dots,n$.

We consider the following hypotheses in the sequel.

(A_0) The mapping $x\mapsto x-f(t,x)$ is quasi-monotone increasing for each $t\in J$, and

(B_0) The mapping $x\mapsto g(t,x)$ is quasi-monotone nondecreasing for each $t\in J$.

THEOREM 2.1. *Let $x,y\in C_p(J,\mathbb{R}^n)$ be two locally Hölder continuous with an exponent λq, $0<\lambda<1$ and let hypotheses (A_0) and (B_0) hold. Suppose that*

$$\left.\begin{aligned} &D^q[x(t)-f(t,x(t))\leqslant g(t,x(t)),\ t\in J\\ &[x(t)-f(t,x(t))](t-t_0)^{q-1}\Big|_{t=t_0}=X^o,\end{aligned}\right\}\tag{2.1}$$

and

$$\left.\begin{aligned} &D^q[y(t)-f(t,y(t))] \geqslant g(t,y(t)),\ t\in J \\ &[y(t)-f(t,y(t))](t-t_0)^{q-1}\Big|_{t=t_0} = Y^o. \end{aligned}\right\} \tag{2.2}$$

If one of the inequalities (2.1) *and* (2.2) *is strict and*

$$X^o < Y^o \tag{2.3}$$

then

$$x(t) < y(t),\ t\in J. \tag{2.4}$$

Proof. Suppose that inequality (2.4) is not true. Define

$$X(t) = x(t) - f(t,x(t))$$

and

$$Y(t) = y(t) - f(t,y(t))$$

for each $t\in J$. Then from the continuity of the functions X and Y it follows that there exists an index j, $1\leqslant j\leqslant n$ and $t_0\leqslant t_1\leqslant t_0+a$ such that

$$X_j(t_1) = Y_j(t_1),\quad X_j(t)\leqslant Y_j(t),\ t_0\leqslant t<t_1$$

and

$$X_i(t)\leqslant Y_i(t),\quad i\neq j.$$

Setting

$$M_j(t) = X_j(t)Y_j(t),$$

we obtain

$$M_j(t_1)=0,\quad M_j(t)\leqslant 0,\ t_0\leqslant t\leqslant t_1,$$

and

$$M_i(t)\leqslant 0,\quad i\neq j.$$

Applying a standard result we obtain,

$$D^qM_j(t_1)\geqslant 0. \tag{2.5}$$

Now, assuming the strict inequality (2.2), we obtain from hypotheses (A_0) and (B_0) that

$$\begin{aligned} g_j(t,x_1(t_1),\cdots,x_n(t_1)) &\geqslant D^q\left[x_j(t_1)-f_j(t_1,x_1(t_1),\cdots,x_n(t_1)))\right] \\ &\geqslant D^q\left[y_j(t_1)-f_j(t_1,y_1(t_1),\cdots,y_n(t_1)))\right] \\ &> g_j(t,y_1(t_1),\cdots,y_n(t_1)) \end{aligned} \tag{2.6}$$

The above relation (2.6) is a contradiction and hence the relation (2.4) holds on J. This completes the proof. □

The next result is a nonstrict inequality for the hybrid FDE (1.1) on J. This result is proved under a one-sided Lipschitz condition.

THEOREM 2.2. *Assume that the inequalities* (2.1) *and* (2.2) *with nonstrict inequalities and that the hypotheses* (A_0) *and* (B_0) *hold. Further suppose that there exists a constant* $L > 0$ *such that*

$$g_i(t,x) - g_i(t,y) \leqslant L(x_i - y_i), \tag{2.7}$$

for each i, $1 \leqslant i \leqslant n$, *where* $x,y \in C(J,\mathbb{R})$ *with* $x \geqslant y$.

Then,

$$X^o = [x(t) - f(t,x(t))](t-t_0)^{1-q} \leqslant [y(t) - f(t,y(t))](t-t_0)^{1-q} = Y^o \tag{2.8}$$

implies

$$x(t) \leqslant y(t),\ t \in J. \tag{2.9}$$

Proof. We set

$$Y_\varepsilon(t) = y_\varepsilon(t) - f(t, y_\varepsilon(t))$$

and

$$Y_\varepsilon(t) = Y(t) + \varepsilon\lambda(t) \tag{2.10}$$

for each $\varepsilon > 0$, $\varepsilon \in \mathbb{R}^n$, where $\lambda(t) = (t-t_0)^{1-q}E_{q,q}2L(t-t_0)^q$. This shows that

$$Y_\varepsilon^o > Y^o > X^o$$

which yields that

$$Y_\varepsilon(t) > Y(t).$$

Now employing the Lipschitz condition,

$$\begin{aligned} D^q\left[y_\varepsilon(t) - f(t,y_\varepsilon(t))\right] &= D^qY(t) + \varepsilon D^q\lambda(t) \\ &\geqslant g(t,y(t)) + 2\varepsilon L\lambda(t) \\ &\geqslant g(t,y_\varepsilon(t)) - L\varepsilon\lambda(t) + 2L\varepsilon\lambda(t) \\ &> g(t,y_\varepsilon(t)). \end{aligned}$$

Here, we have employed the fact that $\lambda(t)$ is a solution of the IVP

$$D^q\lambda(t) = 2L\lambda(t),\ \lambda(t)(t-t_0)^{1-q}\Big|_{t=t_0} = \lambda^o$$

with $\lambda^o = 1$. Now we apply Theorem 2.1 to $Y_\varepsilon(t)$ and $X(t)$ to get

$$Y_\varepsilon(t) > X(t),\ t \in J.$$

When $\varepsilon \to 0$, we obtain

$$Y(t) \geqslant X(t) \quad \text{or} \quad y(t) - f(t,y(t)) \geqslant x(t) - f(t,x(t))$$

for each $t \in J$. Finally, from hypothesis (A_0) we get the desired conclusion (2.9). □

3. Existence and comparison Theorems

The importance of the mathematical inequalities lies in their applications to allied areas of mathematics. Similarly, differential inequalities proved in Theorem 2.1 and 2.2 are very much useful for proving the other aspects for the hybrid FDE (1.1) on J. Next, we prove the comparison theorems for FDE (1.1), since comparison theorems are powerful tools for proving global existence and uniqueness results for differential and integral equations. Hence, differential and integral inequalities have important place in the theory of differential and integral equations.

Before stating our comparison result, we list some basic hypotheses concerning the functions involved in the FDE (1.1). These hypotheses are needed for proving the existence theorem for the FDE (1.1). We only sketch the main steps involved in the proof of existence result, because its proof is similar to that of a scalar case treated in Dhage and Mugale [3].

(A_1) There exist constants $L > 0$ and $M > 0$ such that

$$|f(t,x) - f(t,y)| \leqslant \frac{L|x-y|_n}{M + |x-y|_n}$$

for all $t \in J$, where $|\cdot|_n$ is a norm in $\mathbb{R}^n$. Moreover, we assume that $L \leqslant M$.

(B_1) The function g is bounded on $J \times \mathbb{R}^n$ with bound M_g.

THEOREM 3.1. (Existence theorem) *Assume that hypotheses* (A_0)–(A_1) *and* (B_0)–(B_1) *hold. Then the hybrid FDE (1.1) admits a solution.*

Proof. The hybrid HFDE (1.1) is equivalent to the HFIE

$$x(t) = X^0(t) + f(t,x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\,ds \tag{3.1}$$

We place the HFIE in the space $X = C(J,\mathbb{R}^n)$ and define a subset S of X by

$$S = \left\{ x \in C(J,\mathbb{R}^n) \mid \|x\| \leqslant M \right\} \tag{3.2}$$

where, $\sup_{t\in J} |f(t,0)|_n = F_0$ and $M = |X^0|_n + \frac{M_g a}{q} + L + F_0$

Define two operators $A : C(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ and $B : S \to C(J,\mathbb{R}^n)$ by

$$Ax(t) = f(t,x(t)),\ t \in J \tag{3.3}$$

and

$$Bx(t) = X^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\,ds,\ t \in J. \tag{3.4}$$

Then the FIE (3.1) is transformed into the following equivalent operator equation

$$Ax(t) + Bx(t) = x(t),\ t \in J \tag{3.5}$$

The rest of the proof is similar to Theorem 3.2 given in Dhage and Mugale [3] and can be obtained by an application of a hybrid fixed point theorem of Dhage [1] in a Banach space $C(J,\mathbb{R}^n)$ with appropriate modifications. □

THEOREM 3.2. (Comparison theorem) *Assume that hypotheses* $m \in C_p(J,\mathbb{R}^n)$ *is locally Hölder continuous and*

$$D^q[m(t) - f(t,m(t))] \leqslant g(t,m(t)) \tag{3.6}$$

for all $t \in J$. *Let* $r(t)$ *be the maximal solution of the IVP*

$$\left.\begin{aligned} D^q[u(t) - f(t,u(t))] &= g(t,u(t)),\ t \in J \\ [u(t) - f(t,u(t))](t-t_0)^{q-1}\Big|_{t=t_0} &= U^o, \end{aligned}\right\} \tag{3.7}$$

existing on J *such that*

$$M^o = [m(t) - f(t,m(t))](t-t_0)^{1-q}\Big|_{t=t_0} \leqslant U^o. \tag{3.8}$$

Then, we have

$$m(t) \leqslant r(t), \quad t \in J. \tag{3.9}$$

Proof. From the notion of a maximal solution $r(t)$, it is enough to prove that

$$m(t) \leqslant r(t,\varepsilon), \quad t \in J \tag{3.10}$$

where $r(t,\varepsilon)$ is any solution of the hybrid FDE

$$\left.\begin{aligned} D^q[u(t) - f(t,u(t))] &= g(t,u(t)) + \varepsilon, \\ [u(t) - f(t,u(t))](t-t_0)^{q-1}\Big|_{t=t_0} &= U^o + \varepsilon, \end{aligned}\right\} \tag{3.11}$$

for all $t \in J$, where $\varepsilon > 0$ is small number in $\mathbb{R}^n$.

Now the expression in (3.11) yields

$$\begin{aligned} D^q[u(t) - f(t,u(t))] &= g(t,u(t)) + \varepsilon \\ &> g(t,u(t)) \end{aligned}$$

Applying strict inequality formulated in Theorem 2.1, we obtain

$$m(t) < r(t,\varepsilon), \quad t \in J.$$

Since,

$$\lim_{\varepsilon \to 0} r(t,\varepsilon) = r(t) \tag{3.12}$$

uniformly on J. Hence, taking the limit as $\varepsilon \to 0$ in (3.12) yields (3.9). This completes the proof. □

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