Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 461–475
doi:10.7153/jmi-07-41

GEOMETRIC PROPERTIES OF BANACH SPACE VALUED BOCHNER–LEBESGUE SPACES WITH VARIABLE EXPONENT

CHEN CHENG AND JINGSHI XU

(*Communicated by L. Diening*)

Abstract. In this paper, the Banach space valued Bochner-Lebesgue spaces with variable exponent are introduced. Then the dual space, the reflexivity, uniformly convexity and uniformly smoothness of these new spaces are obtained. Finally the properties of the Banach valued Bochner-Sobolev spaces with variable exponent are also given. Those are a generalization of scalar valued Lebesgue and Sobolev spaces with variable exponent.

1. Introduction

Since 1991, variable exponent spaces, including variable exponent Lebesgue, Sobolev, Besov, Triebel-Lizorkin and Morrey spaces, have attracted many attentions; see [2, 4, 7, 8, 10, 11, 15, 16] and references therein. These spaces have also many applications. It is well known that Banach space valued Bochner-Lebesgue and Sobolev spaces have been used in analysis, for example, see [3, 13]. Motivated by mentioned references, we will discuss Banach space valued Bochner-Lebesgue spaces with variable exponent. In what follows, $(A,\mathscr{A},\mu)$ will be a σ-finite complete measure space. Suppose D is a subset of A, let χ_D be the indicator function on D. Let E be a Banach space with norm $\|\cdot\|$. The dual space of E is the vector space E^* of all continuous linear mappings from E to $\mathbb{R}$ or $\mathbb{C}$. To avoid a double definition we let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by S_E its unit sphere $\{x \in E : \|x\| = 1\}$. Let $\mathscr{P}(A,\mu)$ denote the set of all μ-measurable functions $p(\cdot) : A \to [1,\infty]$ which are called variable exponents on A.

For a function $p(\cdot) \in \mathscr{P}(A,\mu)$, we denote $p^- := \operatorname{essinf}_{y\in A} p(y)$, $p^+ := \operatorname{esssup}_{y\in A} p(y)$ and $p'(\cdot) \in \mathscr{P}(A,\mu)$ by $1/p(y)+1/p'(y) = 1$.

A convex, left-continuous function $\varphi : [0,\infty) \to [0,\infty]$ with $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$, and $\lim_{t\to 0^+} \varphi(t) = 0$ is called a φ-function. Then we recall the definition of generalized φ-function.

DEFINITION 1. Let $(A,\mathscr{A},\mu)$ be a σ-finite complete measure space. A real function $\varphi : A\times[0,\infty) \to [0,\infty]$ is said to be a generalized φ-function on (A,μ) if

(a) $\varphi(y,\cdot)$ is a φ-function for all $y \in A$.

(b) $y \mapsto \varphi(y,t)$ is measurable for all $t \geqslant 0$.

Mathematics subject classification (2010): Primary 46G10; Secondary 46A80, 46B22.
Keywords and phrases: Variable exponent, Bochner-Lebesgue space, Radon-Nikodym property, reflexivity, uniformly convexity.

© ELEMENT, Zagreb
Paper JMI-07-41

If φ is a generalized φ-function on $(A,\mathscr{A},\mu)$, we write $\varphi\in\Phi(A,\mu)$.

DEFINITION 2. A function $f:A\to E$ is strongly μ-measurable if there exists a sequence $\{f_n\}_{n\geqslant 1}$ of μ-simple functions converging to f μ-almost everywhere.

Then by $L^0(A,E)$ we denote the space of all E-valued strongly μ-measurable functions on A. We set $t^\infty=0$, if $t\in[0,1]$ and $t^\infty=\infty$, if $t\in(1,\infty)$.

DEFINITION 3. The Bochner-Lebesgue spaces with variable exponent $L^{p(\cdot)}(A,E)$ is the collection of all strongly μ-measurable functions $f:A\to E$ endowed with the norm:

$$\|f\|_{L^{p(\cdot)}(A,E)}:=inf\{\lambda>0,\rho_{p(\cdot)}(f/\lambda)\leqslant 1\}$$

where $\rho_{p(\cdot)}(f):=\int_A\varphi_{p(\cdot)}(y,\|f(y)\|)\mathrm{d}\mu(y)$, $\varphi_{p(\cdot)}(y,\|f(y)\|):=\|f(y)\|^{p(y)}$ and $p(\cdot)\in\mathscr{P}(A,\mu)$.

It is easy to see that $\varphi_{p(\cdot)}\in\Phi(A,\mu)$.

Our first result is that $L^{p(\cdot)}(A,E)$ is complete.

THEOREM 1. *Let $p(\cdot)\in\mathscr{P}(A,\mu)$. Then $L^{p(\cdot)}(A,E)$ is a Banach space.*

Secondly, we consider the dual of $L^{p(\cdot)}(A,E)$. To do so, we need to recall some definitions.

DEFINITION 4. A mapping $F:\mathscr{A}\to E$ is called an E-valued measure on $\mathscr{A}$ if for all disjoint unions $A=\bigcup_{n=1}^{\infty}A_n$ in $\mathscr{A}$, $F(A)=\sum_{n=1}^{\infty}F(A_n)$ with convergence in the norm of E. The variation of an E-valued measure F is the mapping $\|F\|:\mathscr{A}\to[0,\infty]$ defined by

$$\|F\|(A)=\sup_{\pi}\sum_{B\in\pi}\|F(B)\|,$$

where the supremum is taken of all finite disjoint partitions π of A. An E-valued measure F is of bounded variation if $\|F\|(A)<\infty$.

DEFINITION 5. A Banach space E has the Radon-Nikodym property with respect to a σ-finite measure space $(A,\mathscr{A},\mu)$ if for every E-valued measure F of bounded variation on $\mathscr{A}$ which is absolutely continuous with respect to μ there exists a function $\phi\in L^1(A,E)$ such that

$$F(A_1)=\int_{A_1}\phi\,\mathrm{d}\mu,\ \forall A_1\in\mathscr{A}.$$

Our second result is the following.

THEOREM 2. *Let* $(A,\mathscr{A},\mu)$ *be a* σ*-finite complete measure space,* E^* *have the Radon-Nikodym property with respect to* $(A,\mathscr{A},\mu)$*, and* $p(\cdot)\in\mathscr{P}(A,\mu)$ *with* $p^+<\infty$*, then the mapping* $g\mapsto\phi_g, L^{p'(\cdot)}(A,E^*)\to(L^{p(\cdot)}(A,E))^*$ *which is defined by*

$$\langle\phi_g,f\rangle=\int_A\langle g,f\rangle\mathrm{d}\mu,\forall f\in L^{p(\cdot)}(A,E)$$

is a linear isomorphism and

$$\|g\|_{L^{p'(\cdot)}(A,E^*)}\leqslant\|\phi_g\|_{(L^{p(\cdot)}(A,E))^*}\leqslant 2\|g\|_{L^{p'(\cdot)}(A,E^*)}.$$

Suppose E is reflexive, then E^* is also reflexive. From [6], E^* has the Radon-Nikodym property, thus by Theorem 2, we have the following corollary.

COROLLARY 1. *If* E *is reflexive and* $p(\cdot)\in\mathscr{P}(A,E)$ *with* $1<p^-\leqslant p^+<\infty$*, then* $L^{p(\cdot)}(A,E)$ *is reflexive.*

Thirdly, we shall consider the uniformly convexity of these spaces. Let us recall the definition of the uniformly convexity.

DEFINITION 6. Let $(X,\|\cdot\|)$ be a normed space. For every $0\leqslant\varepsilon\leqslant 2$, let $\delta_X(\varepsilon)=\inf\{1-\|(f+g)/2\|:\ \|f\|,\|g\|\leqslant 1,\ \|f-g\|\geqslant\varepsilon\}$, then $\delta_X(\varepsilon)$ is called the modulus of convexity of X. $\|\cdot\|$ is uniformly convex if $\delta_X(\varepsilon)>0$ whenever $0<\varepsilon\leqslant 2$.

A Banach space X is called uniformly convex, if there exists a uniformly convex norm $\|\cdot\|'$, which is equivalent to the original norm of X.

THEOREM 3. *Let* $p(\cdot)\in\mathscr{P}(A,\mu)$ *with* $1<p^-\leqslant p^+<\infty$ *and* E *be a uniformly convex Banach space. Then* $L^{p(\cdot)}(A,E)$ *is uniformly convex.*

Another geometric property which we will discuses for $L^{p(\cdot)}(A,E)$ is uniformly smoothness.

DEFINITION 7. Let X be a normed space. Define

$$\rho_X:(0,+\infty)\to[0,+\infty)$$

by

$$\rho_X(t)=\sup\left\{\frac{1}{2}(\|x+ty\|+\|x-ty\|)-1:x,y\in S_X\right\}$$

if $X\neq\{0\}$. If $X=0$,

$$\rho_X(t)=\begin{cases}0 & \text{if } 0<t<1;\\ t-1 & \text{if } t\geqslant 1.\end{cases}$$

Then ρ_X is called uniformly smoothness of X. If $\lim\limits_{t\to 0^+}\dfrac{\rho_X(t)}{t}=0$, then X is called uniformly smooth.

COROLLARY 2. *Suppose* $p(\cdot) \in \mathscr{P}(A,\mu)$ *with* $1 < p^- <$ *and* $p^+ < +\infty$. *If* E *is a uniformly smooth Banach space, then* $L^{p(\cdot)}(A,E)$ *is also a uniformly smooth Banach space.*

Finally, we shall discuss the properties of the Banach space valued Bochner-Sobolev spaces with variable exponent.

Let Ω be a open set of $\mathbb{R}^n$, and μ be the Lebesgue measure on $\mathbb{R}^n$. In this case we denote by $L^{p(\cdot)}(\Omega,E)$ the Bochner-Lebesgue space with values in Banach space E and variable exponent $p(\cdot)$, and $\mathscr{P}(\Omega,\mu)$ by $\mathscr{P}(\Omega)$ for simplicity.

DEFINITION 8. Let $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$ be multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ as the multiplicity of α. Suppose $u \in L^1_{\text{loc}}(\Omega,E)$ (all local integrable functions valued in E over Ω). If there exists $g \in L^1_{\text{loc}}(\Omega,E)$ for any $\phi \in C_0^\infty(\Omega)$ such that

$$\int_\Omega u\partial^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\phi \, \mathrm{d}x,$$

then g is called a weak derivative of u with respect to α. Denote g by $\partial^\alpha u$. For simplicity, denote $\frac{\partial u}{\partial x_j}$ by $\partial_j u$.

DEFINITION 9. Let $k \in \mathbb{N}_0$, $p(\cdot) \in \mathscr{P}(\Omega)$. Suppose $u \in L^{p(\cdot)}(\Omega,E)$. If u's weak derivatives $\partial^\alpha u$ belong to $L^{p(\cdot)}(\Omega,E)$, $|\alpha| \leqslant k$, then u is called a Sobolev function, and denote by $u \in W^{k,p(\cdot)}(\Omega,E)$. Define the modular of u in $W^{k,p(\cdot)}(\Omega,E)$ as

$$\rho_{W^{k,p(\cdot)}(\Omega,E)}(u) = \sum_{0 \leqslant |\alpha| \leqslant k} \rho_{p(\cdot)}(\partial^\alpha u).$$

Define the norm of u in $W^{k,p(\cdot)}(\Omega,E)$ as

$$\|u\|_{W^{k,p(\cdot)}(\Omega,E)} = \inf\{\lambda > 0 : \rho_{W^{k,p(\cdot)}(\Omega,E)}(u/\lambda) \leqslant 1\}.$$

It is easy to see that $W^{0,p(\cdot)}(\Omega,E) = L^{p(\cdot)}(\Omega,E)$.

For the Sobolev space $W^{k,p(\cdot)}(\Omega,E)$, we have the following result.

THEOREM 4. *Let* E *be a Banach space.*

(i) *If* $p(\cdot) \in \mathscr{P}(\Omega)$, *then* $W^{k,p(\cdot)}(\Omega,E)$ *is also a Banach space.*

(ii) *If* $p(\cdot)$ *is bounded and* E *is separable, then* $W^{k,p(\cdot)}(\Omega,E)$ *is also separable.*

(iii) *If* $1 < p^- \leqslant p^+ < \infty$ *and* E *is reflexive, then* $W^{k,p(\cdot)}(\Omega,E)$ *is also reflexive.*

(iv) *If* $1 < p^- \leqslant p^+ < \infty$ *and* E *is uniformly convex, then* $W^{k,p(\cdot)}(\Omega,E)$ *is also uniformly convex.*

DEFINITION 10. Let X be a normed space. If every sequence u_i converges weakly to u implies $\frac{1}{m}\sum_{i=1}^{m} u_i$ converges to u in X, then it is called that X has Banach-Saks property.

From [9], we know every uniformly convex space has the Banach-Sakes property. Thus we have the following corollary.

COROLLARY 3. *Let E be a uniformly convex Banach space.*

(i) *Suppose $p(\cdot) \in \mathscr{P}(A,\mu)$ with $1 < p^- \leqslant p^+ < \infty$. then $L^{p(\cdot)}(A,E)$ has the Banach-Saks property.*

(ii) *If $k \in \mathbb{N}$, and $p(\cdot) \in \mathscr{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$, then $W^{k,p(\cdot)}(\Omega,E)$ has the Banach-Saks property.*

DEFINITION 11. A normed space X has the Radon-Riesz property, if whenever a sequence (u_i) converges weakly to u and $\|u_i\|$ converges to $\|u\|$, it follows that u_i converges to u in X.

It is well known that every uniformly convex normed space has the Radon-Riesz property, for example, see Theorem 5.2.18 in [14]. Therefore we have the following corollary.

COROLLARY 4. *Let E be a uniformly convex Banach space.*

(i) *Suppose $p(\cdot) \in \mathscr{P}(A,\mu)$ with $1 < p^- \leqslant p^+ < \infty$. then $L^{p(\cdot)}(A,E)$ has the Radon-Riesz property.*

(ii) *If $k \in \mathbb{N}$, and $p(\cdot) \in \mathscr{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$, then $W^{k,p(\cdot)}(\Omega,E)$ has the Radon-Riesz property.*

Finally, we point out that the notation $\square$ means the proof is finished.

2. Proofs of the main results

To give the proofs, we will use the ideas for the scalar valued setting; see [5]. First, we need preliminaries.

DEFINITION 12. Let X be a $\mathbb{K}$-vector space. A function $\rho : X \to [0,\infty]$ is called a convex semimodular on X if the following properties hold.

(a) $\rho(0) = 0$.

(b) $\rho(\lambda x) = \rho(x)$ for all $x \in X, \lambda \in \mathbb{K}$ with $|\lambda| = 1$.

(c) ρ is convex.

(d) ρ is left-continuous.

(e) $\rho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

LEMMA 1. *Let ρ be a convex semimodular on X. Denote $\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leqslant 1\}$.*

(a) *$\|x\|_\rho \leqslant 1$ and $\rho(x) \leqslant 1$ are equivalent.*

(b) *If $\|x\|_\rho \leqslant 1$, then $\rho(x) \leqslant \|x\|_\rho$.*

Lemma 1 is Lemma 2.1.14 and Corollary 2.1.15 in [5]. The following Lemma is Lemma 2.1.9 in [5].

LEMMA 2. *Let ρ be a convex semimodular on X and $x_k \in X_\rho$. Then $x_k \to 0$ for $k \to \infty$ if and only if $\lim_{k\to\infty} \rho(\lambda x_k) = 0$ for all $\lambda > 0$.*

LEMMA 3. *If $f \in L^0(A,E)$, then $y \mapsto \varphi_{p(\cdot)}(y, \|f(y)\|)$ is μ-measurable and $\rho_{p(\cdot)}(f) = \int_A \varphi_{p(\cdot)}(y, \|f(y)\|)\mathrm{d}\mu(y)$ is a convex semimodular on $L^0(A,E)$.*

Proof. Let $f_k \to f$, where f_k are simple functions. Then

$$\varphi_{p(\cdot)}(y, \|f_k(y)\|) = \sum_j \varphi_{p(\cdot)}(y, \|x_{k_j}\|)\chi_{A_j^k}(y)$$

which is measurable and $\varphi_{p(\cdot)}(y, \|f_k(y)\|) \to \varphi_{p(\cdot)}(y, \|f(y)\|)$.

Thus, $\varphi_{p(\cdot)}(\cdot, \|f(\cdot)\|)$ is measurable.

We next show that $\rho_{p(\cdot)}$ is a convex semimodular on $L^0(A,E)$:

(a) Obviously, $\rho_{p(\cdot)}(0) = 0$.

(b) $\rho_{p(\cdot)}(\lambda f) = \int_A \varphi_{p(\cdot)}(y, \|f(y)\|)\mathrm{d}\mu(y) = \rho_{p(\cdot)}(f)$ for $|\lambda| = 1$.

(c) The convexity of $\rho_{p(\cdot)}$ is a direct consequence of convexity of $\varphi_{p(\cdot)}$.

(d) If $\lambda_k \uparrow 1$ and $y \in A$, thus by the left-continuity and monotonicity of $\varphi_{p(\cdot)}(y, \cdot)$, it follows

$$0 \leqslant \varphi_{p(\cdot)}(y, \|\lambda_k f(y)\|) \uparrow \varphi_{p(\cdot)}(y, \|f(y)\|).$$

Hence $\lim_{k\to\infty} \int_A \varphi_{p(\cdot)}(y, \|\lambda_k f(y)\|)\mathrm{d}\mu(y) = \int_A \varphi_{p(\cdot)}(y, \|f(y)\|)\mathrm{d}\mu(y)$ by the theorem of monotone convergence, i.e $\rho_{p(\cdot)}(\lambda_k f) \to \rho_{p(\cdot)}(f)$.

(e) Assume $f \in L^0(A,E)$ such that $\rho_{p(\cdot)}(\lambda f) = 0$ for all $\lambda > 0$. So for any $k \in \mathbb{N}$ we have $\varphi_{p(\cdot)}(y, k\|f(y)\|) = 0$ for almost all $y \in A$. Since $\mathbb{N}$ is countable we deduce that $\varphi_{p(\cdot)}(y, k\|f(y)\|) = 0$ for almost all $y \in A$ and all $k \in \mathbb{N}$. The convexity of $\varphi_{p(\cdot)}$ and $\varphi_{p(\cdot)}(y, 0) = 0$ imply that $\varphi_{p(\cdot)}(y, \lambda\|f(y)\|) = 0$ for almost all $y \in A$ and all $\lambda > 0$. Since $\lim_{t\to\infty} \varphi_{p(\cdot)}(y,t) = \infty$ for all $y \in A$, this implies that $\|f(y)\| = 0$ for almost all $y \in A$, hence $f = 0$. So $\rho_{p(\cdot)}$ is a convex semimodular on $L^0(A,E)$. □

LEMMA 4. *Let $\mu(A) < \infty$. Then every $\|\cdot\|_{L^{p(\cdot)}}$-Cauchy sequence is also a Cauchy sequence with respect to convergence in measure. Moreover, if $f_k \in L^{p(\cdot)}(A,E)$ with $\|f_k\|_{L^{p(\cdot)}} \to 0$, then $f_k \to 0$ in measure.*

Proof. Fix $\varepsilon > 0$ and let $V_t := \{y \in A, \varphi_{p(\cdot)}(y,t) = 0\}$, for $t > 0$. Then V_t is measurable. For all $y \in A$ the function $t \to \varphi_{p(\cdot)}(y,t)$ is non-decreasing and $\lim_{t\to\infty} \varphi_{p(\cdot)}(y,t) = \infty$, so $V_t \downarrow \emptyset$ as $t \to \infty$. Since $\mu(A) < \infty$, we have $\lim_{k\to\infty} \mu(V_k) = \mu(\emptyset) = 0$. Thus, there exists $K \in \mathbb{N}$, such that $\mu(V_K) < \varepsilon$. For a μ-measurable set $U \subset A$, define

$$\nu_K(U) := \rho_{p(\cdot)}(K\chi_U) = \int_U \varphi_{p(\cdot)}(y,K)\mathrm{d}\mu(y).$$

If U is μ-measurable with $\nu_K(U) = 0$, then $\varphi_{p(\cdot)}(y,K) = 0$ for μ-almost every $y \in U$. Thus $\mu(U \backslash V_K) = 0$ by the definition of V_K. Hence, U is a $\mu|_{A\backslash V_K}$-null set, which means that $\mu|_{A\setminus V_K}$ is absolutely continuous with respect to ν_K. Since $\mu|_{A\setminus V_K} \leqslant \mu(A) < \infty$, and $\mu|_{A\setminus V_K}$ is absolutely continuous with respect to ν_K, there exists $\delta \in (0,1)$ such that $\nu_K(U) \leqslant \delta$ implies $\mu(U\setminus V_K) \leqslant \varepsilon$.

Since f_k is a $\|\cdot\|_{L^{p(\cdot)}}$-Cauchy sequence, there exists $k_0 \in \mathbb{N}$ such that $\|K\varepsilon^{-1}\delta^{-1}(f_m - f_k)\|_{L^{p(\cdot)}} \leqslant 1$ for all $m,k \geqslant k_0$.

Assume in the following $m,k \geqslant k_0$, then by convexity and Lemma 1,

$$\rho_{p(\cdot)}(K\varepsilon^{-1}(f_m - f_k)) \leqslant \delta\rho_{p(\cdot)}(K\varepsilon^{-1}\delta^{-1}(f_m - f_k)) \leqslant \delta.$$

We write $E_{m,k,\varepsilon} := \{y \in A : \|f_m(y) - f_k(y)\| \geqslant \varepsilon\}$. Then

$$\nu_K(E_{m,k,\varepsilon}) = \int_{E_{m,k,\varepsilon}} \varphi_{p(\cdot)}(y,K)\mathrm{d}\mu(y) \leqslant \rho_{p(\cdot)}(K\varepsilon^{-1}(f_m - f_k)) \leqslant \delta.$$

By the choice of δ, this implies that $\mu(E_{m,k,\varepsilon}\setminus V_k) \leqslant \varepsilon$. With $\mu(V_k) < \varepsilon$ we have $\mu(E_{m,k,\varepsilon}) \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that f_k is a Cauchy sequence with respect to convergence in measure. If $\|f_k\|_{L^{p(\cdot)}} \to 0$, then exists $K \in \mathbb{N}$ such that $\mu\{\|f_k\| \geqslant \varepsilon\} \leqslant 2\varepsilon$ for all $k \geqslant K$. This proves $f_k \to 0$ in measure. □

LEMMA 5. *Every $\|\cdot\|_{L^{p(\cdot)}}$-Cauchy sequence $(f_k) \subset L^{p(\cdot)}(A,E)$ has a subsequence which converges μ-almost everywhere to a measurable function f.*

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$, with A_i pairwise disjoint and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$. Then by Lemma 4, f_k is a Cauchy sequence with respect to convergence in measure on A_1. Therefore there exists a measurable function $f : A_1 \to E$ and a subsequence of f_k which converges to f μ-almost everywhere. Repeating this argument for every A_i and passing to the diagonal sequence, we get a subsequence (f_{k_j}) and a μ-measurable function $f : A \to E$ such that $f_{k_j} \to f$ μ-almost everywhere. □

After these preparation, we can give the proof of Theorem 1.

Proof of Theorem 1. Let (f_k) be a Cauchy sequence. By Lemma 5 there exists a subsequence f_{k_j} and a μ-measurable function $f : A \to E$ such that $f_{k_j} \to f$ for μ-almost every $y \in A$. This implies $\varphi_{p(\cdot)}(y, \|f_{k_j}(y) - f(y)\|) \to 0$ μ-almost everywhere.

Let $\lambda > 0$ and $0 < \varepsilon < 1$, since (f_k) is a Cauchy sequence, there exists $K = K(\lambda,\varepsilon) \in \mathbb{N}$, such that $\|\lambda(f_m - f_k)\|_{L^{p(\cdot)}} < \varepsilon$ for all $m,k \geqslant K$ which implies $\rho_{p(\cdot)}(\lambda(f_m - f_k)) \leqslant \varepsilon$ by Lemma 1(b). Therefore by Fatou's lemma

$$\begin{aligned}
\rho_{p(\cdot)}(\lambda(f_m - f)) &= \int_A \varphi_{p(\cdot)}(y, \lambda\|f_m(y) - \lim_{j\to\infty} f_{k_j}(y)\|)\mathrm{d}\mu(y) \\
&= \int_A \lim_{j\to\infty} \varphi_{p(\cdot)}(y, \lambda\|f_m(y) - f_{k_j}(y)\|)\mathrm{d}\mu(y) \\
&\leqslant \liminf_{j\to\infty} \int_A \varphi_{p(\cdot)}(y, \lambda\|f_m(y) - f_{k_j}(y)\|)\mathrm{d}\mu(y) \\
&= \liminf_{j\to\infty} \rho_{p(\cdot)}(\lambda(f_m - f_{k_j})) \leqslant \varepsilon.
\end{aligned}$$

Then for $m \to \infty$ and all $\lambda > 0$, $\rho_{p(\cdot)}(\lambda(f_m - f)) \to 0$. By Lemma 2, $\|f_k - f\|_{L^{p(\cdot)}} \to 0$. Thus every Cauchy sequence converges in $L^{p(\cdot)}(A,E)$. □

Now we turn to prove Theorem 2.

A linear subspace F of E^* is called norming for a subset S of E if for all $x \in S$ we have

$$\|x\| = \sup_{x^* \in F, \|x^*\| \leqslant 1} |\langle x, x^* \rangle|.$$

LEMMA 6. *Let $p(\cdot) \in \mathscr{P}(A,\mu)$, Y be a closed subspace of E^* which is norming for E. Then the mapping $g \mapsto \phi_g$ which is defined by*

$$\langle \phi_g, f \rangle = \int_A \langle g, f \rangle \mathrm{d}\mu, \forall f \in L^{p(\cdot)}(A,E)$$

is a linear isomorphism from $L^{p'(\cdot)}(A,Y)$ to a closed subspace of $(L^{p(\cdot)}(A,E))^*$ which is norming for $L^{p(\cdot)}(A,E)$, and

$$\|g\|_{L^{p'(\cdot)}(A,Y)} \leqslant \|\phi_g\|_{(L^{p(\cdot)}(A,E))^*} \leqslant 2\|g\|_{L^{p'(\cdot)}(A,Y)}$$

for all $g \in L^{p'(\cdot)}(A,Y)$.

Proof. By the Hölder inequality, it remains to prove that

$$\|g\|_{L^{p'(\cdot)}(A,Y)} \leqslant \|\phi_g\|_{(L^{p(\cdot)}(A,E))^*}.$$

Without loss of generality, we may assume that $\|g\|_{L^{p'(\cdot)}(A,Y)} = 1$. It suffices to prove $\|\phi_g\|_{(L^{p(\cdot)}(A,E))^*} \geqslant 1$.

Assume $\lim_{n\to\infty} g_n = g$ on $L^{p'(\cdot)}(A,Y)$, then for $\varepsilon > 0$ be arbitrary, there exist $N_0 > 0$ such that

$$\|\phi_g\| \geqslant \|\phi_{g_n}\| - \|\phi_{g-g_n}\| \geqslant \|\phi_{g_n}\| - \|g - g_n\|_{L^{p'(\cdot)}(A,Y)} \geqslant \|\phi_{g_n}\| - \varepsilon$$

for all $n \geqslant N_0$.

Hence, it suffices to prove for a simple function g satisfying $\|g\|_{L^{p'(\cdot)}(A,Y)} = 1$ such that $\|\phi_g\| \geqslant 1$.

Now, let $g = \sum_{n=1}^{N} \chi_{A_n} \otimes x_n^*$ with A_n pairwise disjoint and $0 < \mu(A_n) < \infty$, $x_n^* \in Y$ are not zero.

For $\varepsilon > 0$, by choosing unit vector $x_n \in E$ such that $\langle x_n^*, x_n\rangle \geqslant (1-\varepsilon)\|x_n^*\|$, we denote $f = \sum_{n=1}^{N} \chi_{A_n} \otimes \|x_n^*\|^{p'(\cdot)-1} x_n$. This implies

$$\begin{aligned}\rho_{p(\cdot)}(f) &= \int_A \varphi_{p(\cdot)}(y, \|f(y)\|)\mathrm{d}\mu(y) = \sum_{n=1}^{N} \int_{A_n} \varphi_{p(\cdot)}(y, \|x_n^*\|^{p'(y)-1})\mathrm{d}\mu(y) \\ &= \sum_{n=1}^{N} \int_{A_n} \|x_n^*\|^{(p'(y)-1)p(y)}\mathrm{d}\mu(y) = \sum_{n=1}^{N} \int_{A_n} \|x_n^*\|^{p'(y)}\mathrm{d}\mu(y) \\ &= \int_A \varphi_{p'(\cdot)}(y, \|g(y)\|)\mathrm{d}\mu(y) = \rho_{p'(\cdot)}(g) \leqslant 1.\end{aligned}$$

By Lemma 1 again, we have that $\|f\|_{L^{p(\cdot)},E)} \leqslant 1$. But

$$\langle \phi_g, f\rangle = \int_A \sum_{n=1}^{N} \chi_{A_n} \|x_n^*\|^{(p'(\cdot)-1)} \langle x_n^*, x_n\rangle \mathrm{d}\mu(y) \geqslant (1-\varepsilon) \int_A \|x_n^*\|^{p'(\cdot)}\mathrm{d}\mu(y) = 1-\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\|\phi_g\| \geqslant 1$. □

Now, it is the position to prove Theorem 2.

Proof of Theorem 2. Let $p \in \mathscr{P}(A,\mu)$ with $p^+ < \infty$. For a fixed $\Lambda \in (L^{p(\cdot)}(A,E))^*$, let $(S^{(n)})_{n\geqslant 1}$ be an approximate sequence of A, we denote the mapping $F_n : \mathscr{A} \to E^*$ by

$$\langle F_n(S), x\rangle = \langle \Lambda, \chi_{S\cap S^{(n)}} \otimes x\rangle, S \in \mathscr{A}, x \in E.$$

This implies

$$\begin{aligned}|\langle F_n(S), x\rangle| &\leqslant \|\Lambda\| \|x \otimes \chi_{S\cap S^{(n)}}\|_{L^{p(\cdot)}(A,E)} \\ &\leqslant \|x\| |\Lambda\| \min\{(\mu(S\cap S^n))^{1/p^-}, (\mu(S\cap S^{(n)}))^{1/p^+}\}\end{aligned}$$

where it follows $\|\chi_{S\cap S^{(n)}}\|_{L^{p(\cdot)}(A,E)} = inf\{\lambda > 0 : \int_A (\frac{1}{\lambda})^{p(y)} \|\chi_{S\cap S^{(n)}}\|^{p(y)}\mathrm{d}\mu(y)\}$. Hence, F_n is absolutely continuous with respect to μ. Moreover, for unit vectors $x_j \in E$ and $S = \bigcup_{j=1}^{k} A_j$ with A_j pairwise disjoint, it has

$$\begin{aligned}\left|\sum_{j=1}^{k} \langle F_n(A_j), x_j\rangle\right| &= \left|\langle \Lambda, \sum_{j=1}^{k} \chi_{A_j\cap S^{(n)}} \otimes x_j\rangle\right| \\ &\leqslant \|\Lambda\| \left\|\sum_{j=1}^{k} \chi_{A_j\cap S^{(n)}} \otimes x_j\right\|_{L^{p(\cdot)}(A,E)} \\ &\leqslant \|\Lambda\| \min\{(\mu(S\cap S^{(n)}))^{1/p^-}, (\mu(S\cap S^{(n)}))^{1/p^+}\},\end{aligned}$$

where the third step follows from the definition of $\|\cdot\|_{L^{p(\cdot)}(A,E)}$.

By taking the supremum of the inequality over unit vectors $x_j \in E$ and all pairwise disjoint measurable sets A_j such that $S = \bigcup_{j=1}^{k} A_j$, it follows

$$\|F_n\|(S) \leqslant \|\Lambda\| \min\{(\mu(S\cap S^{(n)}))^{1/p^-}, (\mu(S\cap S^{(n)}))^{1/p^+}\},$$

In particular, every F_n is of bounded variation.

Since E^* has the Radon-Nikodym property with respect to $(A, \mathscr{A}, \mu)$ and F_n is of bounded variation, these imply that every F_n has a Radon-Nikodym derivative $g_n \in L^1(A, E^*)$, such that

$$F_n(S) = \int_S g_n \mathrm{d}\mu, \forall S \in \mathscr{A}.$$

The function g_n support in $S^{(n)}$ and $g_n|_{S^{(m)}} = g_m|_{S^{(m)}}$, where $m \leqslant n$. Therefore the limit $g = \lim_{n\to\infty} g_n$ is a strongly E^*-value measurable function, and we have

$$\langle \Lambda, \chi_{S\cap S^{(n)}} \otimes x\rangle = |\langle F_n(S), x\rangle| = \int_S \langle g_n, x\rangle \mathrm{d}\mu = \langle g, \chi_{S\cap S^{(n)}} \otimes x\rangle.$$

At last, we will prove that $g \in L^{p'(\cdot)}(A, E^*)$. Note the mapping

$$\phi_g^n : f \mapsto \int_{A_n} \langle g, f\rangle \mathrm{d}\mu, \mathrm{where} A_n = S^{(n)} \cup \{\|g\| \leqslant n\}$$

is a bounded linear functional on $L^{p(\cdot)}(A, E)$.

The functional ϕ_g^n is the same as the simple function which supports in A_n, then they are same on $L^{p(\cdot)}(A_n, E)$ by denseness. Hence, the bounded function $g|_{A_n}$ as an element in $L^{p'(\cdot)}(A_n, E^*)$ can represent functional $\Lambda|_{L^{p(\cdot)}(A_n,E)}$. Using Lemma 6, we have

$$\|g\|_{L^{p'(\cdot)}(A_n,E^*)} \leqslant \sup_{\|f\|_{L^{p(\cdot)}(A_n,E)} \leqslant 1} |\langle \phi_g, f\rangle| = \|\Lambda\|_{L^{p'(\cdot)}(A_n,E^*)} \leqslant \|\Lambda\|.$$

By the monotone convergence theorem,

$$\|g\|_{L^{p'(\cdot)}(A,E^*)} \leqslant \|\Lambda\|,$$

for $A_n \uparrow A$ as $n \to \infty$. Note that $\langle \Lambda, \chi_{A_n} f\rangle = \langle g, \chi_{A_n} f\rangle$, using dominated convergence theorem and Hölder inequality, we have $\langle \Lambda, f\rangle = \langle g, f\rangle$, for all $f \in L^{p(\cdot)}(A, E)$. Thus we have proved $g \in L^{p'(\cdot)}(A, E^*)$. □

Now, we turn to prove Theorem 3. To do so, we need the following lemma; see Lemma 5.2.9 in [14].

LEMMA 7. *For each p such that $1 < p < \infty$ and each function $\lambda : (0,2] \to (0,1]$, there is a function $\gamma_{p,\lambda} : (0,2] \to (0,1]$ such that if X is a uniformly convexity normed space whose modulus of convexity δ_X has the property that $\lambda(\varepsilon) \leqslant \delta_X(\varepsilon)$ when $0 < \varepsilon \leqslant 2$, then*

$$\|(x+y)/2\|^p \leqslant (1-\gamma_{p,\lambda(t)})(\|x\|^p + \|y\|^p)/2,$$

whenever $0 < t \leqslant 2$ and x and y are members of X such that $\|x-y\| \geqslant t \max\{\|x\|, \|y\|\}$.

DEFINITION 13. A semimodular ρ on X is called uniformly convex if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho\left(\frac{f-g}{2}\right) \leqslant \varepsilon \frac{\rho(f)+\rho(g)}{2} \text{ or } \rho\left(\frac{f+g}{2}\right) \leqslant (1-\delta)\frac{\rho(f)+\rho(g)}{2}$$

for all $f,g \in X_\rho$.

LEMMA 8. *Let E be a uniformly convex Banach space. Let*

$$\rho_{p(\cdot)}(f) := \int_A \|f(y)\|^{p(y)} \mathrm{d}\mu(y)$$

for any $f \in L^0(A,E)$. Then $\rho_{p(\cdot)}$ is uniformly convex on $L^0(A,E)$.

Proof. Let $0 < \varepsilon < 1$. There is nothing to show if $\rho_{p(\cdot)}(f) = \infty$ or $\rho_{p(\cdot)}(g) = \infty$. So in the following let $\rho_{p(\cdot)}(f)$, $\rho_{p(\cdot)}(g) < \infty$, which implies by convexity $\rho_{p(\cdot)}(\frac{f+g}{2})$, $\rho_{p(\cdot)}(\frac{f-g}{2}) < \infty$. Assume that $\rho_{p(\cdot)}(\frac{f-g}{2}) > \varepsilon\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2}$. we show that

$$\rho_{p(\cdot)}\left(\frac{f+g}{2}\right) \leqslant \left(1-\frac{\delta_1\varepsilon}{2}\right)\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2}, \quad \text{for some } 0<\delta_1<1,$$

which proves that $\rho_{p(\cdot)}$ is uniformly convex. Define

$$D := \{y \in A : \ \|f(y)-g(y)\| > \frac{\varepsilon}{2}\max\{\|f(y)\|, \|g(y)\|\}\}.$$

Since $p^- > 1$, by Lemma 7, there exists $0 < \delta_1 < 1$ such that

$$\left\|\frac{f(y)+g(y)}{2}\right\|^{p^-} \leqslant (1-\delta_1)\frac{\|f(y)\|^{p^-}+\|g(y)\|^{p^-}}{2}, \text{ for all } y \in D.$$

This and the convexity of $s \to s^{\frac{p(y)}{p^-}}$, $s \geqslant 0$, for $y \in A$, imply

$$\begin{aligned}\left\|\frac{f(y)+g(y)}{2}\right\|^{p(y)} &\leqslant \left((1-\delta_1)\frac{\|f(y)\|^{p^-}+\|g(y)\|^{p^-}}{2}\right)^{\frac{p(y)}{p^-}} \\ &\leqslant (1-\delta_1)\frac{\|f(y)\|^{p(y)}+\|g(y)\|^{p(y)}}{2}.\end{aligned}$$

Thus

$$\rho_{p(\cdot)}\left(\chi_D\frac{f+g}{2}\right) \leqslant (1-\delta_1)\frac{\rho_{p(\cdot)}(\chi_D f)+\rho_{p(\cdot)}(\chi_D g)}{2} \tag{1}$$

Then for almost all $y \in A \backslash D$, we have

$$\|\frac{f(y)-g(y)}{2}\| \leqslant \frac{\varepsilon}{4}\max\{\|f(y)\|, \|g(y)\|\} \leqslant \frac{\varepsilon}{2}\frac{\|f(y)\|+\|g(y)\|}{2}.$$

Thus we obtain that

$$\rho_{p(\cdot)}\left(\chi_{A\backslash D}\frac{f-g}{2}\right) \leqslant \frac{\varepsilon}{2}\frac{\rho_{p(\cdot)}(\chi_{A\backslash D}f)+\rho_{p(\cdot)}(\chi_{A\backslash D}g)}{2} \leqslant \frac{\varepsilon}{2}\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2}.$$

This and $\rho_{p(\cdot)}(\frac{f-g}{2}) > \varepsilon\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2}$ imply

$$\rho_{p(\cdot)}\left(\chi_D\frac{f-g}{2}\right) = \rho_{p(\cdot)}\left(\frac{f-g}{2}\right) - \rho_{p(\cdot)}\left(\chi_{A\backslash D}\frac{f-g}{2}\right) > \frac{\varepsilon}{2}\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2} \tag{2}$$

Since $\frac{1}{2}[\rho_{p(\cdot)}(\chi_{A\backslash D}f)+\rho_{p(\cdot)}(\chi_{A\backslash D}g)] - \rho_{p(\cdot)}(\chi_{A\backslash D}\frac{f+g}{2}) \geqslant 0$, by splitting the domain of integrals into the sets D and $A\backslash D$, the convexity and (1) and (2), we have

$$\begin{aligned}\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2} - \rho_{p(\cdot)}\left(\frac{f+g}{2}\right) &\geqslant \delta_1\frac{\rho_{p(\cdot)}(\chi_D f)+\rho_{p(\cdot)}(\chi_D g)}{2}\\ &\geqslant \delta_1\rho_{p(\cdot)}\left(\chi_D\frac{f-g}{2}\right)\\ &\geqslant \frac{\delta_1\varepsilon}{2}\frac{\rho_{p(\cdot)}(f)+\rho_{p(\cdot)}(g)}{2}.\end{aligned}$$

This finishes the proof. □

DEFINITION 14. A semimodular ρ on X is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that $\rho(2f) \leqslant K\rho(f)$ for all $y \in X_\rho$.

The following lemma is Theorem 2.4.14 in [5].

LEMMA 9. *If a semimodular ρ on X is uniformly convex and satisfies the Δ_2-condition, then $(X_\rho, \|\cdot\|_\rho)$ is uniformly convex.*

Thus, we can give the proof of Theorem 3.

Proof of Theorem 3. Since $\rho_{p(\cdot)}$ satisfies the Δ_2-condition, Theorem 3 follows from this, Lemma 8 and Lemma 9. □

To prove Corollary 2, we need the following lemmas, which are well known; see [14].

LEMMA 10. *Every normed space that is isometrically isomorphic to a uniformly smooth normed space is itselfy uniformly smooth.*

LEMMA 11. *A normed space is uniformly convexity if and only if its dual space is uniformly smooth, and is uniformly smooth if and only if its dual space is uniformly convexity.*

LEMMA 12. *Every uniformly smooth Banach space is reflexive.*

Therefore, we can give the proof of Corollary 2.

Proof of Corollary 2. Since E is uniformly smooth, E^* is uniformly convex by Lemma 11. Thus $L^{p'(\cdot)}(A,E^*)$ is uniformly convex by Theorem 3. By Lemma 11 again, $(L^{p'(\cdot)}(A,E^*))^*$ is uniformly smooth. From Lemma 10, $L^{p(\cdot)}(A,(E^*)^*)$ is uniformly smooth. By Lemma 12, $(E^*)^* = E$. Therefore $L^{p(\cdot)}(A,E)$ is uniformly smooth. □

Finally we turn to prove Theorem 4. To this end, we need some lemmas. The following one is well known, for example, see [14].

LEMMA 13. *Let X be a Banach space and Y be either a closed subspace of X or a Cartesian product X^N, $N \in \mathbb{N}$.*

(i) *If X is reflexive, then Y is also reflexive.*

(ii) *If X is separable, then Y is also separable.*

(iii) *If X is uniformly convex, then Y is also uniformly convex.*

REMARK 1. Let X_j be Banach spaces for $j = 1,\ldots,N$. If for $j = 1,\ldots,N$, X_j are reflexive, separable and uniformly convex, respectively, then Lemma 13 holds also for Cartesian product $X_1 \times \ldots \times X_N$. There are many equivalent norms on $X_1 \times \ldots \times X_N$. For example, let $1 \leqslant p \leqslant \infty$, for any $(x_1,\ldots,x_N) \in X_1 \times \ldots \times X_N$, define

$$\|(x_1,\ldots,x_N)\|_p = \left(\sum_{j=1}^{N} \|x_j\|_{X_j}^p\right)^{1/p},$$

then $\|\cdot\|_p$ are equivalent norms on $X_1 \times \ldots \times X_N$. Indeed, if $(X_j, \|\cdot\|_j)$ are uniformly convex for $j = 1,\ldots,N$, then $(X_1 \times \ldots \times X_N, \|\cdot\|_2)$ is uniformly convex, see Theorem 5.2.25 in [14].

The following lemma is Lemma 2.4.16 in [5].

LEMMA 14. *If ρ_1,ρ_2 are uniformly convex modular on X, then $\rho := \rho_1 + \rho_2$ is also a uniformly convex modular on X.*

Finally, we can give the proof of Theorem 4.

Proof of Theorem 4. We only prove the results for the case $k = 1$, for $k > 1$ the proof is similar.

(i) First we show the Sobolev space is a Banach space. Let (u_i) be a Cauchy sequence in $W^{1,p(\cdot)}(\Omega,E)$. We show there exists $u \in W^{1,p(\cdot)}(\Omega,E)$ such that $u_i \to u$ in $W^{1,p(\cdot)}(\Omega,E)$ as $i \to \infty$.

By Theorem 1, $L^{p(\cdot)}(\Omega,E)$ is a Banach space, therefore there exist $u, g_1,\ldots,g_n \in L^{p(\cdot)}(\Omega,E)$ such that $u_i \to u$, and for $j = 1,\ldots,n$, $\partial_j u_i \to g_j$. Suppose $\psi \in C_0^\infty(\Omega)$. Since $u_i \in W^{1,p(\cdot)}(\Omega,E)$, we have

$$\int_\Omega u_i \partial_j \psi \mathrm{d}x = -\int_\Omega \psi \partial_j u_i \mathrm{d}x.$$

Since strong convergence in $L^{p(\cdot)}(\Omega,E)$ implies weak convergence in $L^{p(\cdot)}(\Omega,E)$, so if $i\to\infty$, we have

$$\int_\Omega u_i\partial_j\psi \mathrm{d}x \to \int_\Omega u\partial_j\psi \mathrm{d}x$$

and

$$\int_\Omega \psi\partial_j u_i \mathrm{d}x \to \int_\Omega \psi g_j \mathrm{d}x,$$

these mean $(g_1,\dots g_n)$ are u's weak derivatives. Thus $u\in W^{1,p(\cdot)}(\Omega,E)$ and $u_j\to u$ in $W^{1,p(\cdot)}(\Omega,E)$.

(ii) Let $p(\cdot)$ is bounded and E is separable. Note that the Lebesgue measure over $\mathbb{R}^n$ is separable, then similar to the constant exponent case we can obtain that $L^{p(\cdot)}(\Omega,E)$ is also separable and we omit the detail here.

(iii) If $1<p^-\leqslant p^+<\infty$, and E is reflexive, by Corollary 1, $L^{p(\cdot)}(\Omega,E)$ is reflexive. By the mapping $u\mapsto(u,\nabla u)$, $W^{1,p(\cdot)}(\Omega,E)$ is isomorphic a closed subspace of $L^{p(\cdot)}(\Omega,E)\times(L^{p(\cdot)}(\Omega,E))^n$. From Lemma 13, $W^{1,p(\cdot)}(\Omega,E)$ is also reflexive.

(iv) Let $1<p^-\leqslant p^+<\infty$ and E is uniformly convex. Remark that the semimodular on $W^{1,p(\cdot)}(\Omega,E)$ satisfies the $\triangle_2$ in this case. By Lemma 8, the semimodular ρ on $L^{p(\cdot)}(\Omega,E)$ is uniformly convex, again from Lemma 14, the semimodular on $W^{1,p(\cdot)}(\Omega,E)$ is uniformly convex. Thus by Lemma 9 $W^{1,p(\cdot)}(\Omega,E)$ is uniformly convex. □

REMARK 2. The (iv) of Theorem 4 can also obtained by Lemma 13.

Acknowledgements. The corresponding author Jing Shi XU is supported by the National Natural Science Foundation of China (Grant No. 11071064 and 11226167) and the National Natural Science Foundation of Hainan Providence (111006 and 131004). The authors would like to thank the referee for his valuable comments and suggestions for the improvement of the manuscript.

REFERENCES

[1] A. ALMEIDA, P. HÄSTÖ, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258** (2010), 1628–1655.

[2] A. ALMEIDA, J. HASANOV, S. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J. **15** (2008), 195–208.

[3] H. AMANN, *Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory*, Birkhäuser, Basel, 1995.

[4] L. DIENING, P. HÄSTÖ, S. ROUDENKO, *Function spaces of variable smoothness and integrablity*, J. Funct. Anal. **256** (2009), 1731–1768.

[5] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, M. RŮŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics 2017, Springer-Verlag, Berlin, 2011.

[6] J. DIESTEL, J. J. UHL, *Vector Measures*, Amer. Math. Soc., 1977.

[7] P. HÄSTÖ, *Local-to-global results in variable exponent spaces*, Math. Res. Lett. **16** (2009), 263–278.

[8] P. HARJULEHTO, P. HÄSTÖ, U. V. LE, M. NUORTIO, *Overview of differential equations with nonstandard growth*, Nonlinear Anal. **72** (2010), 4551–4574.

[9] S. KAKUTANI, *Weak convergence in uniformly convex spaces*, Tohoku Math. J. **45** (1938), 188–193.

[10] H. KEMPKA, *2-microlocal Besov and Triebel-Lizorkin spaces of variable integrability*, Rev. Mat. Complut, **22** (2009), 227–251.

[11] H. KEMPKA, *Atomic, molecular and wavelet decomposition of generalized 2-microlocal Besov spaces*, J. Funct. Spaces Appl. **8** (2010), 129–165.

[12] O. KOVÁČIK, J. RÁKOSNÍK, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czech Math. J. **41** (1991), 592–618.

[13] P. C. KUNSTMANN, L. WEIS, *Maximal-regularity for parabolic equations, Fourier multiplier theorems and* H^∞*-functional calculus, Functional analytic methods for evolution equations*, Lecture Notes in Math. 1855, Springer, Berlin, 65–311, 2004.

[14] R. E. MEGGINSON, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics 183, Springer-Verlag, 1998.

[15] J. XU, *Variable Besov and Triebel-Lizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 511–522.

[16] J. XU, *An atomic decomposition of variable Besov and Triebel-Lizorkin spaces*, Armenian J. Math. **2** (2009), 1–12.

(Received April 19, 2012)

Chen Cheng
Liuzhou Railway Vocational Technical College
Liuzhou 545007, China

Jingshi Xu
Department of Mathematics, Hainan Normal University
Haikou 571158, China
e-mail: jingshixu@126.com
(the corresponding author)