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THE EQUIVALENCE OF CONVERGENCE THEOREMS OF ISHIKAWA–MANN ITERATIONS WITH ERRORS FOR Φ–CONTRACTIVE MAPPINGS IN UNIFORMLY SMOOTH BANACH SPACES

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Abstract. Let E be an arbitrary uniformly smooth real Banach space, D be a nonempty closed convex subset of E, and $T : D \to D$ a generalized Lipschitz Φ-contractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions: (i) $a_n + c_n \leqslant 1$, $b_n + d_n \leqslant 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$; (iii) $c_n = o(a_n)$; (iv) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some $x_0, z_0 \in D$, let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be any bounded sequences in D, and $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.1) and (1.2), respecitvely. Then the convergence of (1.1) is equivalent to that of (1.2).

1. Introduction and preliminary

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that

(i) If E is a smooth Banach space, then the mapping J is single-valued and $J(\alpha x) = \alpha J(x)$ for all $x \in E$ and $\alpha \in \Re$;

(ii) If E is a uniformly smooth Banach space, then the mapping J is uniformly continuous on any bounded subset of E. We denote the single-valued normalized duality mapping by j.

DEFINITION 1.1. [1] Let D be a nonempty closed convex subset of E, $T : D \to D$ be a mapping.

(1) T is called strongly pseudocontractive if there is a constant $k \in (0,1)$ such that for all $x, y \in D$,

$$\langle Tx - Ty, j(x-y)\rangle \leqslant k\|x-y\|^2.$$

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(2) T is called ϕ-strongly pseudocontractive if for all $x,y \in D$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

(3) T is called Φ-pseudocontractive if for all $x,y \in D$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing continuous function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \|x-y\|^2 - \Phi(\|x-y\|).$$

It is obvious that Φ-pseudocontractive mappings not only include ϕ-strongly pseudocontractive mappings, but also include strongly pseudocontractive mappings. Closely related to the class of pseudocontractive-type mappings are those of accretive types.

DEFINITION 1.2. [1] The mapping $T : E \to E$ is called strongly accretive if and only if $I - T$ is strongly pseudocontractive; T is called ϕ-strongly accretive if and only if $I - T$ is ϕ-strongly pseudocontractive; T is called Φ-accretive if and only if $I - T$ is Φ-pseudocontractive.

DEFINITION 1.3. [4] For arbitrary given $x_0 \in D$, Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} y_n = (1 - b_n - d_n)x_n + b_n T x_n + d_n w_n, \\ x_{n+1} = (1 - a_n - c_n)x_n + a_n T y_n + c_n v_n, \ \forall n \geqslant 0, \end{cases} \tag{1.1}$$

where $\{v_n\}$, $\{w_n\}$ are any bounded sequences in D; $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0,1]$ and satisfy $a_n + c_n \leqslant 1$, $b_n + d_n \leqslant 1$ for all $n \geqslant 0$. If $b_n = d_n = 0$, we define Mann iterative process with errors $\{z_n\}$ by

$$z_{n+1} = (1 - a_n - c_n)z_n + a_n T z_n + c_n u_n, n \geqslant 0, \tag{1.2}$$

where $\{u_n\}$ is any bounded sequence in D.

DEFINITION 1.4. [5] A mapping $T : D \to D$ is called generalized Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leqslant L(1 + \|x - y\|)$, $\forall x,y \in D$.

The aim of this paper is to prove the equivalence of convergent results of above Ishikawa and Mann iterations with errors for generalized Lipschitz Φ-contractive mappings in uniformly smooth real Banach spaces. For this, we need the following lemmas.

LEMMA 1.5. [1] *Let E be a uniformly smooth real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, J(x+y)\rangle \tag{1.3}$$

for all $x,y \in E$.

LEMMA 1.6. [3] *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality*

$$\rho_{n+1} \leqslant (1 - \lambda_n)\rho_n + \sigma_n, n \geqslant 0, \tag{1.4}$$

where $\lambda_n \in [0,1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sigma_n = o(\lambda_n)$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main results

THEOREM 2.1. *Let E be an arbitrary uniformly smooth real Banach space, D be a nonempty closed convex subset of E, and $T: D \to D$ be a generalized Lipschitz Φ-pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions (i) $a_n + c_n \leqslant 1$, $b_n + d_n \leqslant 1$; (ii) a_n, b_n, $d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (ii) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some x_0, $z_0 \in D$, let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be any bounded sequences in D, and $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.1) and (1.2), respectively. Then the following conclusions are the equivalent*

(i) (1.1) converges strongly to the unique fixed point q of T;

(ii) (1.2) converges strongly to the unique fixed point q of T.

Proof. (i)$\Rightarrow$(ii) is obvious. We only show that (ii)$\Rightarrow$(i). Since $T: D \to D$ is a generalized Lipschitz Φ-pseudocontractive mapping, there exists a strictly increasing continuous function $\Phi: [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, J(x-y)\rangle \leqslant \|x-y\|^2 - \Phi(\|x-y\|),$$

i.e.,

$$\langle (I-T)x - (I-T)y, J(x-y)\rangle \geqslant \Phi(\|x-y\|) \tag{2.1}$$

and

$$\|Tx - Ty\| \leqslant L(1 + \|x-y\|)$$

for any $x, y \in D$.

Step 1. There exists $x_0 \in D$ with $x_0 \neq Tx_0$ such that $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$ (range of Φ). Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r): r \in [0,+\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then for $q \in D$, there exists a sequence $\{v_n\}$ in D such that $v_n \to q$ as $n \to \infty$ with $v_n \neq q$. Furthermore, we obtain that $\{v_n - Tv_n\}$ is bounded. Hence there exists a natural number n_0 such that $\|v_n - Tv_n\| \cdot \|v_n - q\| < \frac{r_1}{2}$ for $n \geqslant n_0$, then we redefine $x_0 = v_{n_0}$ and $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$.

Step 2. For any $n \geqslant 0$, $\{x_n\}$ is bounded. Set $R = \Phi^{-1}(r_0)$. From (2.1), we have

$$\langle x_0 - Tx_0, J(x_0 - q)\rangle \geqslant \Phi(\|x_0 - q\|), \tag{2.2}$$

i.e., $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \geqslant \Phi(\|x_0 - q\|)$. Thus we obtain that $\|x_0 - q\| \leqslant R$. Denote $B_1 = \{x \in D: \|x - q\| \leqslant R\}$, $B_2 = \{x \in D: \|x - q\| \leqslant 2R\}$. Since T is generalized Lipschitz, then T is bounded. We may define

$$M = \sup_{x \in B_2}\{\|Tx - q\| + 1\} + \sup_n\{\|v_n - q\|\} + \sup_n\{\|w_n - q\|\}.$$

Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now we assume that it holds for some n, i.e., $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose that it is not the case, then $\|x_{n+1} - q\| > R$. Since J is uniformly continuous on bounded subset of E, then for $\varepsilon_0 = \frac{\Phi(\frac{R}{4})}{24L(1+2R)}$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \varepsilon$ when $\|x - y\| < \delta$, $\forall x, y \in B_2$. Now denote

$$\tau_0 = \min\left\{\frac{R}{2[L(1+2R)+2R+M]}, \frac{R}{4[L(1+R)+2R+M]}, \frac{\delta}{2[L(1+2R)+2R+M]}, \frac{\Phi(\frac{R}{4})}{24R^2}, \frac{\Phi(\frac{R}{4})}{24L(1+2R)}, \frac{\Phi(\frac{R}{4})}{48MR}\right\}.$$

Since a_n, b_n, c_n, $d_n \to 0$ as $n \to \infty$, without loss of generality, we assume that $0 \leqslant a_n, b_n, c_n, d_n \leqslant \tau_0$ for any $n \geqslant 0$. Since $c_n = o(a_n)$, denote $c_n < a_n\tau_0$. So we have

$$\|Tx_n - q\| \leqslant L(1 + \|x_n - q\|) \leqslant L(1+R), \tag{2.3}$$

$$\begin{aligned}\|y_n - q\| &\leqslant (1 - b_n - d_n)\|x_n - q\| + b_n\|Tx_n - q\| + d_n\|w_n - q\| \\ &\leqslant R + b_nL(1 + \|x_n - q\|) + d_nM \leqslant R + b_nL(1+R) + d_nM \\ &\leqslant R + \tau_0[L(1+R)+M] \leqslant 2R,\end{aligned} \tag{2.4}$$

$$\|Ty_n - q\| \leqslant L(1 + \|y_n - q\|) \leqslant L(1+2R), \tag{2.5}$$

$$\|x_n - Tx_n\| \leqslant L + (1+L)\|x_n - q\| \leqslant L + (1+L)R \tag{2.6}$$

and

$$\begin{aligned}\|(x_n - q) - (y_n - q)\| &\leqslant b_n\|x_n - Tx_n\| + d_n[\|w_n - q\| + \|x_n - q\|] \\ &\leqslant b_n[L + (1+L)R] + d_n(M+R) \leqslant \tau_0[L(1+R) + 2R + M] \\ &\leqslant \tau_0[L(1+2R) + 2R + M] \leqslant \frac{\delta}{2} < \delta,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\|x_n - q\| &\geqslant \|x_{n+1} - q\| - a_n\|Ty_n - x_n\| - c_n\|v_n - x_n\| \\ &\geqslant \|x_{n+1} - q\| - a_n[\|Ty_n - q\| + \|x_n - q\|] - c_n[\|x_n - q\| + \|v_n - q\|] \\ &\geqslant R - a_n[L(1+2R) + R] - c_n(R+M) \\ &\geqslant R - \tau_0[L(1+2R) + M + 2R] \geqslant R - \frac{R}{2} = \frac{R}{2},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\|y_n - q\| &\geqslant \|x_n - q\| - b_n\|Tx_n - x_n\| - d_n\|x_n - w_n\| \\ &\geqslant \|x_n - q\| - b_n[L + (1+L)R] - d_n[\|x_n - q\| + \|w_n - q\|] \\ &\geqslant \|x_n - q\| - b_n[L + (1+L)R] - d_n(R+M) \\ &\geqslant \|x_n - q\| - \tau_0[L + (2+L)R + M] > \frac{R}{2} - \frac{R}{4} = \frac{R}{4},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\|x_{n+1} - q\| &\leqslant (1 - a_n - c_n)\|x_n - q\| + a_n\|Ty_n - q\| + c_n\|v_n - q\| \\ &\leqslant R + \tau_0[L(1+2R) + M] \leqslant 2R,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\|(x_{n+1} - q) - (x_n - q)\| &\leqslant a_n\|Ty_n - x_n\| + c_n\|u_n - x_n\| \\ &\leqslant a_n[\|Ty_n - q\| + \|x_n - q\|] + c_n[\|v_n - q\| + \|x_n - q\|] \\ &\leqslant a_n[L(1+2R) + R] + c_n(M+R) \\ &\leqslant \tau_0[L(1+2R) + 2R + M] \leqslant \frac{\delta}{2} < \delta.\end{aligned} \tag{2.11}$$

Therefore,

$$\|J(x_n-q)-J(y_n-q)\|<\varepsilon_0,$$
$$\|J(x_{n+1}-q)-J(x_n-q)\|<\varepsilon_0.$$

Using Lemma 1.5 and above formulas, we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leqslant (1-a_n-c_n)^2\|x_n-q\|^2+2a_n\langle Ty_n-q,J(x_{n+1}-q)\rangle \\
&\quad +2c_n\langle u_n-q,J(x_{n+1}-q)\rangle \\
&\leqslant (1-a_n)^2\|x_n-q\|^2+2a_n\langle Ty_n-q,J(x_{n+1}-q)-J(x_n-q)\rangle \\
&\quad +2a_n\langle Ty_n-q,J(x_n-q)-J(y_n-q)\rangle \\
&\quad +2a_n\langle Ty_n-q,J(y_n-q)\rangle+2c_n\langle u_n-q,J(x_{n+1}-q)\rangle \qquad (2.12)\\
&\leqslant (1-a_n)^2\|x_n-q\|^2+2a_n\|Ty_n-q\|\cdot\|J(x_{n+1}-q)-J(x_n-q)\| \\
&\quad +2a_n\|Ty_n-q\|\cdot\|J(x_n-q)-J(y_n-q)\| \\
&\quad +2a_n[\|y_n-q\|^2-\Phi(\|y_n-q\|)]+2c_n\|u_n-q\|\cdot\|x_{n+1}-q\| \\
&\leqslant (1-a_n)^2R^2+4a_nL(1+2R)\varepsilon_0+2a_n[\|y_n-q\|^2-\Phi(\|y_n-q\|)]+4c_nMR
\end{aligned}$$

and

$$\begin{aligned}
\|y_n-q\|^2 &\leqslant (1-b_n-d_n)^2\|x_n-q\|^2+2b_n\langle Tx_n-q,J(y_n-q)\rangle \\
&\quad +2d_n\langle w_n-q,J(y_n-q)\rangle \\
&\leqslant \|x_n-q\|^2+2b_n\langle Tx_n-q,J(y_n-q)-J(x_n-q)\rangle \\
&\quad +2b_n\langle Tx_n-q,J(x_n-q)\rangle+2d_n\|w_n-q\|\cdot\|y_n-q\| \qquad (2.13)\\
&\leqslant \|x_n-q\|^2+2b_n\|Tx_n-q\|\cdot\|J(y_n-q)-J(x_n-q)\| \\
&\quad +2b_n[\|x_n-q\|^2-\Phi(\|x_n-q\|)]+2d_n\|w_n-q\|\cdot\|y_n-q\| \\
&\leqslant R^2+2b_nL(1+R)\varepsilon_0+2b_nR^2+4d_nMR.
\end{aligned}$$

Substitute (2.13) into (2.12)

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leqslant (1-a_n)^2R^2+4a_nL(1+2R)\varepsilon_0+2a_n[R^2+2b_nL(1+R)\varepsilon_0 \\
&\quad +2b_nR^2+4d_nMR]-2a_n\Phi(\|y_n-q\|)+4c_nMR \\
&\leqslant R^2+a_n^2R^2+4a_nL(1+2R)\varepsilon_0+2a_n[2b_nL(1+R)\varepsilon_0 \\
&\quad +2b_nR^2+4d_nMR]-2a_n\Phi\left(\frac{R}{4}\right)+4c_nMR \\
&= R^2+2a_n\left[\frac{a_n}{2}R^2+2L(1+2R)\varepsilon_0+2b_nL(1+R)\varepsilon_0\right. \qquad (2.14)\\
&\quad \left.+2b_nR^2+4d_nMR+\frac{2c_nMR}{a_n}\right]-2a_n\Phi\left(\frac{R}{4}\right) \\
&\leqslant R^2+2a_n\left[\frac{\Phi\left(\frac{R}{4}\right)}{2}-\Phi\left(\frac{R}{4}\right)\right] \\
&\leqslant R^2-\Phi\left(\frac{R}{4}\right)a_n\leqslant R^2
\end{aligned}$$

which is a contradiction. Thus $x_{n+1} \in B_1$, i.e., $\{x_n\}$ is a bounded sequence. So $\{y_n\}$, $\{Ty_n\}$, $\{Tx_n\}$ are all bounded sequences. Since $\|z_n - q\| \to 0$ as $n \to \infty$, without loss of generality, we let $\|z_n - q\| \leqslant 1$. Therefore $\|x_n - z_n\|$ is also bounded.

Step 3. We want to prove that $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Set $M_0 = \max\{\sup_n \|Ty_n - Tz_n\|,\ \sup_n \|v_n - u_n\|,\ \sup_n \|x_n - z_n\|,\ \sup_n \|Tx_n - x_n\|,\ \sup_n \|w_n - x_n\|,\ \sup_n \|y_n - z_n\|,\ \sup_n \|v_n - x_n\|\}$. By Lemma 1.5, we have

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\|^2 &\leqslant (1 - a_n - c_n)^2 \|x_n - z_n\|^2 + 2a_n \langle Ty_n - Tz_n, J(x_{n+1} - z_{n+1})\rangle \\
&\quad + 2c_n \langle v_n - u_n, J(x_{n+1} - z_{n+1})\rangle \\
&\leqslant (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n \langle Ty_n - Tz_n, J(x_{n+1} - z_{n+1}) - J(x_n - z_n)\rangle \\
&\quad + 2a_n \langle Ty_n - Tz_n, J(x_n - z_n) - J(y_n - z_n)\rangle \\
&\quad + 2a_n \langle Ty_n - Tz_n, J(y_n - z_n)\rangle + 2c_n \|v_n - u_n\| \cdot \|x_{n+1} - z_{n+1}\| \\
&\leqslant (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n \\
&\quad + 2a_n [\|y_n - z_n\|^2 - \Phi(\|y_n - z_n\|)] + 2c_n M_0^2
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
\|y_n - z_n\|^2 &\leqslant \|x_n - z_n\|^2 + 2b_n \langle Tx_n - x_n, J(y_n - z_n)\rangle + 2d_n \langle w_n - x_n, J(y_n - z_n)\rangle \\
&\leqslant \|x_n - z_n\|^2 + 2b_n M_0^2 + 2d_n M_0^2,
\end{aligned} \tag{2.16}$$

where $A_n = \|J(x_{n+1} - z_{n+1}) - J(x_n - z_n)\|, B_n = \|J(x_n - z_n) - J(y_n - z_n)\|$ with $A_n, B_n \to 0$ as $n \to \infty$.

Taking place (2.16) into (2.15),

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\|^2 &\leqslant (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n \\
&\quad + 2a_n [\|x_n - z_n\|^2 + 2b_n M_0^2 + 2d_n M_0^2 - \Phi(\|y_n - z_n\|)] + 2c_n M_0^2 \\
&\leqslant \|x_n - z_n\|^2 + a_n^2 M_0^2 + 2a_n M_0 A_n + 2a_n M_0 B_n + 4a_n b_n M_0^2 + 4a_n d_n M_0^2 \\
&\quad - 2a_n \Phi(\|y_n - z_n\|) + 2c_n M_0^2 \\
&= \|x_n - z_n\|^2 + 2a_n [C_n - 2a_n \Phi(\|y_n - z_n\|)],
\end{aligned} \tag{2.17}$$

where $C_n = \frac{a_n M_0^2}{2} + M_0 A_n + M_0 B_n + 2b_n M_0^2 + 2d_n M_0^2 + \frac{c_n M_0^2}{a_n} \to 0$ as $n \to \infty$.

Set $\inf_{n \geqslant 0} \frac{\Phi(\|y_n - z_n\|)}{1 + \|x_{n+1} - z_{n+1}\|^2} = \lambda$, then $\lambda = 0$. If it is not the case, we assume that $\lambda > 0$. Let $0 < \gamma < \min\{1, \lambda\}$, then $\frac{\Phi(\|y_n - z_n\|)}{1 + \|x_{n+1} - z_{n+1}\|^2} \geqslant \gamma$, i.e., $\Phi(\|y_n - z_n\|) \geqslant \gamma + \gamma \|x_{n+1} - z_{n+1}\|^2 \geqslant \gamma \|x_{n+1} - z_{n+1}\|^2$. Thus, from (2.17) that

$$\|x_{n+1} - z_{n+1}\|^2 \leqslant \|x_n - z_n\|^2 + 2a_n [C_n - \gamma \|x_{n+1} - z_{n+1}\|^2]. \tag{2.18}$$

It implies that

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\|^2 &\leqslant \frac{1}{1 + 2a_n \gamma} \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma} \\
&= \left(1 - \frac{2a_n \gamma}{1 + 2a_n \gamma}\right) \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma}.
\end{aligned} \tag{2.19}$$

Let $\rho_n = \|x_n - z_n\|^2$, $\lambda_n = \frac{2a_n\gamma}{1+2a_n\gamma}$, $\sigma_n = \frac{2a_nC_n}{1+2a_n\gamma}$. Then we obtain that

$$\rho_{n+1} \leqslant (1-\lambda_n)\rho_n + \sigma_n.$$

Applying Lemma 1.6, then $\rho_n \to 0$ as $n \to \infty$. This is a contradiction and so $\lambda = 0$. Therefore, there exists an infinite subsequence such that $\frac{\Phi(\|y_{n_i}-z_{n_i}\|)}{1+\|x_{n_i+1}-z_{n+1}\|^2} \to 0$ as $i \to \infty$. Since $0 \leqslant \frac{\Phi(\|y_{n_i}-z_{n_i}\|)}{1+M_0^2} \leqslant \frac{\Phi(\|y_{n_i}-z_{n_i}\|)}{1+\|x_{n_i+1}-z_{n+1}\|^2}$, then $\Phi(\|y_{n_i}-z_{n_i}\|) \to 0$ as $i \to \infty$. In view of the strictly increasing continuity of Φ, we have $\|y_{n_i}-z_{n_i}\| \to 0$ as $i \to \infty$. From (1.1), we have

$$\|x_{n_i}-z_{n_i}\| \leqslant \|y_{n_i}-z_{n_i}\| + b_{n_i}\|x_{n_i}-Tx_{n_i}\| + c_{n_i}\|x_{n_i}-w_{n_i}\| \to 0$$

as $i \to \infty$. Next we want to prove that $\|x_n - z_n\| \to 0$ as $n \to \infty$. Let $\forall \varepsilon \in (0,1)$, there exists n_{i_0} such that $\|x_{n_i}-z_{n_i}\| < \varepsilon$, $a_n < \min\{\frac{\varepsilon}{4L(1+M_0)}, \frac{\varepsilon}{8M_0}\}$, $c_n < \frac{\varepsilon}{16M_0}$, b_n, $d_n < \frac{\varepsilon}{8M_0}$, $C_n < \frac{\Phi(\frac{\varepsilon}{4})}{2}$ for any n_i, $n \geqslant n_{i_0}$. First, we want to prove $\|x_{n_i+1}-z_{n_i+1}\| < \varepsilon$. Suppose that it is not this case, then $\|x_{n_i+1}-z_{n_i+1}\| \geqslant \varepsilon$. Using (1.1), we may get the following estimates:

$$\begin{aligned}\|x_{n_i}-z_{n_i}\| &\geqslant \|x_{n_i+1}-z_{n_i+1}\| - a_{n_i}\|Ty_{n_i}-Tz_{n_i}\| - a_{n_i}\|x_{n_i}-z_{n_i}\| \\ &\quad -c_{n_i}\|v_{n_i}-u_{n_i}\| - c_{n_i}\|x_{n_i}-z_{n_i}\| \\ &\geqslant \varepsilon - a_{n_i}L(1+M_0) - (a_{n_i}+2c_{n_i})M_0 \\ &> \frac{\varepsilon}{2}, \qquad (2.20) \\ \|y_{n_i}-z_{n_i}\| &\geqslant \|x_{n_i}-z_{n_i}\| - b_{n_i}\|Tx_{n_i}-x_{n_i}\| - d_{n_i}\|v_{n_i}-x_{n_i}\| \\ &\geqslant \frac{\varepsilon}{2} - (b_{n_i}+d_{n_i})M_0 \\ &> \frac{\varepsilon}{4}. \qquad (2.21)\end{aligned}$$

Since Φ is strictly increasing, then (2.21) leads to $\Phi(\|y_{n_i}-z_{n_i}\|) \geqslant \Phi(\frac{\varepsilon}{4})$. From (2.17), we have

$$\begin{aligned}\|x_{n_i+1}-z_{n_i+1}\|^2 &\leqslant \|x_{n_i}-z_{n_i}\|^2 + 2a_{n_i}[C_{n_i} - \Phi(\|y_{n_i}-z_{n_i}\|)] \\ &< \varepsilon^2 + 2a_{n_i}\Big[\frac{1}{2}\Phi\Big(\frac{\varepsilon}{4}\Big) - \Phi\Big(\frac{\varepsilon}{4}\Big)\Big] \\ &\leqslant \varepsilon^2 - \Phi\Big(\frac{\varepsilon}{4}\Big)a_{n_i} \\ &\leqslant \varepsilon^2, \qquad (2.22)\end{aligned}$$

which is a contradiction. Hence, $\|x_{n_i+1}-z_{n_i+1}\| < \varepsilon$. Suppose that $\|x_{n_i+m}-z_{n_i+m}\| < \varepsilon$ holds. Repeating the above course, we can easily prove that $\|x_{n_i+m+1}-z_{n_i+m+1}\| < \varepsilon$ holds. Therefore, for any m, we obtain that $\|x_{n_i+m}-z_{n_i+m}\| < \varepsilon$, which means $\|x_n - z_n\| \to 0$ as $n \to \infty$. This completes the proof. □

THEOREM 2.2. *Let E be an arbitrary uniformly smooth real Banach space, and $T : E \to E$ be a generalized Lipschitz Φ-accretive mapping with $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ which satisfy the conditions (i) $a_n + c_n \leqslant 1$, $b_n + d_n \leqslant 1$; (ii) a_n, b_n, $d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some x_0, $z_0 \in D$, let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be any bounded sequences in E, and $\{x_n\}$, $\{z_n\}$ be Ishikawa and Mann iterations with errors defined by*

$$\begin{cases} y_n = (1 - b_n - d_n)x_n + b_n S x_n + d_n w_n, \\ x_{n+1} = (1 - a_n - c_n)x_n + a_n S y_n + c_n v_n, \quad n \geqslant 0 \end{cases} \tag{2.23}$$

and

$$z_{n+1} = (1 - a_n - c_n)z_n + a_n S z_n + c_n u_n, \tag{2.24}$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for any $x \in E$. Then the following two assertions are equivalent:

(i) $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$;

(ii) $\{z_n\}$ converges strongly to the unique solution of the equation $Tx = 0$.

Proof. Since T is generalized Lipschitz and Φ-accretive mapping, it follows that

$$\|Tx - Ty\| \leqslant L(1 + \|x - y\|),$$

i.e.,

$$\|Sx - Sy\| \leqslant L_1(1 + \|x - y\|), L_1 = 1 + L;$$

$$\langle Tx - Ty, J(x - y)\rangle \geqslant \Phi(\|x - y\|),$$

i.e.,

$$\langle Sx - Sy, J(x - y)\rangle \leqslant \|x - y\|^2 - \Phi(\|x - y\|),$$

for all $x, y \in E$. Then S is the generalized Lipschitz Φ-pseudocontractive mapping. By Theorem 2.1, we obtain the conclusion of Theorem 2.2. □

REMARK 2.3. It is mentioned that, in 2006, C. E. Chidume and C. O. Chidume [1] proved the approximation theorem for zeros of generalized Lipschitz generalized Φ-quasi-accretive operators. This result provided significant improvements of past known corresponding results. However, there exists a gap in the proof process of Theorem 3.1 of [1], i.e., $\Sigma_{n=0}^{\infty} c_n < +\infty$ *does not implies* $c_n = o(b_n)$. For example, set the iteration parameters: $a_n = 1 - b_n - c_n$, where $\{b_n\}$: $b_1 = 0$, $b_n = \frac{1}{n}$, $n \geqslant 2$; $\{c_n\}$: $1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4}, \frac{1}{5^2}, \cdots, \frac{1}{8^2}, \frac{1}{9}, \frac{1}{10^2}, \cdots, \frac{1}{15^2}, \frac{1}{16}, \frac{1}{17^2}, \cdots, \frac{1}{24^2}, \frac{1}{25}, \frac{1}{26^2}, \cdots$. Up to now, we do not know whether the condition $c_n = o(a_n)$ is taken place by $\Sigma_{n=0}^{\infty} c_n < +\infty$ in Theorem 2.1 and Theorem 2.2.

REFERENCES

[1] C. E. CHIDUME AND C. O. CHIDUME, *Convergence theorem for zeros of generalized Lipschitz generalized Φ-quasi-accretive operators*, Proc. Amer. Math. Soc. **134** (1) (2006) 243–251.

[2] S. S. CHANG, Y. J. CHO AND J. K. KIM, *Some results for uniformly L-Lipschitzian mappings in Banach spaces*, Appl. Math. Lett. doi:10:1016/j.mal.2008.02.016.

[3] X. WENG, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc. **113** (1991) 727–731.

[4] Y. G. XU, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998) 91–101.

[5] H. Y. ZHOU AND D. Q. CHEN, *Iterative approximations of fixed points for nonlinear mappings of ϕ-hemicontractive type in normed linear spaces*, Math. Appl, **11** (3) (1998) 118–121.

[6] K. DEIMLING, *Nonlinear functional analysis*, Springer-Verlag, Berlin, Heiderberg, New York, Tokyo (1980).

[7] S. ISHIKAWA, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1) (1974) 147–150.

[8] W. R. MANN, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953) 506–510.

[9] L. S. LIU, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995) 114–125.

[10] Z. Q. XUE, H. Y. ZHOU AND Y. J. CHO, *Iterative solutions of nonlinear equations for m-accretive operators in Banach space*, J. Nonlinear and Convex Analysis, **1** (3) (2003) 313–320.

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