THE EQUIVALENCE OF CONVERGENCE THEOREMS OF
ISHIKAWA–MANN ITERATIONS WITH ERRORS FOR Φ–CONTRACTION
MAPPINGS IN UNIFORMLY SMOOTH BANACH SPACES

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Abstract. Let $E$ be an arbitrary uniformly smooth real Banach space, $D$ be a nonempty closed convex subset of $E$, and $T : D \to D$ a generalized Lipschitz $\Phi$-contractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions:

(i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$; (iii) $c_n = o(a_n)$; (iv) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0, z_0 \in D$, let $\{u_n\}, \{v_n\}, \{w_n\}$ be any bounded sequences in $D$, and $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.1) and (1.2), respectively. Then the convergence of (1.1) is equivalent to that of (1.2).

1. Introduction and preliminary

Let $E$ be a real Banach space and $E^* \equiv$ its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that

(i) If $E$ is a smooth Banach space, then the mapping $J$ is single-valued and $J(\alpha x) = \alpha J(x)$ for all $x \in E$ and $\alpha \in \mathbb{R}$;

(ii) If $E$ is a uniformly smooth Banach space, then the mapping $J$ is uniformly continuous on any bounded subset of $E$. We denote the single-valued normalized duality mapping by $j$.

DEFINITION 1.1. [1] Let $D$ be a nonempty closed convex subset of $E$, $T : D \to D$ be a mapping.

(1) $T$ is called strongly pseudocontractive if there is a constant $k \in (0,1)$ such that for all $x, y \in D$,

$$\langle Tx - Ty, j(x - y) \rangle \leq k \|x - y\|^2.$$


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(2) $T$ is called $\phi$-strongly pseudocontractive if for all $x, y \in D$, there exist $j(x - y) \in J(x - y)$ and a strictly increasing continuous function $\phi: [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|.$$  

(3) $T$ is called $\Phi$-pseudocontractive if for all $x, y \in D$, there exist $j(x - y) \in J(x - y)$ and a strictly increasing continuous function $\Phi: [0, +\infty) \rightarrow [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|).$$

It is obvious that $\Phi$-pseudocontractive mappings not only include $\phi$-strongly pseudocontractive mappings, but also include strongly pseudocontractive mappings. Closely related to the class of pseudocontractive-type mappings are those of accretive types.

**Definition 1.2.** [1] The mapping $T: E \rightarrow E$ is called strongly accretive if and only if $I - T$ is strongly pseudocontractive; $T$ is called $\phi$-strongly accretive if and only if $I - T$ is $\phi$-strongly pseudocontractive; $T$ is called $\Phi$-accretive if and only if $I - T$ is $\Phi$-pseudocontractive.

**Definition 1.3.** [4] For arbitrary given $x_0 \in D$, Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{align*}
y_n &= (1 - b_n - d_n)x_n + b_nTx_n + d_nw_n, \\
x_{n+1} &= (1 - a_n - c_n)x_n + a_nTy_n + c_nv_n, \quad \forall n \geq 0,
\end{align*}$$  

where $\{v_n\}, \{w_n\}$ are any bounded sequences in $D$; $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are four real sequences in $[0,1]$ and satisfy $a_n + c_n \leq 1$, $b_n + d_n \leq 1$ for all $n \geq 0$. If $b_n = d_n = 0$, we define Mann iterative process with errors $\{z_n\}$ by

$$z_{n+1} = (1 - a_n - c_n)z_n + a_nTz_n + c_nu_n, \quad \forall n \geq 0,$$

where $\{u_n\}$ is any bounded sequence in $D$.

**Definition 1.4.** [5] A mapping $T: D \rightarrow D$ is called generalized Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|x - y\|)$, $\forall x, y \in D$.

The aim of this paper is to prove the equivalence of convergent results of above Ishikawa and Mann iterations with errors for generalized Lipschitz $\Phi$-contractive mappings in uniformly smooth real Banach spaces. For this, we need the following lemmas.

**Lemma 1.5.** [1] Let $E$ be a uniformly smooth real Banach space and let $J: E \rightarrow 2^{E^*}$ be a normalized duality mapping. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$$  

for all $x, y \in E$.

**Lemma 1.6.** [3] Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, \quad n \geq 0,$$

where $\lambda_n \in [0,1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sigma_n = o(\lambda_n)$. Then $\rho_n \rightarrow 0$ as $n \rightarrow \infty$. 

2. Main results

**Theorem 2.1.** Let $E$ be an arbitrary uniformly smooth real Banach space, $D$ be a nonempty closed convex subset of $E$, and $T : D \to D$ be a generalized Lipschitz $\Phi$-pseudocontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n, b_n, d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0, z_0 \in D$, let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be any bounded sequences in $D$, and $\{x_n\}$ and $\{z_n\}$ be Ishikawa and Mann iterative sequences with errors defined by (1.1) and (1.2), respectively. Then the following conclusions are the equivalent

(i) (1.1) converges strongly to the unique fixed point $q$ of $T$;
(ii) (1.2) converges strongly to the unique fixed point $q$ of $T$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious. We only show that (ii) $\Rightarrow$ (i). Since $T : D \to D$ is a generalized Lipschitz $\Phi$-pseudocontractive mapping, there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, J(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|),$$

i.e.,

$$\langle (I - T)x - (I - T)y, J(x - y) \rangle \geq \Phi(\|x - y\|) \tag{2.1}$$

and

$$\|Tx - Ty\| \leq L(1 + \|x - y\|)$$

for any $x, y \in D$.

**Step 1.** There exists $x_0 \in D$ with $x_0 \neq Tx_0$ such that $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$ (range of $\Phi$). Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, +\infty]\} = r_1 < +\infty$ with $r_1 < r_0$, then for $q \in D$, there exists a sequence $\{v_n\}$ in $D$ such that $v_n \to q$ as $n \to \infty$ with $v_n \neq q$. Furthermore, we obtain that $\{v_n - Tv_n\}$ is bounded. Hence there exists a natural number $n_0$ such that $\|v_n - Tv_n\| \cdot \|v_n - q\| < \frac{r_1}{2}$ for $n \geq n_0$, then we redefine $x_0 = v_{n_0}$ and $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$.

**Step 2.** For any $n \geq 0$, $\{x_n\}$ is bounded. Set $R = \Phi^{-1}(r_0)$. From (2.1), we have

$$\langle x_0 - Tx_0, J(x_0 - q) \rangle \geq \Phi(\|x_0 - q\|), \tag{2.2}$$

i.e., $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \geq \Phi(\|x_0 - q\|)$. Thus we obtain that $\|x_0 - q\| \leq R$. Denote $B_1 = \{x \in D : \|x - q\| \leq R\}$, $B_2 = \{x \in D : \|x - q\| \leq 2R\}$. Since $T$ is generalized Lipschitz, then $T$ is bounded. We may define

$$M = \sup_{x \in B_2} \{\|Tx - q\| + 1\} + \sup_n \{\|v_n - q\|\} + \sup_n \{\|w_n - q\|\}.$$

Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now we assume that it holds for some $n$, i.e., $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose that it is not the case, then $\|x_{n+1} - q\| > R$. Since $J$ is uniformly continuous on bounded subset of $E$,
then for $\varepsilon_0 = \frac{\Phi(\frac{\delta}{4})}{24L(1+2R)}$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \varepsilon$ when $\|x - y\| < \delta, \forall x, y \in B_2$. Now denote
\[
\tau_0 = \min \left\{ \frac{R}{2[L(1+2R) + 2R + M]}, \frac{R}{4[L(1+R) + 2R + M]} \right\}.
\]

Since $a_n, b_n, c_n, d_n \to 0$ as $n \to \infty$, without loss of generality, we assume that $0 < a_n, b_n, c_n, d_n \leq \tau_0$ for any $n \geq 0$. Since $c_n = o(a_n)$, denote $c_n < a_n \tau_0$. So we have
\[
\begin{align*}
\|Tx_n - q\| &\leq L(1 + \|x_n - q\|) \leq L(1 + R), \quad (2.3) \\
\|y_n - q\| &\leq (1 - b_n - d_n)\|x_n - q\| + b_n\|Tx_n - q\| + d_n\|w_n - q\|
\leq R + b_n L(1 + \|x_n - q\|) + d_n M \leq R + b_n L(1 + R) + d_n M \quad (2.4) \\
\|Ty_n - q\| &\leq L(1 + \|y_n - q\|) \leq L(1 + 2R), \quad (2.5) \\
\|x_n - Tx_n\| &\leq L + (1 + L)\|x_n - q\| \leq L + (1 + L)R \quad (2.6)
\end{align*}
\]

and
\[
\begin{align*}
\|x_n - q\| &\geq \|x_{n+1} - q\| - a_n\|Ty_n - x_n\| - c_n\|\v_n - x_n\|
\geq \|x_{n+1} - q\| - a_n\|[\|Ty_n - q\| + \|x_n - q\|] - c_n\|[\|x_n - q\| + \|\v_n - q\|]
\geq R - a_n\|L(1 + 2R) + R - c_n\|(R + M) \quad (2.7)
\geq R - \tau_0[L(1 + 2R) + M + 2R] > R - \frac{R}{2} = \frac{R}{2},
\end{align*}
\]
\[
\begin{align*}
\|y_n - q\| &\geq \|x_n - q\| - b_n\|Tx_n - x_n\| - d_n\|x_n - w_n\|
\geq \|x_n - q\| - b_n\|[L + (1 + L)R] - d_n\|[\|x_n - q\| + \|w_n - q\|]
\geq \|x_n - q\| - b_n\|[L + (1 + L)R] - d_n\|(R + M)
\geq \|x_n - q\| - \tau_0[L + (2 + L)R + M] > \frac{R}{2} - \frac{R}{4} = \frac{R}{4},
\end{align*}
\]
\[
\begin{align*}
\|x_{n+1} - q\| &\leq (1 - a_n - c_n)\|x_n - q\| + a_n\|Ty_n - q\| + c_n\|\v_n - q\|
\leq R + \tau_0[L(1 + 2R) + M] \leq 2R, \quad (2.9)
\end{align*}
\]
\[
\begin{align*}
\|(x_{n+1} - q) - (x_n - q)\| &\leq a_n\|Ty_n - x_n\| + c_n\|u_n - x_n\|
\leq a_n\|[\|Ty_n - q\| + \|x_n - q\|] + c_n\|[\|\v_n - q\| + \|x_n - q\|]
\leq a_n\|[L(1 + 2R) + R] + c_n\|(M + R)
\leq \tau_0[L(1 + 2R) + 2R + M] \leq \frac{\delta}{2} < \delta. \quad (2.10)
\end{align*}
\]
Therefore,
\[
\|J(x_n - q) - J(y_n - q)\| < \varepsilon_0,
\]
\[
\|J(x_{n+1} - q) - J(x_n - q)\| < \varepsilon_0.
\]

Using Lemma 1.5 and above formulas, we obtain
\[
\|x_{n+1} - q\|^2 \leq (1 - a_n - c_n)^2 \|x_n - q\|^2 + 2a_n \langle Ty_n - q, J(x_{n+1} - q) \rangle \\
+ 2c_n \langle u_n - q, J(x_{n+1} - q) \rangle \\
\leq (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \langle Ty_n - q, J(x_{n+1} - q) - J(x_n - q) \rangle \\
+ 2a_n \langle Ty_n - q, J(x_n - q) - J(y_n - q) \rangle \\
+ 2a_n \langle Ty_n - q, J(y_n - q) \rangle + 2c_n \langle u_n - q, J(x_{n+1} - q) \rangle \\
\leq (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \|Ty_n - q\| \cdot \|J(x_{n+1} - q) - J(x_n - q)\| \\
+ 2a_n \|Ty_n - q\| \cdot \|J(x_n - q) - J(y_n - q)\| \\
+ 2a_n [\|y_n - q\|^2 - \Phi(\|y_n - q\|)] + 2c_n \|u_n - q\| \cdot \|x_{n+1} - q\| \\
\leq (1 - a_n)^2 R^2 + 4a_n L(1 + 2R) \varepsilon_0 + 2a_n [\|y_n - q\|^2 - \Phi(\|y_n - q\|)] + 4c_n MR
\]

and
\[
\|y_n - q\|^2 \leq (1 - b_n - d_n)^2 \|x_n - q\|^2 + 2b_n \langle Tx_n - q, J(y_n - q) \rangle \\
+ 2d_n \langle w_n - q, J(y_n - q) \rangle \\
\leq \|x_n - q\|^2 + 2b_n \langle Tx_n - q, J(y_n - q) - J(x_n - q) \rangle \\
+ 2b_n \langle Tx_n - q, J(x_n - q) \rangle + 2d_n \|w_n - q\| \cdot \|y_n - q\| \quad (2.13)
\]
\[
\|x_{n+1} - q\|^2 \leq (1 - a_n)^2 R^2 + 4a_n L(1 + 2R) \varepsilon_0 + 2a_n [R^2 + 2b_n L(1 + R) \varepsilon_0] \\
+ 2b_n R^2 + 4d_n MR - 2a_n \Phi(\|y_n - q\|) + 4c_n MR \\
\leq R^2 + a_n^2 R^2 + 4a_n L(1 + 2R) \varepsilon_0 + 2a_n^2 L(1 + R) \varepsilon_0 \\
+ 2b_n R^2 + 4d_n MR - 2a_n \Phi\left(\frac{R}{4}\right) + 4c_n MR \\
= R^2 + 2a_n \left[ \frac{a_n}{2} R^2 + 2L(1 + 2R) \varepsilon_0 + 2b_n L(1 + R) \varepsilon_0 \right] \\
+ 2b_n R^2 + 4d_n MR + \frac{2c_n MR}{a_n} - 2a_n \Phi\left(\frac{R}{4}\right) \\
\leq R^2 + 2a_n \left[ \frac{\Phi\left(\frac{R}{4}\right)}{2} - \Phi\left(\frac{R}{4}\right) \right] \\
\leq R^2 - \Phi\left(\frac{R}{4}\right) a_n \leq R^2
\]

Substitute (2.13) into (2.12)
which is a contradiction. Thus $x_{n+1} \in B_1$, i.e., $\{x_n\}$ is a bounded sequence. So $\{y_n\}$, $\{T y_n\}$, $\{T x_n\}$ are all bounded sequences. Since $\|z_n - q\| \to 0$ as $n \to \infty$, without loss of generality, we let $\|z_n - q\| \leq 1$. Therefore $\|x_n - z_n\|$ is also bounded.

**Step 3.** We want to prove that $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Set $M_0 = \max\{\sup_n \|T y_n - T z_n\|, \sup_n \|v_n - u_n\|, \sup_n \|x_n - z_n\|, \sup_n \|T x_n - x_n\|, \sup_n \|w_n - x_n\|, \sup_n \|y_n - z_n\|, \sup_n \|v_n - x_n\|\}$. By Lemma 1.5, we have

$$\|x_{n+1} - z_{n+1}\|^2 \leq (1 - a_n - c_n)^2 \|x_n - z_n\|^2 + 2a_n \langle T y_n - T z_n, J(x_{n+1} - z_{n+1}) \rangle + 2c_n \langle v_n - u_n, J(x_{n+1} - z_{n+1}) \rangle$$

$$\leq (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n \langle T y_n - T z_n, J(x_{n+1} - z_{n+1}) \rangle - J(x_n - z_n) \rangle + 2a_n \langle T y_n - T z_n, J(y_n - z_n) \rangle + 2c_n \|v_n - u_n\| \cdot \|x_{n+1} - z_{n+1}\|$$

$$\leq (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n + 2a_n \|y_n - z_n\|^2 - \Phi(\|y_n - z_n\|) + 2c_n M_0^2$$

and

$$\|y_n - z_n\|^2 \leq \|x_n - z_n\|^2 + 2b_n \langle T x_n - x_n, J(y_n - z_n) \rangle + 2d_n \langle w_n - x_n, J(y_n - z_n) \rangle$$

$$\leq \|x_n - z_n\|^2 + 2b_n M_0^2 + 2d_n M_0^2 ,$$

(2.16)

where $A_n = \|J(x_{n+1} - z_{n+1}) - J(x_n - z_n)\|$, $B_n = \|J(x_n - z_n) - J(y_n - z_n)\|$ with $A_n, B_n \to 0$ as $n \to \infty$.

Taking place (2.16) into (2.15),

$$\|x_{n+1} - z_{n+1}\|^2 \leq (1 - a_n)^2 \|x_n - z_n\|^2 + 2a_n M_0 A_n + 2a_n M_0 B_n + 2a_n \|y_n - z_n\|^2 - \Phi(\|y_n - z_n\|) + 2c_n M_0^2$$

$$\leq \|x_n - z_n\|^2 + a_n M_0^2 + 2a_n M_0 A_n + 2a_n M_0 B_n + 4a_n b_n M_0^2 + 4a_n d_n M_0^2$$

$$= \|x_n - z_n\|^2 + 2a_n \|C_n - a_n \Phi(\|y_n - z_n\|)\|.$$

(2.17)

where $C_n = \frac{a_n M_0^2}{2} + M_0 A_n + M_0 B_n + 2b_n M_0^2 + 2d_n M_0^2 + \frac{c_n M_0^3}{a_n} \to 0$ as $n \to \infty$.

Set $\inf_{n \geq 0} \|\Phi(\|y_n - z_n\|)\| = \lambda$, then $\lambda = 0$. If it is not the case, we assume that $\lambda > 0$. Let $0 < \gamma < \min\{1, \lambda\}$, then $\|\Phi(\|y_n - z_n\|)\| \geq \gamma + \gamma \|x_{n+1} - z_{n+1}\|^2 \geq \gamma \|x_{n+1} - z_{n+1}\|^2$. Thus, from (2.17) that

$$\|x_{n+1} - z_{n+1}\|^2 \leq \|x_n - z_n\|^2 + 2a_n [C_n - \gamma \|x_{n+1} - z_{n+1}\|^2].$$

(2.18)

It implies that

$$\|x_{n+1} - z_{n+1}\|^2 \leq \frac{1}{1 + 2a_n \gamma} \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma}$$

$$= \left(1 - \frac{2a_n \gamma}{1 + 2a_n \gamma}\right) \|x_n - z_n\|^2 + \frac{2a_n C_n}{1 + 2a_n \gamma}.$$
Let \( \rho_n = \|x_n - z_n\|^2 \), \( \lambda_n = \frac{2a_n \gamma}{1 + 2a_n \gamma} \), \( \sigma_n = \frac{2a_n C_n}{1 + 2a_n \gamma} \). Then we obtain that
\[
\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n.
\]
Applying Lemma 1.6, then \( \rho_n \to 0 \) as \( n \to \infty \). This is a contradiction and so \( \lambda = 0 \). Therefore, there exists an infinite subsequence such that \( \Phi(\|y_{n_d} - z_{n_d}\|) \to 0 \) as \( i \to \infty \).

Since \( 0 \leq \Phi(\|y_{n_d} - z_{n_d}\|) \leq \frac{\Phi(\|y_{n_d} - z_{n_d}\|)}{1 + M_0} \) \( \|y_{n_d} - z_{n_d}\| \to 0 \) as \( i \to \infty \). In view of the strictly increasing continuity of \( \Phi \), we have \( \|y_{n_d} - z_{n_d}\| \to 0 \) as \( i \to \infty \). From (1.1), we have
\[
\|x_{n_d} - z_{n_d}\| \leq \|y_{n_d} - z_{n_d}\| + b_{n_d}\|x_{n_d} - T_{\lambda_n} z_{n_d}\| + c_{n_d}\|x_{n_d} - w_{n_d}\| \to 0
\]
as \( i \to \infty \). Next we want to prove that \( \|x_n - z_n\| \to 0 \) as \( n \to \infty \). Let \( \forall \varepsilon \in (0, 1) \), there exists \( n_{i_0} \) such that \( \|x_n - z_n\| < \varepsilon \), \( a_n < \min\{\frac{\varepsilon}{4(1 + M_0)}, \frac{\varepsilon}{8M_0}\} \), \( c_n < \frac{\varepsilon}{16M_0} \), \( b_n \), \( d_n < \frac{\varepsilon}{8M_0} \), \( C_n < \frac{\Phi(\varepsilon/2)}{2} \) for any \( n_i \), \( n \geq n_{i_0} \). First, we want to prove \( \|x_{n+1} - z_{n+1}\| < \varepsilon \).

Suppose that it is not this case, then \( \|x_{n+1} - z_{n+1}\| \geq \varepsilon \). Using (1.1), we may get the following estimates:
\[
\|x_{n_d} - z_{n_d}\| \geq \|x_{n_d+1} - z_{n_d+1}\| - a_{n_d}\|T_{\lambda_n} y_{n_d} - T_{\lambda_n} z_{n_d}\| - a_{n_d}\|x_{n_d} - z_{n_d}\|
\]
\[
- c_{n_d}\|v_{n_d} - u_{n_d}\| - c_{n_d}\|x_{n_d} - z_{n_d}\|
\]
\[
\geq \varepsilon - a_{n_d}L(1 + M_0) - (a_{n_d} + 2c_{n_d})M_0
\]
\[
> \frac{\varepsilon}{2},
\]
(2.20)
\[
\|y_{n_d} - z_{n_d}\| \geq \|x_{n_d} - z_{n_d}\| - b_{n_d}\|T_{\lambda_n} x_{n_d} - x_{n_d}\| - d_{n_d}\|v_{n_d} - x_{n_d}\|
\]
\[
\geq \frac{\varepsilon}{2} - (b_{n_d} + d_{n_d})M_0
\]
\[
> \frac{\varepsilon}{4}.
\]
(2.21)

Since \( \Phi \) is strictly increasing, then (2.21) leads to \( \Phi(\|y_{n_d} - z_{n_d}\|) \geq \Phi(\varepsilon/4) \). From (2.17), we have
\[
\|x_{n+1} - z_{n+1}\|^2 \leq \|x_{n_d} - z_{n_d}\|^2 + 2a_{n_d}[C_{n_d} - \Phi(\|y_{n_d} - z_{n_d}\|)]
\]
\[
< \varepsilon^2 + 2a_{n_d}\left[ \frac{1}{2}\Phi\left( \frac{\varepsilon}{4} \right) - \Phi\left( \frac{\varepsilon}{4} \right) \right]
\]
\[
\leq \varepsilon^2 - \Phi\left( \frac{\varepsilon}{4} \right) a_{n_d}
\]
\[
\leq \varepsilon^2,
\]
(2.22)
which is a contradiction. Hence, \( \|x_{n+1} - z_{n+1}\| < \varepsilon \). Suppose that \( \|x_{n+m} - z_{n+m}\| < \varepsilon \) holds. Repeating the above course, we can easily prove that \( \|x_{n+m+1} - z_{n+m+1}\| < \varepsilon \) holds. Therefore, for any \( m \), we obtain that \( \|x_{n+m} - z_{n+m}\| < \varepsilon \), which means \( \|x_n - z_n\| \to 0 \) as \( n \to \infty \). This completes the proof. \( \square \)
THEOREM 2.2. Let $E$ be an arbitrary uniformly smooth real Banach space, and $T : E \to E$ be a generalized Lipschitz $\Phi$-accretive mapping with $N(T) = \{ x \in E : Tx = 0 \} \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ be four real sequences in $[0,1]$ which satisfy the conditions (i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$; (ii) $a_n$, $b_n$, $d_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0$, $z_0 \in D$, let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be any bounded sequences in $E$, and $\{x_n\}$, $\{z_n\}$ be Ishikawa and Mann iterations with errors defined by

$$
\begin{align*}
&y_n = (1 - b_n - d_n)x_n + b_nSx_n + d_nw_n, \\
&x_{n+1} = (1 - a_n - c_n)x_n + a_nSy_n + c_nv_n, \quad n \geq 0
\end{align*}
$$

and

$$
\begin{align*}
&z_{n+1} = (1 - a_n - c_n)z_n + a_nSz_n + c_nu_n,
\end{align*}
$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for any $x \in E$. Then the following two assertions are equivalent:

(i) $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$;

(ii) $\{z_n\}$ converges strongly to the unique solution of the equation $Tx = 0$.

Proof. Since $T$ is generalized Lipschitz and $\Phi$-accretive mapping, it follows that

$$
\|Tx - Ty\| \leq L(1 + \|x - y\|),
$$

i.e.,

$$
\|Sx - Sy\| \leq L_1(1 + \|x - y\|), L_1 = 1 + L;
$$

$$
\langle Tx - Ty, J(x - y) \rangle \geq \Phi(\|x - y\|),
$$

i.e.,

$$
\langle Sx - Sy, J(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|),
$$

for all $x, y \in E$. Then $S$ is the generalized Lipschitz $\Phi$-pseudocontractive mapping. By Theorem 2.1, we obtain the conclusion of Theorem 2.2. \(\square\)

REMARK 2.3. It is mentioned that, in 2006, C. E. Chidume and C. O. Chidume \cite{1} proved the approximation theorem for zeros of generalized Lipschitz generalized $\Phi$-quasi-accretive operators. This result provided significant improvements of past known corresponding results. However, there exists a gap in the proof process of Theorem 3.1 of \cite{1}, i.e., $\sum_{n=0}^{\infty} c_n < +\infty$ does not imply $c_n = o(b_n)$. For example, set the iteration parameters: $a_n = 1 - b_n - c_n$, where $\{b_n\}$: $b_1 = 0$, $b_n = \frac{1}{n}$, $n \geq 2$; $\{c_n\}$: $1$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\cdots$, $\frac{1}{10^2}$, $\cdots$, $\frac{1}{15^2}$, $\cdots$, $\frac{1}{24^2}$, $\frac{1}{25}$, $\frac{1}{26^2}$, $\cdots$. Up to now, we do not know whether the condition $c_n = o(a_n)$ is taken place by $\sum_{n=0}^{\infty} c_n < +\infty$ in Theorem 2.1 and Theorem 2.2.

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