Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 487–493 doi:10.7153/jmi-07-43

ON A KY FAN TYPE INEQUALITY DUE TO H. ALZER

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(*Communicated by P. R. Mercer*)

Abstract. Let A_n and H_n (respectively, A'_n and H'_n) be the weighted arithmetic and harmonic means of $x_1, x_2, \dots, x_n$ (respectively, $1-x_1$, $1-x_2, \dots, 1-x_n$), where $x_i \in (0,1/2]$ $(i=1,2,\dots, n;\ n \geqslant 2)$. We mainly show that, if not all of the x_i's are equal, then

$$\min_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i} < \frac{A'_n - H'_n}{A_n - H_n} < \max_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i},$$

which is a refinement and converse of the Ky Fan type inequality $A'_n - H'_n \leqslant A_n - H_n$ due to H. Alzer. Some parallel and related results are also discussed.

1. Introduction

Throughout this article, let $n \geqslant 2$ and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, such that $\sum_{i=1}^n \lambda_i = 1$. Given n arbitrary real numbers $x_1, \dots, x_n > 0$, we denote by A_n, G_n and H_n the arithmetic, geometric and harmonic means of $x_1, \dots, x_n$ respectively, i.e.

$$A_n = \sum_{i=1}^n \lambda_i x_i, \qquad G_n = \prod_{i=1}^n x_i^{\lambda_i}, \qquad H_n = \frac{1}{\sum_{i=1}^n \lambda_i \frac{1}{x_i}}. \tag{1}$$

Also, when $x_i \in (0,1/2]$, we denote by A'_n, G'_n, and H'_n the arithmetic, geometric and harmonic means of $1-x_1, \dots, 1-x_n$ respectively, i.e.

$$A'_n = \sum_{i=1}^n \lambda_i (1-x_i), \qquad G'_n = \prod_{i=1}^n (1-x_i)^{\lambda_i}, \qquad H'_n = \frac{1}{\sum_{i=1}^n \lambda_i \frac{1}{1-x_i}}. \tag{2}$$

If $k = 1,2,\dots,n$, then A_k, G_k and H_k (respectively, A'_k, G'_k and H'_k) are taken to be the arithmetic, geometric and harmonic means of $x_1, \dots, x_k$ (respectively, $1-x_1, \dots, 1-x_k$) with respect to the weights $\lambda_1 (\sum_{i=1}^k \lambda_i)^{-1}, \dots, \lambda_k (\sum_{i=1}^k \lambda_i)^{-1}$. When emphasizing, we write $A_k(x_1, \cdots, x_k)$ instead of A_k, and so on.

Mathematics subject classification (2010): 26D15.
Keywords and phrases: A-G-H inequality, Ky Fan's inequality.

Paper JMI-07-43

The Ky Fan's inequality

$$\frac{A'_n}{G'_n} \leqslant \frac{A_n}{G_n}, \tag{3}$$

was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [5, p. 5], and from then it has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [3] and the references therein, see also [9] for some new approaches.

In 1988, an additive analogue of (3) presented by H. Alzer [4] as

$$A'_n - G'_n \leqslant A_n - G_n. \tag{4}$$

In both of (3) and (4), equality holds if and only if $x_1 = \cdots = x_n$.

Later, H. Alzer in [2], showed that if not all of x_i's are equal, then

$$\min_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i} < \frac{A'_n - G'_n}{A_n - G_n} < \max_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i}, \tag{5}$$

which is a refinement and converse for (4).

In [8], P. R. Mercer in a short note showed that (5) follows rather easily from the result of D. I. Cartwright and M. J. Field [7].

Another interesting additive analogue of Ky Fan's inequality was discovered by H. Alzer [1] in 1993, as follows:

$$A'_n - H'_n \leqslant A_n - H_n, \tag{6}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Using differentiation and Tchebyschef inequality, the proof of H. Alzer for (6) is elementary, but technical. Besides, the proof of Alzer is only for the case of equal weights $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$.

In this article, using a recursive identity together with the A-G-H inequality, we give a simple proof of (6) in the case of arbitrary weights, by establishing an analogue of (5) for the inequality (6).

2. The main results

In this section, first we establish a recursive identity concerning arithmetic and harmonic means of positive numbers, which in turn yields a representation of $A_n - H_n$ as a finite series of nonnegative terms. Then, using this representation with A-G-H inequality, we get the desired conclusion.

LEMMA 2.1. *We have*

$$A_n - H_n = (1-\lambda_n)\left[A_{n-1} - H_{n-1} + \lambda_n \frac{H_n(x_n - H_{n-1})^2}{x_n H_{n-1}}\right]. \tag{7}$$

As a consequence,

$$A_n - H_n = \sum_{k=2}^{n} \frac{\lambda_k \sum_{i=1}^{k-1} \lambda_i}{\sum_{i=1}^{k} \lambda_i} \cdot \frac{H_k (x_k - H_{k-1})^2}{x_k H_{k-1}}. \tag{8}$$

Proof. We have

$$\begin{aligned} A_n - H_n &= (1-\lambda_n)(A_{n-1} - H_{n-1}) + (1-\lambda_n) H_{n-1} + \lambda_n x_n - \frac{x_n H_{n-1}}{(1-\lambda_n) x_n + \lambda_n H_{n-1}} \\ &= (1-\lambda_n)(A_{n-1} - H_{n-1}) + (1-\lambda_n)\lambda_n \frac{(x_n - H_{n-1})^2}{(1-\lambda_n) x_n + \lambda_n H_{n-1}} \\ &= (1-\lambda_n)(A_{n-1} - H_{n-1}) + (1-\lambda_n)\lambda_n \frac{H_n (x_n - H_{n-1})^2}{x_n H_{n-1}}, \end{aligned}$$

and (7) is obtained.

By (7), we have

$$A_k - H_k = \left(1 - \frac{\lambda_k}{\sum_{i=1}^{k} \lambda_i}\right) \left[A_{k-1} - H_{k-1} + \frac{\lambda_k}{\sum_{i=1}^{k} \lambda_i} \cdot \frac{H_k (x_k - H_{k-1})^2}{x_k H_{k-1}}\right] \quad (2 \leqslant k \leqslant n). \tag{9}$$

Now, multiplying both sides of (9) by $\sum_{i=1}^{k} \lambda_i$ and taking summation from $k = 2$ to $k = n$, we get (8), and the proof is complete. □

COROLLARY 2.2. *We have*

$$\frac{1}{H_n} - \frac{1}{A_n} = (1-\lambda_n) \left[\frac{1}{H_{n-1}} - \frac{1}{A_{n-1}} + \lambda_n \frac{(A_{n-1} - x_n)^2}{x_n A_{n-1} A_n}\right], \tag{10}$$

and,

$$\frac{1}{H_n} - \frac{1}{A_n} = \sum_{k=2}^{n} \frac{\lambda_k \sum_{i=1}^{k-1} \lambda_i}{\sum_{i=1}^{k} \lambda_i} \cdot \frac{(A_{k-1} - x_k)^2}{x_k A_{k-1} A_k}. \tag{11}$$

Proof. Clearly for each $k = 1, 2, \dots, n$, we have

$$A_k \left(\frac{1}{x_1}, \dots, \frac{1}{x_k}\right) = \frac{1}{H_k (x_1, \dots, x_k)},$$

and

$$H_k \left(\frac{1}{x_1}, \dots, \frac{1}{x_k}\right) = \frac{1}{A_k (x_1, \dots, x_k)}.$$

Now, changing the roles of x_i's by $\frac{1}{x_i}$'s, the identities (10) and (11) follow from (7) and (8) respectively by replacing A_k, H_k and x_k, by $\frac{1}{H_k}, \frac{1}{A_k}$ and $\frac{1}{x_k}$ $(k = 1, 2, \dots, n)$, respectively. □

THEOREM 2.3. *If* $x_i \in (0,1/2]$ $(i=1,\dots,n; n \geqslant 2)$ *not all are equal, then*

$$\min_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i} < \frac{A_n'-H_n'}{A_n-H_n} < \max_{1\leqslant i\leqslant n} \frac{x_i}{1-x_i}, \tag{12}$$

The inequalities in (12) *give a refinement and converse for* (6).

Proof. Applying the identity (8) for $(1-x_i)$'s instead of x_i's; $i=1,\dots,n$, we have

$$A_n'-H_n' = \sum_{k=2}^{n} \frac{\lambda_k \sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i} \cdot \frac{H_k'(1-x_k-H_{k-1}')^2}{(1-x_k)H_{k-1}'}. \tag{13}$$

Let $x_{r_1} \leqslant x_{r_2} \leqslant \cdots \leqslant x_{r_n}$ be a rearrangement of x_i's in increasing order. Let also for each $1\leqslant k\leqslant n$, $A_{r,k}$ and $H_{r,k}$, respectively $A_{r,k}'$ and $H_{r,k}'$ be the arithmetic and harmonic means of $x_{r_1},\dots,x_{r_k}$, respectively $1-x_{r_1},\dots,1-x_{r_k}$, with respect to the weights $\lambda_{r_1}(\sum_{i=1}^{k}\lambda_{r_i})^{-1},\dots,\lambda_{r_k}(\sum_{i=1}^{k}\lambda_{r_i})^{-1}$. Now, since $x_{r_k}\leqslant x_{r_n}$, $H_{r,k-1}<x_{r_n}$, $1-x_{r_n}\leqslant 1-x_{r_k}$ and $1-x_{r_n}<H_{r,k-1}'$ $(2\leqslant k\leqslant n)$, by (8) and (13), we get

$$\frac{A_n'-H_n'}{A_n-H_n} = \frac{A_{r,n}'-H_{r,n}'}{A_{r,n}-H_{r,n}} < \frac{x_{r_n}}{1-x_{r_n}} \frac{\sum_{k=2}^{n} \frac{\lambda_{r_k}\sum_{i=1}^{k-1}\lambda_{r_i}}{\sum_{i=1}^{k}\lambda_{r_i}}(H_{r,k-1}'-1+x_{r_k})^2}{\sum_{k=2}^{n} \frac{\lambda_{r_k}\sum_{i=1}^{k-1}\lambda_{r_i}}{\sum_{i=1}^{k}\lambda_{r_i}}(x_{r_k}-H_{r,k-1})^2}. \tag{14}$$

But,

$$H_{r,k-1}'+H_{r,k-1} \leqslant A_{r,k-1}'+A_{r,k-1} = 1 = (1-x_{r_k})+x_{r_k} \qquad (2\leqslant k\leqslant n),$$

or $x_{r_k}-H_{r,k-1}\geqslant H_{r,k-1}'-(1-x_{r_k})\geqslant 0$, which yields $(x_{r_k}-H_{r,k-1})^2 \geqslant (H_{r,k-1}'-(1-x_{r_k}))^2$. So, the right hand of (14) is less than or equal to $\frac{x_{r_n}}{1-x_{r_n}} = \max\limits_{1\leqslant i\leqslant n}\frac{x_i}{1-x_i}$, and the right hand inequality in (12) is established.

For the left hand inequality of (12), rearrange x_i's in a decreasing order $x_{s_1}\geqslant x_{s_2}\geqslant\cdots\geqslant x_{s_n}$ and let for each $1\leqslant k\leqslant n$, $A_{s,k}$ and $H_{s,k}$, respectively $A_{s,k}'$ and $H_{s,k}'$, be the arithmetic and harmonic means of $x_{s_1},\dots,x_{s_k}$, respectively $1-x_{s_1},\dots,1-x_{s_k}$, with respect to the weights $\lambda_{s_1}(\sum_{i=1}^{k}\lambda_{s_i})^{-1},\dots,\lambda_{s_k}(\sum_{i=1}^{k}\lambda_{s_i})^{-1}$. Now since $x_{s_k}\geqslant x_{s_n}$, $H_{s,k-1}>x_{s_n}$, $1-x_{s_n}\geqslant 1-x_{s_k}$, $1-x_{s_n}>H_{s,k-1}'$, and $(H_{s,k-1}-x_{s_k})^2\leqslant(1-x_{s_k}-H_{s,k-1}')^2$ $(2\leqslant k\leqslant n)$, we have

$$\begin{aligned}\frac{A_n'-H_n'}{A_n-H_n} &= \frac{A_{s,n}'-H_{s,n}'}{A_{s,n}-H_{s,n}} > \frac{x_{s_n}}{1-x_{s_n}} \frac{\sum_{k=2}^{n} \frac{\lambda_{s_k}\sum_{i=1}^{k-1}\lambda_{s_i}}{\sum_{i=1}^{k}\lambda_{s_i}}(H_{s,k-1}'-1+x_{s_k})^2}{\sum_{k=2}^{n} \frac{\lambda_{s_k}\sum_{i=1}^{k-1}\lambda_{s_i}}{\sum_{i=1}^{k}\lambda_{s_i}}(x_{s_k}-H_{s,k-1})^2} \\ &\geqslant \frac{x_{s_n}}{1-x_{s_n}} = \min_{1\leqslant i\leqslant n}\frac{x_i}{1-x_i}.\end{aligned}$$

This completes the proof. □

THEOREM 2.4. *If* $x_i \in (0,1/2]$ $(i=1,\dots,n;\ n \geqslant 2)$ *not all are equal, then*

$$\min_{1\leqslant i\leqslant n}\left(\frac{x_i}{1-x_i}\right)^3 < \frac{\frac{1}{H_n'}-\frac{1}{A_n'}}{\frac{1}{H_n}-\frac{1}{A_n}} < \max_{1\leqslant i\leqslant n}\left(\frac{x_i}{1-x_i}\right)^3. \tag{15}$$

Proof. Renaming x_i's and corresponding λ_i's, we can assume that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Using the identity (11) for $(1-x_i)$'s instead of x_i's, $i=1,\dots,n$, we have

$$\frac{1}{H_n'}-\frac{1}{A_n'} = \sum_{k=2}^{n} \frac{\lambda_k \sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i}\cdot\frac{(A_{k-1}'-1+x_k)^2}{(1-x_k)A_{k-1}'A_k'}.$$

So, considering $A_{k-1}'-1+x_k = x_k - A_{k-1}$, $k=2,\cdots,n$, we get

$$\frac{\frac{1}{H_n'}-\frac{1}{A_n'}}{\frac{1}{H_n}-\frac{1}{A_n}} < \frac{\frac{1}{(1-x_n)^3}\sum_{k=2}^{n}\frac{\lambda_k\sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i}\cdot(A_{k-1}'-1+x_k)^2}{\frac{1}{x_n^3}\sum_{k=2}^{n}\frac{\lambda_k\sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i}\cdot(A_{k-1}-x_k)^2} = \left(\frac{x_n}{1-x_n}\right)^3,$$

and

$$\frac{\frac{1}{H_n'}-\frac{1}{A_n'}}{\frac{1}{H_n}-\frac{1}{A_n}} > \frac{\frac{1}{(1-x_1)^3}\sum_{k=2}^{n}\frac{\lambda_k\sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i}\cdot(A_{k-1}'-1+x_k)^2}{\frac{1}{x_1^3}\sum_{k=2}^{n}\frac{\lambda_k\sum_{i=1}^{k-1}\lambda_i}{\sum_{i=1}^{k}\lambda_i}\cdot(A_{k-1}-x_k)^2} = \left(\frac{x_1}{1-x_1}\right)^3.$$

This completes the proof. □

REMARK 2.5. (i) The identities (7) and (10), yield the following Rado type inequalities [6],

$$A_n - H_n \geqslant (1-\lambda_n)(A_{n-1}-H_{n-1}), \tag{16}$$

and

$$\frac{1}{H_n}-\frac{1}{A_n} \geqslant (1-\lambda_n)\left[\frac{1}{H_{n-1}}-\frac{1}{A_{n-1}}\right], \tag{17}$$

with equality holding in (16), respectively (17), if and only if $x_n = H_{n-1}$, respectively $x_n = A_{n-1}$.

(ii) The inequalities in (15) give us a refinement and converse of the inequality

$$\frac{1}{H_n'}-\frac{1}{A_n'} \leqslant \frac{1}{H_n}-\frac{1}{A_n}, \tag{18}$$

due to J. Sandor [11]. Clearly, (18) is a consequence of (6), since,

$$\frac{1}{H_n'}-\frac{1}{A_n'} = \frac{A_n'-H_n'}{A_n'H_n'} \leqslant \frac{A_n-H_n}{A_nH_n} = \frac{1}{H_n}-\frac{1}{A_n}.$$

Indeed, since by the Lemma 3.1 of [10], for each $a > b \geqslant c > d > 0$, the function

$$f(x) = \frac{a^x - b^x}{c^x - d^x} \qquad (-\infty < x < +\infty),$$

is strictly increasing on the real line, taking $a = A'_n, b = H'_n, c = A_n$ and $d = H_n$ in the case of not all x_i's equal, we have $f(x) < f(1) < 1$ for each $x < 1$, which in particular case of $x = -1$ yields (18) with strict inequality.

The Figure 1 shows the behavior of this function f drawn for the special case $n = 3$; $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $x_1 = 1/2$, $x_2 = 1/3$ and $x_3 = 1/4$. As it is seen the behavior of this function is not clear for $x > 1$.

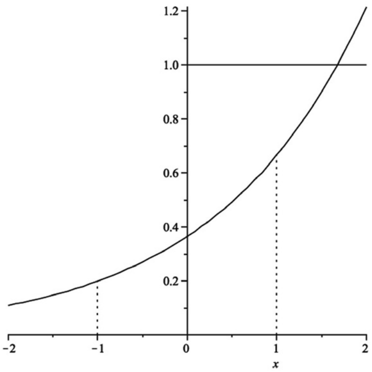

Figure 1: $y = f(x) = \frac{A_n'^x - H_n'^x}{A_n^x - H_n^x}$

We conclude the paper with the following open problem:

Open problem: With the above notations, sre the values $A_n^n - H_n^n$ and $A_n'^n - H_n'^n$ comparable, at least in the case of equal weights?

Acknowledgement. I would like to sincerely thank Professor Peter R. Mercer and anonymous referees for exact evaluation of the results of this paper.

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(Received July 5, 2012)

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