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SOME GENERALIZED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we generalize some integral inequalities to more general situations. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of integral equations and differential equations. Applications are given to illustrate the usefulness of the inequalities.

1. Introduction

Over the years integral inequalities have become a major tool in the analysis of various differential and integral equations that occur in nature or are built by man (see $[1-13]$). In studying the boundedness behavior of the solutions of certain differential and integral equations, Ou-Iang $[1]$ and Pachpatte $[2,3]$ gave some new integral inequalities. We list them as follows.

THEOREM A. (Ou-Iang) (See [1]) *If u and f are non-negative functions on $[0,\infty)$ satisfying*

$$u^2(t) \leqslant k^2 + 2\int_0^t f(s)u(s)ds, \;\; for \;\; all \;\; t \in [0,\infty),$$

where $k \geqslant 0$ is a constant, then

$$u(t) \leqslant k + \int_0^t f(s)ds. \;\; t \in [0,\infty).$$

THEOREM B. (Pachpatte) (See [2]) *Suppose that u, f, g are continuous non-negative functions on $[0,\infty)$ and ω is a continuous non-decreasing function on $[0,\infty)$ with $\omega(r) > 0$ for $r > 0$. If*

$$u^2(t) \leqslant k^2 + 2\int_o^t [f(s)u(s) + g(s)u(s)\omega(u(s))]ds, \;\; for \;\; all \;\; t \in [0,\infty),$$

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where k is a constant, then

$$u(t) \leqslant \Omega^{-1}\left[\Omega\left(k+\int_0^t f(s)ds\right)+\int_0^t g(s)ds\right], \ for\ all\ t\in[0,t_1],$$

where

$$\Omega(t) := \int_1^t \frac{1}{\omega(s)}ds, \ t>0,$$

Ω^{-1} is the inverse of Ω, and $t_1\in[0,\infty)$ is chosen in such a way that $\Omega(k+\int_0^t f(s)ds)+\int_0^t g(s)ds \in Dom(\Omega^{-1})$, for all $t\in[0,t_1]$.

THEOREM C. (Pachpatte) (See [3]) *Let u,f,g be real-valued non-negative continuous functions defined on R_+, and c_1, c_2 be non-negative constants. If*

$$u(t)\leqslant\left(c_1+\int_0^t f(s)u(s)ds\right)\left(c_2+\int_0^t g(s)u(s)ds\right)$$

and

$$c_1c_2\int_0^t R(s)Q(s)ds<1, \ \ for\ all\ t\in R_+,$$

then

$$u(t)\leqslant\frac{c_1c_2Q(t)}{1-c_1c_2\int_0^t R(s)Q(s)ds}, \quad t\in R_+,$$

where

$$R(t)=\int_0^t [f(t)g(s)+f(s)g(t)]ds,$$

$$Q(t)=exp\left(\int_0^t [c_1g(s)+c_2f(s)]ds\right).$$

The main aim of the present is to generalize some integral inequalities to more general situations, which can be used as ready and powerful tools in the study of qualitative as well as quantitative properties of solutions of integral equations and differential equations. We also illustrate the usefulness of these inequalities.

2. Main results

In the next Theorems and Corollaries, for any $\varphi,\psi\in C(R_+,R_+)$ and any constants $p,q\geqslant 0$, define

$$\Phi_p(r):=\int_1^r\frac{ds}{\varphi(s^{\frac{1}{p}})}, \qquad \Phi_p(0)=\lim_{r\to 0^+}\Phi_p(r);$$

$$\Psi_q(r):=\int_1^r\frac{ds}{\psi(s^{\frac{1}{q}})}, \qquad \Psi_q(0)=\lim_{r\to 0^+}\Psi_q(r).$$

To prove our Theorem 2.1, we need the following lemma.

LEMMA 2.1. *Let $f_i \in C(R_+ \times R_+, R_+)$ with $(t,s) \mapsto \partial_t f_i(t,s)$, $\partial_t f_i(t,s) \in C(R_+ \times R_+, R_+)$, $i = 1,2,3,4$, $p > 0$ be a constant. Assume in addition that $\varphi, k \in C(R_+, R_+)$, $\alpha \in C^1(R_+, R_+)$ are non-decreasing functions with $\alpha(t) \leqslant t$ for $t \geqslant 0$, $\varphi(t) > 0$, and $\int_1^\infty \frac{dt}{\varphi(t)} = \infty$. If $u \in C(R_+, R_+)$ satisfies*

$$u^p(t) \leqslant k(t) + \int_0^{\alpha(t)} f_1(t,s)\varphi(u(s))ds + \int_0^t f_2(t,s)\varphi(u(s))ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds, \quad t \geqslant 0,$$

then

$$u(t) \leqslant \left\{\Phi_p^{-1}[\Phi_p(k(t)) + A(t)]\right\}^{\frac{1}{p}}, \quad t \in \Delta, \tag{2.1}$$

where

$$A(t) = \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds$$

Φ_p^{-1} is the inverse of Φ_p, and $t \in \Delta$ is chosen in such a way that, $\Phi_p(k(t)) + A(t) \in Dom(\Phi_p^{-1})$.

Proof. Let $T \in \Delta$, $T \geqslant 0$ be fixed and denote

$$x(t) = \int_0^{\alpha(t)} f_1(t,s)\varphi(u(s))ds + \int_0^t f_2(t,s)\varphi(u(s))ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds,$$

then

$$u(t) \leqslant [k(t) + x(t)]^{\frac{1}{p}},$$

our assumption on f_i, φ, α imply that x is non-decreasing on R_+, $i = 1,2,3,4$. Hence for $t \in [0,T]$, by calculations we have

$$\begin{aligned} x'(t) = & \left[f_1(t,\alpha(t))\varphi(u(\alpha(t)))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_1(t,s)\varphi(u(s))ds\right] \\ & + \left[f_2(t,t)\varphi(u(t)) + \int_0^t \partial_t f_2(t,s)\varphi(u(s))ds\right] \\ & + \left[f_3(t,\alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_3(t,s)ds\right] \cdot \int_0^t f_4(t,s)\varphi(u(s))ds \\ & + \left[f_4(t,t)\varphi(u(t)) + \int_0^t \partial_t f_4(t,s)\varphi(u(s))ds\right] \cdot \int_0^{\alpha(t)} f_3(t,s)ds \end{aligned}$$

$$\leqslant \varphi\left[(k(T)+x(t))^{\frac{1}{p}}\right]\cdot\frac{d}{dt}\left[\int_0^{\alpha(t)} f_1(t,s)ds+\int_0^t f_2(t,s)ds\right.$$
$$\left.+\int_0^{\alpha(t)} f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds\right],$$

then we get

$$\frac{x'(t)}{\varphi\left[(k(T)+x(t))^{\frac{1}{p}}\right]}\leqslant\frac{d}{dt}\left[\int_0^{\alpha(t)} f_1(t,s)ds\right.$$
$$\left.+\int_0^t f_2(t,s)ds+\int_0^{\alpha(t)} f_3(t.s)ds\cdot\int_0^t f_4(t,s)ds\right].$$

Considering the definition of Φ and the integral on the interval $[0,t]$, yields

$$\Phi_p(x(t)+k(T))\leqslant\Phi_p(k(T))+\int_0^{\alpha(t)} f_1(t,s)ds$$
$$+\int_0^t f_2(t,s)ds+\int_0^{\alpha(t)} f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds,\quad t\in[0,T].$$

As Φ_p^{-1} is increasing on $Dom(\Phi_p^{-1})$, then

$$x(t)+k(T)\leqslant\Phi_p^{-1}\left[\Phi_p(k(T))+\int_0^{\alpha(t)} f_1(t,s)ds+\int_0^t f_2(t,s)ds\right.$$
$$\left.+\int_0^{\alpha(t)} f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds\right],\quad t\in[0,T].$$

Let $t=T$ in the above relation, since $T\geqslant 0$ was arbitrarily chosen, considering $u(t)\leqslant (x(t)+k(t))^{\frac{1}{p}}$, we get(2.1). $\square$

THEOREM 2.1. *Let f_i,φ,α,k be as in Lemma 2.1, $i=1,2,3,4$. Assume in addition that $g_j\in C(R_+\times R_+,R_+)$ with $(t,s)\mapsto\partial_t g_j(t,s)$, $\partial_t g_j(t,s)\in C(R_+\times R_+,R_+)$, $j=1,2$, and $p>1$ be a constant. If $u\in C(R_+,R_+)$ satisfies*

$$u^p(t)\leqslant k(t)+\frac{p}{p-1}\int_0^{\alpha(t)}[f_1(t,s)u(s)+g_1(t,s)u(s)\varphi(u(s))]\,ds$$
$$+\frac{p}{p-1}\int_0^t[f_2(t,s)u(s)+g_2(t,s)u(s)\varphi(u(s))]\,ds$$
$$+\frac{p}{p-1}\int_0^{\alpha(t)} f_3(t,s)u(s)ds\cdot\int_0^t f_4(t,s)\varphi(u(s))ds,\quad for\ \ t\geqslant 0,$$

then

$$u(t)\leqslant\left\{\Phi^{-1}_{1-\frac{1}{p}}\left[\Phi_{1-\frac{1}{p}}(z(t)+B(t))+C(t)\right]\right\}^{\frac{1}{p-1}},\ \ t\in\Delta,\tag{2.2}$$

where

$$z(t) = (k(t))^{1-\frac{1}{p}},$$
$$B(t) = (k(t))^{1-\frac{1}{p}} + \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds,$$
$$C(t) = \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds,$$

$\Phi^{-1}_{1-\frac{1}{p}}$ *is the inverse of* $\Phi_{1-\frac{1}{p}}$, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{1-\frac{1}{p}}(B(t)) \in Dom\left(\Phi^{-1}_{1-\frac{1}{p}}\right)$.

Proof. Let $T \in \Delta$, $T \geqslant 0$ be fixed and denote

$$\begin{aligned} x(t) = {} & \frac{p}{p-1}\int_0^{\alpha(t)} [f_1(t,s)u(s) + g_1(t,s)u(s)\varphi(u(s))]ds \\ & + \frac{p}{p-1}\int_0^t [f_2(t,s)u(s) + g_2(t,s)u(s)\varphi(u(s))]ds \\ & + \frac{p}{p-1}\int_0^{\alpha(t)} f_3(t,s)u(s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds, \end{aligned}$$

then

$$u(t) \leqslant (k(t) + x(t))^{\frac{1}{p}}.$$

Our assumption on f_i, g_j, φ, α, $(i = 1,2,3,4, j = 1,2)$ imply that x is non-decreasing on R_+. Hence, for $t \in [0,T]$, by calculations we have

$$\begin{aligned} x'(t) = {} & \frac{p}{p-1}\Big[f_1(t,\alpha(t))u(\alpha(t))\alpha'(t) + g_1(t,\alpha(t))u(\alpha(t))\varphi(u(\alpha(t)))\alpha'(t) \\ & + \int_0^{\alpha(t)} (\partial_t f_1(t,s)u(s) + \partial_t g_1(t,s)u(s)\varphi(u(s)))ds\Big] \\ & + \frac{p}{p-1}\Big[f_2(t,t)u(t) + g_2(t,t)u(t)\varphi(u(t)) \\ & + \int_0^t (\partial_t f_2(t,s)u(s) + \partial_t g_2(t,s)u(s)\varphi(u(s)))ds\Big] \\ & + \frac{p}{p-1}\left[f_3(t,\alpha(t))u(\alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_3(t,s)u(s)ds\right] \cdot \int_0^t f_4(t,s)\varphi(u(s))ds \\ & + \frac{p}{p-1}\left[f_4(t,t)\varphi(u(t)) + \int_0^t \partial_t f_4(t,s)\varphi(u(s))ds\right] \cdot \int_0^{\alpha(t)} f_3(t,s)u(s)ds \\ \leqslant {} & \frac{p}{p-1}\frac{d}{dt}\left[\int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds\right] \cdot [k(T) + x(t)]^{\frac{1}{p}} \\ & + \frac{p}{p-1}\frac{d}{dt}\Big[\int_0^{\alpha(t)} g_1(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds + \int_0^t g_2(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \\ & + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds\Big] \cdot [k(T)+x(t)]^{\frac{1}{p}}, \quad t \in [0,T], \end{aligned}$$

then we get

$$\begin{aligned}\frac{p-1}{p}\frac{x'(t)}{[k(T)+x(t)]^{\frac{1}{p}}} \leqslant & \frac{d}{dt}\left[\int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds\right] \\ & + \frac{d}{dt}\left[\int_0^{\alpha(t)} g_1(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds\right. \\ & + \int_0^t g_2(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \\ & \left. + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds\right].\end{aligned}$$

Considering the integration on $[0,T]$ to obtain

$$\begin{aligned}[k(T)+x(t)]^{1-\frac{1}{p}} \leqslant & (k(T))^{1-\frac{1}{p}} + \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds \\ & + \int_0^{\alpha(t)} g_1(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds + \int_0^t g_2(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \\ & + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds, \ \ t\in[0,T].\end{aligned}$$

Let

$$\begin{aligned}z(T) &= (k(T))^{1-\frac{1}{p}}, \\ B(t) &= \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds, \\ C(t) &= \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds,\end{aligned}$$

then from Lemma 2.1 we can easily get

$$[k(T)+x(t)]^{1-\frac{1}{p}} \leqslant \Phi^{-1}_{1-\frac{1}{p}}\left[\Phi_{1-\frac{1}{p}}(z(T)+B(t))+C(t)\right], \ \ t\in[0,T].$$

Let $t=T$ in the above relation, since $T\geqslant 0$ was arbitrarily chosen, considering $u(t)\leqslant [k(t)+x(t)]^{\frac{1}{p}}$, we get (2.2). □

COROLLARY 2.1. *Let φ,α,k,p be as in Lemma 2.1. Assume in addition that f_i, g_j, a_i, $b_j\in C^1(R_+,R_+)$, $i=1,2,3,4$, $j=1,2$. If $u\in C(R_+,R_+)$ satisfies*

$$\begin{aligned}u^p(t) \leqslant k(t) & + \frac{p}{p-1}\int_0^{\alpha(t)} [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u(s)\varphi(u(s))]\,ds \\ & + \frac{p}{p-1}\int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u(s)\varphi(u(s))]\,ds \\ & + \frac{p}{p-1}\int_0^{\alpha(t)} a_3(t)f_3(s)u(s)ds \cdot \int_0^t a_4(t)f_4(s)\varphi(u(s))ds, \quad \text{for } t\geqslant 0,\end{aligned}$$

then

$$u(t) \leqslant \left\{\Phi^{-1}_{1-\frac{1}{p}}\left[\Phi_{1-\frac{1}{p}}(B(t))+C(t)\right]\right\}^{\frac{1}{p-1}}, \ t \in \Delta,$$

where

$$B(t) = (k(t))^{1-\frac{1}{p}} + \int_0^{\alpha(t)} a_1(t)f_1(s)ds + \int_0^t a_2(t)f_2(s)ds,$$

$$C(t) = \int_0^{\alpha(t)} b_1(t)g_1(s)ds + \int_0^t b_2(t)g_2(s)ds + \int_0^{\alpha(t)} a_3(t)f_3(s)ds \cdot \int_0^t a_4(t)f_4(s)ds,$$

and $\Phi^{-1}_{1-\frac{1}{p}}$ *is the inverse of* $\Phi_{1-\frac{1}{p}}$, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{1-\frac{1}{p}}(B(t)) + C(t) \in Dom\left(\Phi^{-1}_{1-\frac{1}{p}}\right)$.

COROLLARY 2.2. *Let* f_i, g_j, φ, α, k $(i=1,2,3,4,\ j=1,2)$ *be as in Theorem 2.1. Assume in addition that* $p=2$ *be a constant. If* $u \in C(R_+,R_+)$ *satisfies*

$$\begin{aligned} u^2(t) \leqslant\ & k(t) + 2\int_0^{\alpha(t)} [f_1(t,s)u(s) + g_1(t,s)u(s)\varphi(u(s))]\, ds \\ & +2\int_0^t [f_2(t,s)u(s) + g_2(t,s)u(s)\varphi(u(s))]\, ds \\ & +2\int_0^{\alpha(t)} f_3(t,s)u(s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds, \quad for\ \ t \geqslant 0, \end{aligned}$$

then

$$u(t) \leqslant \Phi^{-1}_{\frac{1}{2}}\left[\Phi_{\frac{1}{2}}(B(t))+C(t)\right], \ t \in \Delta,$$

where

$$B(t) = (k(t))^{\frac{1}{2}} + \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds,$$

$$C(t) = \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds$$

and $\Phi^{-1}_{\frac{1}{2}}$ *is the inverse of* $\Phi_{\frac{1}{2}}$, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{\frac{1}{2}}(B(t)) + C(t) \in Dom\left(\Phi^{-1}_{\frac{1}{2}}\right)$.

COROLLARY 2.3. *Let* f_i, a_i, g_j, b_j, φ, α, k $(i=1,2,3,4,\ j=1,2)$ *be as in Corollary 2.1. Assume in addition that* $p=2$ *be a constant. If* $u \in C(R_+,R_+)$ *satisfies*

$$\begin{aligned} u^2(t) \leqslant\ & k(t) + 2\int_0^{\alpha(t)} [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u(s)\varphi(u(s))]\, ds \\ & +2\int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u(s)\varphi(u(s))]\, ds \\ & +2\int_0^{\alpha(t)} a_3(t)f_3(s)u(s)ds \cdot \int_0^t a_4(t)f_4(s)\varphi(u(s))ds, \quad for\ \ t \geqslant 0, \end{aligned}$$

then

$$u(t) \leqslant \Phi_{\frac{1}{2}}^{-1}\left[\Phi_{\frac{1}{2}}(B(t))+C(t)\right], \ \ t \in \Delta,$$

where

$$B(t) = (k(t))^{\frac{1}{2}} + \int_0^{\alpha(t)} a_1(t)f_1(s)ds + \int_0^t a_2(t)f_2(s)ds,$$
$$C(t) = \int_0^{\alpha(t)} b_1(t)g_1(s)ds + \int_0^t b_2(t)g_2(s)ds + \int_0^{\alpha(t)} a_3(t)f_3(s)ds \cdot \int_0^t a_4(t)f_4(s)ds,$$

and $\Phi_{\frac{1}{2}}^{-1}$ *is the inverse of* $\Phi_{\frac{1}{2}}$, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{\frac{1}{2}}(B(t))+C(t) \in Dom\left(\Phi_{\frac{1}{2}}^{-1}\right)$.

THEOREM 2.2. *Let* $f_i,\ g_j \in C(R_+ \times R_+, R_+)$ *with* $(t,s) \mapsto \partial_t f_i(t,s),\ \partial_t g_j(t,s) \in C(R_+ \times R_+, R_+),\ i = 1,2,3,4,\ j = 1,2$, *and* $p > 1$ *be a constant. Assume in addition that* $k \in C(R_+,R_+),\ \alpha \in C^1(R_+,R_+)$ *are non-decreasing functions with* $\alpha(t) \leqslant t$ *for* $t \geqslant 0,\ \varphi(t) > 0$, *and* $\int_1^\infty \frac{dt}{\varphi(t)} = \infty$. *If* $u \in C(R_+,R_+)$ *satisfies*

$$\begin{aligned} u^p(t) \leqslant k(t) &+ \frac{p}{p-1}\int_0^{\alpha(t)} [f_1(t,s)u(s)+g_1(t,s)u^p(s)]\,ds \\ &+ \frac{p}{p-1}\int_0^t [f_2(t,s)u(s)+g_2(t,s)u^p(s)]\,ds \\ &+ \frac{p}{p-1}\int_0^{\alpha(t)} f_3(t,s)u(s)ds \cdot \int_0^t f_4(t,s)u^{p-1}(s)ds, \quad for\ \ t \geqslant 0, \end{aligned}$$

then

$$u(t) \leqslant \left\{(k(t))^{1-\frac{1}{p}} + \int_0^t \left(e^{-C(s)}B(s)\right)ds \cdot e^{C(t)}\right\}^{\frac{1}{p-1}}, \ \ t \in \Delta, \tag{2.3}$$

where

$$B(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds\right),$$
$$C(t) = \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds,$$

and $\Phi_{1-\frac{1}{p}}^{-1}$ *is the inverse of* $\Phi_{1-\frac{1}{p}}$, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{1-\frac{1}{p}}(B(t)) \in Dom\left(\Phi_{1-\frac{1}{p}}^{-1}\right)$.

Proof. Let $T \in \Delta$, $T \geqslant 0$ be fixed and denote

$$\begin{aligned} x(t) = & \frac{p}{p-1}\int_0^{\alpha(t)}[f_1(t,s)u(s)+g_1(t,s)u^p(s)]ds \\ & +\frac{p}{p-1}\int_0^{t}[f_2(t,s)u(s)+g_2(t,s)u^p(s)]ds \\ & +\frac{p}{p-1}\int_0^{\alpha(t)}f_3(t,s)u(s)ds\cdot\int_0^t f_4(t,s)u^{p-1}(s)ds, \end{aligned}$$

then

$$u(t) \leqslant (k(t)+x(t))^{\frac{1}{p}}.$$

Our assumption on f_i, g_j, α, $(i=1,2,3,4,\ j=1,2)$ imply that x is non-decreasing on R_+. Hence, for $t\in[0,T]$, by calculations we have

$$\begin{aligned} x'(t) = & \frac{p}{p-1}\Big\{\Big[f_1(t,\alpha(t))u(\alpha(t))\alpha'(t)+g_1(t,\alpha(t))u^p(\alpha(t))\alpha'(t) \\ & +\int_0^{\alpha(t)}[\partial_t f_1(t,s)u(s)+\partial_t g_1(t,s)u^p(s)]ds\Big] \\ & +\Big[f_2(t,t)u(t)+g_2(t,t)u^p(t)+\int_0^t[\partial_t f_2(t,s)u(s)+\partial_t g_2(t,s)u^p(s)]ds\Big] \\ & +\Big[f_3(t,\alpha(t))u(\alpha(t))\alpha'(t)+\int_0^{\alpha(t)}\partial_t f_3(t,s)u(s)ds\Big]\cdot\int_0^t f_4(t,s)u^{p-1}(s)ds \\ & +\Big[f_4(t,t)u^{p-1}(t)+\int_0^t\partial_t f_4(t,s)u^{p-1}(s)ds\Big]\cdot\int_0^{\alpha(t)}f_3(t,s)u(s)ds\Big\} \\ \leqslant & \frac{p}{p-1}\frac{d}{dt}\Big[\int_0^{\alpha(t)}f_1(t,s)ds+\int_0^t f_2(t,s)ds\Big]\cdot[k(T)+x(t)]^{\frac{1}{p}} \\ & +\frac{p}{p-1}\frac{d}{dt}\Big[\int_0^{\alpha(t)}g_1(t,s)ds+\int_0^t g_2(t,s)ds \\ & +\int_0^{\alpha(t)}f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds\Big](k(T)+x(t)),\ for\ all\ t\in[0,T], \end{aligned}$$

then we get

$$\begin{aligned} \frac{p-1}{p}\frac{x'(t)}{[k(T)+x(t)]^{\frac{1}{p}}} & -\frac{d}{dt}\Big[\int_0^{\alpha(t)}g_1(t,s)ds+\int_0^t g_2(t,s)ds \\ & +\int_0^{\alpha(t)}f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds\Big][k(T)+x(t)]^{1-\frac{1}{p}} \\ & \leqslant \frac{d}{dt}\Big[\int_0^{\alpha(t)}f_1(t,s)ds+\int_0^t f_2(t,s)ds\Big]. \end{aligned}$$

Let

$$z(t) = [k(T)+x(t)]^{1-\frac{1}{p}},$$

$$B(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds\right), \tag{2.4}$$
$$C(t) = \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds,$$

then the above relation is equivalent to

$$z'(t) - \frac{d}{dt}C(t)z(t) \leqslant B(t).$$

Multiplying the above inequality by $e^{-C(t)}$ and considering the integration on $[0,t]$ to obtain

$$z(t) \leqslant \left\{(k(T))^{1-\frac{1}{p}} + \int_0^t \left(e^{-C(s)}B(s)\right)ds\right\} e^{C(t)},\ for\ all\ t \in [0,T]. \tag{2.5}$$

Now, using (2.4), (2.5), and let $t = T$, then

$$k(T) + x(T) \leqslant \left\{(k(T))^{1-\frac{1}{p}} + \int_0^T \left(e^{-C(s)}B(s)\right)ds \cdot e^{C(T)}\right\}^{\frac{p}{p-1}}$$

since $T \geqslant 0$ was arbitrarily chosen, considering $u(t) \leqslant [k(t)+x(t)]^{\frac{1}{p}}$, we get(2.3). □

COROLLARY 2.4. *Let f_i, g_j, a_i, b_j, α, k, p be as in Corollary 2.1, $i = 1,2,3,4$, $j = 1,2$. If $u \in C(R_+,R_+)$ satisfies*

$$\begin{aligned} u^p(t) \leqslant k(t) &+ \frac{p}{p-1}\int_0^{\alpha(t)} [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u^p(s)]\,ds \\ &+ \frac{p}{p-1}\int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u^p(s)]\,ds \\ &+ \frac{p}{p-1}\int_0^{\alpha(t)} a_3(t)f_3(s)u(s)ds \cdot \int_0^t a_4(t)f_4(s)u^{p-1}(s)ds,\ \ for\ \ t \geqslant 0, \end{aligned}$$

then

$$u(t) \leqslant \left\{(k(t))^{1-\frac{1}{p}} + \int_0^t \left(e^{-C(s)}B(s)\right)ds \cdot e^{C(t)}\right\}^{\frac{1}{p-1}},\ \ t \in \Delta,$$

where

$$B(t) = \frac{d}{dt}\left(\int_0^{\alpha(t)} a_1(t)f_1(s)ds + \int_0^t a_2(t)f_2(s)ds\right),$$
$$C(t) = \int_0^{\alpha(t)} b_1(t)g_1(s)ds + \int_0^t b_2(t)g_2(s)ds + \int_0^{\alpha(t)} a_3(t)f_3(s)ds \cdot \int_0^t a_4(t)f_4(s)ds.$$

COROLLARY 2.5. *Let f_i, g_j, α, k be as in Theorem 2.2, $i=1,2,3,4$, $j=1,2$. Assume in addition that $p=2$ be a constant. If $u\in C(R_+,R_+)$ satisfies*

$$\begin{aligned} u^2(t) \leqslant k(t) &+ 2\int_0^{\alpha(t)} \left[f_1(t,s)u(s)+g_1(t,s)u^2(s)\right] ds \\ &+2\int_0^t \left[f_2(t,s)u(s)+g_2(t,s)u^2(s)\right] ds \\ &+2\int_0^{\alpha(t)} f_3(t,s)u(s)ds\cdot\int_0^t f_4(t,s)u(s)ds, \quad for\ \ t\geqslant 0, \end{aligned}$$

then

$$u(t)\leqslant (k(t))^{\frac{1}{2}}+\int_0^t \left(e^{-C(s)}B(s)\right) ds\cdot e^{C(t)},\ \ t\geqslant 0,$$

where

$$\begin{aligned} B(t) &= \frac{d}{dt}\left(\int_0^{\alpha(t)} f_1(t,s)ds+\int_0^t f_2(t,s)ds\right), \\ C(t) &= \int_0^{\alpha(t)} g_1(t,s)ds+\int_0^t g_2(t,s)ds+\int_0^{\alpha(t)} f_3(t,s)ds\cdot\int_0^t f_4(t,s)ds. \end{aligned}$$

COROLLARY 2.6. *Let f_i, a_i, g_j, b_j, α, k, be as in Corollary 2.1, $i=1,2,3,4$, $j=1,2$. Assume in addition that $p=2$ be a constant. If $u\in C(R_+,R_+)$ satisfies*

$$\begin{aligned} u^2(t) \leqslant k(t) &+ 2\int_0^{\alpha(t)} [a_1(t)f_1(s)u(s)+b_1(t)g_1(s)u^p(s)]\, ds \\ &+2\int_0^t [a_2(t)f_2(s)u(s)+b_2(t)g_2(s)u^p(s)]\, ds \\ &+2\int_0^{\alpha(t)} a_3(t)f_3(s)u(s)ds\cdot\int_0^t a_4(t)f_4(s)u^p(s)ds, \quad for\ \ t\geqslant 0, \end{aligned}$$

then

$$u(t)\leqslant (k(t))^{\frac{1}{2}}+\int_0^t \left(e^{-C(s)}B(s)\right) ds\cdot e^{C(t)},\ \ t\geqslant 0,$$

where

$$\begin{aligned} B(t) &= \frac{d}{dt}\left(\int_0^{\alpha(t)} a_1(t)f_1(s)ds+\int_0^t a_2(t)f_2(s)ds\right), \\ C(t) &= \int_0^{\alpha(t)} b_1(t)g_1(s)ds+\int_0^t b_2(t)g_2(s)ds+\int_0^{\alpha(t)} a_3(t)f_3(s)ds\cdot\int_0^t a_4(t)f_4(s)ds. \end{aligned}$$

THEOREM 2.3. *Let f_i, $g_j\in C(R_+\times R_+,R_+)$ with $(t,s)\mapsto \partial_t f_i(t,s)$, $\partial_t g_j(t,s)\in C(R_+\times R_+,\ R_+)$, $i=1,2,3,4$, $j=1,2$, $p>q\geqslant 0$ be constants. Assume in addition* *that* $\varphi, k \in C(R_+, R_+)$, $\alpha \in C^1(R_+, R_+)$ *are non-decreasing functions with* $\alpha(t) \leqslant t$ *for* $t \geqslant 0$, $\varphi(t) > 0$, *and* $\int_1^\infty \frac{dt}{\varphi(t)} = \infty$. *If* $u \in C(R_+, R_+)$ *satisfies*

$$\begin{aligned} u^p(t) \leqslant k(t) &+ \frac{p}{p-q} \int_0^{\alpha(t)} [f_1(t,s)u^q(s) + g_1(t,s)u^q(s)\varphi(u(s))]\, ds \\ &+ \frac{p}{p-q} \int_0^t [f_2(t,s)u^q(s) + g_2(t,s)u^q(s)\varphi(u(s))]\, ds \\ &+ \frac{p}{p-q} \int_0^{\alpha(t)} f_3(t,s)u^q(s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds, \quad for \ \ t \geqslant 0, \end{aligned} \tag{2.6}$$

then

$$u(t) \leqslant \left\{\Phi_{p-q}^{-1}[\Phi_{p-q}(B(t)) + C(t)]\right\}^{\frac{1}{p-q}}, \ \ t \in \Delta,$$

where

$$B(t) = (k(t))^{\frac{p}{q}-1} + \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^t f_2(t,s)ds, \tag{2.7}$$

$$C(t) = \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds, \tag{2.8}$$

and Φ_{p-q}^{-1} *is the inverse of* Φ_{p-q}, *and* $t \in \Delta$ *is chosen in such a way that* $\Phi_{p-q}(B(t)) + C(t) \in Dom\left(\Phi_{p-q}^{-1}\right)$.

Proof. For any $r > 0$, define $\psi(r) = \varphi(r^{\frac{1}{q}})$, then our assumption on φ imply that ψ is non-decreasing on R_+, then (2.6) is equivalent to

$$\begin{aligned} u^p(t) \leqslant k(t) &+ \frac{p}{p-q} \int_0^{\alpha(t)} [f_1(t,s)u^q(s) + g_1(t,s)u^q(s)\psi(u^q(s)]ds \\ &+ \frac{p}{p-q} \int_0^{\alpha(t)} [f_2(t,s)u^q(s) + g_2(t,s)u^q(s)\psi(u^q(s))]ds \\ &+ \frac{p}{p-q} \int_0^{\alpha(t)} f_3(t,s)u^q(s)ds \cdot \int_0^t f_4(t,s)\psi(u^q(s))ds. \end{aligned}$$

Let $v(t) = u^q(t)$, then the above inequality is equivalent to

$$\begin{aligned} v^{\frac{p}{q}}(t) \leqslant k(t) &+ \frac{\frac{p}{q}}{\frac{p}{q}-1} \int_0^{\alpha(t)} [f_1(t,s)v(s) + g_1(t,s)v(s)\psi(v(s))]ds \\ &+ \frac{\frac{p}{q}}{\frac{p}{q}-1} \int_0^{\alpha(t)} [f_2(t,s)v(s) + g_2(t,s)v(s)\psi(v(s))]ds \\ &+ \frac{\frac{p}{q}}{\frac{p}{q}-1} \int_0^{\alpha(t)} f_3(t,s)v(s)ds \cdot \int_0^t f_4(t,s)\psi(v(s))ds. \end{aligned}$$

Since $\frac{p}{q} > 1$, it follows from Theorem 2.1 that

$$v(t) \leqslant \left\{\Psi^{-1}_{\frac{p}{q}-1}\left[\Psi_{\frac{p}{q}-1}(B(t))+C(t)\right]\right\}^{\frac{1}{\frac{p}{q}-1}}, \ t \in [0,T],$$

where $B(t)$, $C(t)$ is defined as (2.7), (2.8). Now it is elementary to check by the definition of Ψ, then

$$\Psi_{\frac{p}{q}-1}(r) = \Phi_{p-q}(r),$$

thus we have

$$v(t) \leqslant \left\{\Phi^{-1}_{p-q}\left[\Phi_{p-q}(B(t))+C(t)\right]\right\}^{\frac{q}{p-q}}, \ \ for \ all \ \ t \in [0,T],$$

considering $u(t) = v^{\frac{1}{q}}(t)$, we get

$$u(t) \leqslant \left\{\Phi^{-1}_{p-q}\left[\Phi_{p-q}(B(t))+C(t)\right]\right\}^{\frac{1}{p-q}}, \ \ for \ all \ \ t \in [0,T]. \quad \square$$

COROLLARY 2.7. *Let α, k, p, q be as in Theorem 2.3. Assume in addition that $f_i,\ g_j,\ a_i,\ b_j \in C^1(R_+,R_+)$, $i = 1,2,3,4$, $j = 1,2$. If $u \in C(R_+,R_+)$ satisfies*

$$\begin{aligned} u^p(t) \leqslant k(t) &+ \frac{p}{p-q}\int_0^{\alpha(t)} \left[a_1(t)f_1(s)u^q(s)+b_1(t)g_1(s)u^q(s)\varphi(u(s))\right]ds \\ &+\frac{p}{p-q}\int_0^t \left[a_2(t)f_2(s)u^q(s)+b_2(t)g_2(s)u^q(s)\varphi(u(s))\right]ds \\ &+\frac{p}{p-q}\int_0^{\alpha(t)} a_3(t)f_3(s)u^q(s)ds\cdot\int_0^t a_4(t)f_4(s)\varphi(u(s))ds, \quad for \ \ t \geqslant 0, \end{aligned}$$

then

$$u(t) \leqslant \left\{\Phi^{-1}_{p-q}\left[\Phi_{p-q}(B(t))+C(t)\right]\right\}^{\frac{1}{p-q}}, \ \ t \in \Delta,$$

where

$$B(t) = (k(t))^{\frac{p}{q}-1} + \int_0^{\alpha(t)} a_1(t)f_1(s)ds + \int_0^t a_2(t)f_2(s)ds,$$

$$C(t) = \int_0^{\alpha(t)} b_1(t)g_1(s)ds + \int_0^t b_2(t)g_2(t)(s)ds + \int_0^{\alpha(t)} a_3(t)f_3(s)ds\cdot\int_0^t a_4(t)f_4(s)ds.$$

and Φ^{-1}_{p-q} is the inverse of Φ_{p-q}, and $t \in \Delta$ is chosen in such a way that $\Phi_{p-q}(B(t)) + C(t) \in Dom\left(\Phi^{-1}_{p-q}\right)$.

REMARK. Different choices of k, α, a_i, b_j can give many different inequalities, $i = 1,2,3,4$, $j = 1,2$. For example, let $a_1(t) = 1$, $a_2(t) = b_1(t) = b_2(t) = a_3(t) = 0$, $\alpha(t) = t$, $k(t) = k^2$, $t \in [0,\infty)$, where $k \geqslant 0$ be a constant, our Corollary 2.6 reduces to Theorem A; or let $a_1(t) = b_1(t) = 1$, $a_2(t) = b_2(t) = a_3(t) = 0$, $\alpha(t) = t$, $k(t) = k^2$, $t \in [0,\infty)$, where $k \geqslant 0$ is a constant, our Corollary 2.3 reduces to Theorem B.

3. Examples

Now, we will show that our results are useful in studying the boundedness and stability of solutions to certain integro-differential equations with time delay. These applications are given as examples.

EXAMPLE 1. Consider the nonlinear integro-differential equation with time delay

$$\begin{cases} x'(t) = y - F(t,x) + \int_0^t H(s,x(s-\tau(s)))ds, \\ y'(t) = G(t,x(t-\tau(t))), \end{cases} \tag{3.1}$$

where $F,H,G \in C(R_+ \times R_+, R_+), \tau \in C^1(R_+,R_+)$, and $\tau(t) \leqslant t$ on R_+. If $\alpha(t) = t-\tau(t)$ is an increasing diffeomorphism on R_+, and

$$\begin{aligned} -xF(t,x) &\leqslant a(t)|x|v(|x|), \\ G^2(t,x) &\leqslant b(t)|x|v(|x|), \\ \left(\int_0^t H(s,x)ds\right)^2 &\leqslant |x|\int_0^t c(s)|x|ds, \quad (t,x) \in (R_+ \times R_+), \end{aligned}$$

for $a(t)$, $b(t)$, $c(t) \in C(R_+,R_+)$ and some non-decreasing function $v \in C(R_+,R_+)$ with the properties $v(u) > 0$ for $u > 0$ and $\int_1^\infty \frac{ds}{v(s)} = \infty$, then all the solutions of (3.1) are bounded and global.

In fact, if $(x(t),y(t))$ is a solution of (3.1) defined on the maximal existence interval $[0,T)$, let $u(t) = \sqrt{x^2(t)+y^2(t)}$ and $p(t) = \max\{1,a(t)\} \in C(R_+,R_+)$ for $t \in [0,T)$. From (3.1) and our hypotheses on the functions F,H,G and v, we obtain

$$\begin{aligned} \frac{d}{dt}u^2(t) &= 2xx' + 2yy' \\ &= 2xy - 2xF(t,x) + 2yG(t,x(\alpha)) + 2x\int_0^t H(s,x(\alpha))ds \\ &\leqslant x^2 + y^2 + 2a(t)|x|v(|x|) + y^2 + G^2(t,x(\alpha)) + x^2 + \left(\int_0^t H(s,x(\alpha))ds\right)^2 \\ &\leqslant 2p(t)u^2 + 2p(t)uv(u) + b(t)|x(\alpha)|v(|x(\alpha)|) + |(x(\alpha))|\int_0^t c(s)|(x(\alpha))|ds, \\ &\qquad t \in [0,T). \end{aligned}$$

With $\varphi(u) := u + v(u)$, an integration on $[0,t]$, with $t < T$ yields

$$\begin{aligned} u^2(t) \leqslant\ & u^2(0) + \int_0^t p(s)u(s)\varphi(u(s))ds + 2\int_0^t b(s)|x(\alpha(s))|v(|x(\alpha(s))|)ds \\ &+ \int_0^t |(x(\alpha(s)))|\left(\int_0^s c(r)|(x(\alpha(r)))|dr\right)ds \\ \leqslant\ & u^2(0) + 2\int_0^t p(s)u(s)\varphi(u(s))ds + \int_0^t b(s)u(\alpha(s))\varphi(u(\alpha(s)))ds \\ &+ \int_0^t \varphi(u(s))\left(\int_0^s c(r)u(\alpha(r))dr\right)ds \end{aligned}$$

$$\leqslant u^2(0)+2\int_0^t p(s)u(s)\varphi(u(s))ds+2\int_0^{\alpha(t)} \frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)\varphi(u(s))ds$$
$$+2\int_0^t \varphi(u(s))ds\int_0^{\alpha(t)} \frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)ds,$$

after performing the change of variable $r=\alpha(s)$ and some intermediate steps, where α^{-1} is the inverse of the diffeomorphism α. Our hypotheses on v guarantee that $\int_1^\infty \frac{dr}{\varphi(r)}dr=\infty$ (see [8]). Therefore, $\Phi(r)=\int_1^r \frac{ds}{\varphi(s)}, r>0$, then from Corollary 2.3, we deduce that

$$u(t)\leqslant \Phi_{\frac{1}{2}}^{-1}\left[\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^{\alpha(t)} \frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right.$$
$$\left.+t\int_0^{\alpha(t)} \frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)ds\right],$$
$$=\Phi_{\frac{1}{2}}^{-1}\left[\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^t b(s)ds+t\int_0^t c(s)ds\right],\quad t\in[0,T).$$

This prove that $u(t)$ is bounded on $[0,T)$ if $T<\infty$, and all solutions of (3.1) are global. $T\in\Delta$ is chosen in such a way that $\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^t b(s)ds+t\int_0^t c(s)ds\in Dom\left(\Phi_{\frac{1}{2}}^{-1}\right)$, $\Phi_{\frac{1}{2}}^{-1}$ is the inverse of $\Phi_{\frac{1}{2}}$.

EXAMPLE 2. Consider the nonlinear integro-differential equation with time delay

$$\begin{cases} x'(t)=y+\int_0^t H(s,x(s-\tau(s)))ds, \\ y'(t)=-F(t,y)+G(t,x(t-\tau(t))), \end{cases} \tag{3.2}$$

where $F,H,G\in C(R_+\times R_+,R_+)$, $\tau\in C^1(R_+,R_+)$, and $\tau(t)\leqslant t$ on R_+. If $\alpha(t)=t-\tau(t)$ is an increasing diffeomorphism on R_+, and

$$-yF(t,y)\leqslant a(t)|y|v(|y|),$$
$$G^2(t,x)\leqslant b(t)|x|v(|x|),$$
$$\left(\int_0^t H(s,x)ds\right)^2\leqslant |x|\int_0^t c(s)|x|ds,\quad (t,y),(t,x)\in(R_+\times R_+),$$

for $a(t)$, $b(t)$, $c(t)\in C(R_+,R_+)$ and some non-decreasing function $v\in C(R_+,R_+)$ with the properties $v(u)>0$ for $u>0$ and $\int_1^\infty \frac{ds}{v(s)}=\infty$, then all the solutions of (3.2)are bounded and global.

In fact, if $(x(t),y(t))$ is a solution of (3.2) defined on the maximal existence interval $[0,T)$, let $u(t)=\sqrt{x^2(t)+y^2(t)}$ and $p(t)=\max\{1,a(t)\}\in C(R_+,R_+)$ for $t\in[0,T)$. From (3.2) and our hypotheses on the functions F,H,G and v, we obtain

$$\frac{d}{dt}u^2(t)=2xx'+2yy'$$
$$=2xy-2yF(t,y)+2yG(t,x(\alpha))+2x\int_0^t H(s,x(\alpha))ds$$

$$\leqslant x^2+y^2+2a(t)|y|v(|y|)+y^2+G^2(t,x(\alpha))+x^2+\left(\int_0^t H(s,x(\alpha))ds\right)^2$$
$$\leqslant 2p(t)u^2+2p(t)uv(u)+b(t)|x(\alpha)|v(|x(\alpha)|)$$
$$+|x(\alpha)|\int_0^t c(s)|x(\alpha)|ds, \quad t\in[0,T).$$

With $\varphi(u):=u+v(u)$, an integration on $[0,t]$, with $t<T$ yields

$$u^2(t)\leqslant u^2(0)+2\int_0^t p(s)u(s)\varphi(u(s))ds+\int_0^t b(s)\mid x(\alpha(s))\mid v(\mid x(\alpha(s))\mid)ds$$
$$+2\int_0^t |x(\alpha)|\left(\int_0^s c(r)|x(\alpha)|dr\right)ds$$
$$\leqslant u^2(0)+2\int_0^t p(s)u(s)\varphi(u(s))ds+\int_0^t b(s)u(\alpha(s))\varphi(u(\alpha(s)))ds$$
$$+\int_0^t \varphi(u(s))\left[\int_0^s c(r)u(\alpha(r))dr\right]ds$$
$$\leqslant u^2(0)+2\int_0^t p(s)u(s)\varphi(u(s))ds+2\int_0^{\alpha(t)}\frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)\varphi(u(s))ds$$
$$+2\int_0^t \varphi(u(s))ds\int_0^{\alpha(t)}\frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)ds,$$

after performing the change of variables $r=\alpha(s)$ and some intermediate steps, where α^{-1} is the inverse of the diffeomorphism α. Our hypotheses on v guarantee that $\int_1^\infty \frac{dr}{\varphi(r)}dr=\infty$ (see [8]). Therefore, $\Phi(r)=\int_1^r\frac{ds}{\varphi(s)}, r>0$, then from Corollary 2.3, we deduce that

$$u(t)\leqslant \Phi_{\frac{1}{2}}^{-1}\left[\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^{\alpha(t)}\frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\right.$$
$$\left.+\,t\int_0^{\alpha(t)}\frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}u(s)ds\right]$$
$$=\Phi_{\frac{1}{2}}^{-1}\left[\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^t b(s)ds+t\int_0^t c(s)ds\right], \quad t\in[0,T).$$

This prove that $u(t)$ is bounded on $[0,T)$ if $T<\infty$, and all solutions of (3.2) are global. $T\in\Delta$ is chosen in such a way that $\Phi_{\frac{1}{2}}(u(0))+\int_0^t p(s)ds+\int_0^t b(s)ds+t\int_0^t c(s)ds\in$ Dom $\left(\Phi_{\frac{1}{2}}^{-1}\right)$, $\Phi_{\frac{1}{2}}^{-1}$ is the inverse of $\Phi_{\frac{1}{2}}$.

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