SOME GENERALIZED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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(Communicated by Q.-H. Ma)

Abstract. In this paper, we generalize some integral inequalities to more general situations. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of integral equations and differential equations. Applications are given to illustrate the usefulness of the inequalities.

1. Introduction

Over the years integral inequalities have become a major tool in the analysis of various differential and integral equations that occur in nature or are built by man (see [1 – 13]). In studying the boundedness behavior of the solutions of certain differential and integral equations, Ou-Iang [1] and Pachpatte [2,3] gave some new integral inequalities. We list them as follows.

THEOREM A. (Ou-Iang) (See [1]) If u and f are non-negative functions on \([0, \infty)\) satisfying

\[ u^2(t) \leq k^2 + 2 \int_0^t f(s)u(s)ds, \quad \text{for all } t \in [0, \infty), \]

where \(k \geq 0\) is a constant, then

\[ u(t) \leq k + \int_0^t f(s)ds. \quad t \in [0, \infty). \]

THEOREM B. (Pachpatte) (See [2]) Suppose that \(u, f, g\) are continuous non-negative functions on \([0, \infty)\) and \(\omega\) is a continuous non-decreasing function on \([0, \infty)\) with \(\omega(r) > 0\) for \(r > 0\). If

\[ u^2(t) \leq k^2 + 2 \int_0^t \left[ f(s)u(s) + g(s)u(s)\omega(u(s)) \right] ds, \quad \text{for all } t \in [0, \infty), \]


Keywords and phrases: Integral inequality, integral equation, differential equation, global existence.

This research is supported by the National Natural Science Foundation of China (11171178, 11271225).
where $k$ is a constant, then
\[ u(t) \leq \Omega^{-1} \left[ \Omega \left( k + \int_0^t f(s)ds \right) + \int_0^t g(s)ds \right], \text{ for all } t \in [0, t_1], \]
where
\[ \Omega(t) := \int_1^t \frac{1}{\omega(s)}ds, \quad t > 0, \]
$\Omega^{-1}$ is the inverse of $\Omega$, and $t_1 \in [0, \infty)$ is chosen in such a way that $\Omega(k + \int_0^t f(s)ds + \int_0^t g(s)ds) \in \text{Dom}(\Omega^{-1})$, for all $t \in [0, t_1]$.

**Theorem C.** (Pachpatte) (See [3]) Let $u, f, g$ be real-valued non-negative continuous functions defined on $R_+$, and $c_1, c_2$ be non-negative constants. If
\[ u(t) \leq \left( c_1 + \int_0^t f(s)u(s)ds \right) \left( c_2 + \int_0^t g(s)u(s)ds \right) \]
and
\[ c_1 c_2 \int_0^t R(s)Q(s)ds < 1, \text{ for all } t \in R_+, \]
then
\[ u(t) \leq \frac{c_1 c_2 Q(t)}{1 - c_1 c_2 \int_0^t R(s)Q(s)ds}, \quad t \in R_+, \]
where
\[ R(t) = \int_0^t \left[ f(s)g(s) + f(s)g(t) \right]ds, \]
\[ Q(t) = \exp \left( \int_0^t \left[ c_1 g(s) + c_2 f(s) \right]ds \right). \]

The main aim of the present is to generalize some integral inequalities to more general situations, which can be used as ready and powerful tools in the study of qualitative as well as quantitative properties of solutions of integral equations and differential equations. We also illustrate the usefulness of these inequalities.

### 2. Main results

In the next Theorems and Corollaries, for any $\varphi, \psi \in C(R_+, R_+)$ and any constants $p, q \geq 0$, define
\[ \Phi_p(r) := \int_1^r \frac{ds}{\varphi(s^{\frac{1}{p}})}, \quad \Phi_p(0) = \lim_{r \to 0^+} \Phi_p(r); \]
\[ \Psi_q(r) := \int_1^r \frac{ds}{\psi(s^{\frac{1}{q}})}, \quad \Psi_q(0) = \lim_{r \to 0^+} \Psi_q(r). \]

To prove our Theorem 2.1, we need the following lemma.
Lemma 2.1. Let \( f_i \in C(R_+ \times R_+, R_+)\) with \((t,s) \mapsto \partial_t f_i(t,s), \partial_s f_i(t,s) \in C(R_+ \times R_+, R_+)\), \(i = 1, 2, 3, 4\), \(p > 0\) be a constant. Assume in addition that \( \varphi, k \in C(R_+, R_+)\), \(\alpha \in C^1(R+, R_+)\) are non-decreasing functions with \(\alpha(t) \leq t\) for \(t \geq 0\), \(\varphi(t) > 0\), and 
\[
\int_1^{\infty} \frac{dt}{\varphi(t)} = \infty.
\]
If \(u \in C(R_+, R_+)\) satisfies
\[
\begin{align*}
u^p(t) &\leq k(t) + \int_0^{\alpha(t)} f_1(t,s)\varphi(u(s))ds + \int_0^{t} f_2(t,s)\varphi(u(s))ds \\
&\quad + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^{t} f_4(t,s)\varphi(u(s))ds, \quad t \geq 0,
\end{align*}
\]
then
\[
u(t) \leq \left\{ \Phi^{-1}_p[\Phi_p(k(t)) + A(t)] \right\}^{1/p}, \quad t \in \Delta,
\]
where
\[
A(t) = \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^{t} f_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^{t} f_4(t,s)ds
\]

\(\Phi_p^{-1}\) is the inverse of \(\Phi_p\), and \(t \in \Delta\) is chosen in such a way that, \(\Phi_p(k(t)) + A(t) \in \text{Dom}(\Phi_p^{-1})\).

Proof. Let \(T \in \Delta, T \geq 0\) be fixed and denote
\[
x(t) = \int_0^{\alpha(t)} f_1(t,s)\varphi(u(s))ds + \int_0^{t} f_2(t,s)\varphi(u(s))ds \\
+ \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^{t} f_4(t,s)\varphi(u(s))ds,
\]
then
\[
u(t) \leq [k(t) + x(t)]^{1/p},
\]
our assumption on \(f_i, \varphi, \alpha\) imply that \(x\) is non-decreasing on \(R_+, i = 1, 2, 3, 4\). Hence for \(t \in [0, T]\), by calculations we have
\[
x'(t) = \left[ f_1(t, \alpha(t))\varphi(u(\alpha(t)))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_1(t,s)\varphi(u(s))ds \right]
\]
\[
+ \left[ f_2(t,t)\varphi(u(t)) + \int_0^{t} \partial_t f_2(t,s)\varphi(u(s))ds \right]
\]
\[
+ \left[ f_3(t, \alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_3(t,s)ds \right] \cdot \int_0^{t} f_4(t,s)\varphi(u(s))ds
\]
\[
+ \left[ f_4(t,t)\varphi(u(t)) + \int_0^{t} \partial_t f_4(t,s)\varphi(u(s))ds \right] \cdot \int_0^{\alpha(t)} f_3(t,s)ds
\]
\[
\begin{align*}
&\leq \varphi \left[ (k(T) + x(t))^\frac{1}{p} \right] \cdot \frac{d}{dt} \left[ \int_0^t f_1(t,s)ds + \int_0^t f_2(t,s)ds + \int_0^t f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds \right], \\
&\text{then we get} \\
&\frac{x'(t)}{\varphi \left[ (k(T) + x(t))^\frac{1}{p} \right]} \leq \frac{d}{dt} \left[ \int_0^t f_1(t,s)ds + \int_0^t f_2(t,s)ds + \int_0^t f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds \right].
\end{align*}
\]

Considering the definition of \( \Phi \) and the integral on the interval \([0,t]\), yields

\[
\Phi_p(x(t) + k(T)) \leq \Phi_p(k(T)) + \int_0^t f_1(t,s)ds + \int_0^t f_2(t,s)ds + \int_0^t f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds, \quad t \in [0,T].
\]

As \( \Phi_p^{-1} \) is increasing on \( \text{Dom}(\Phi_p^{-1}) \), then

\[
x(t) + k(T) \leq \Phi_p^{-1} \left[ \Phi_p(k(T)) + \int_0^t f_1(t,s)ds + \int_0^t f_2(t,s)ds + \int_0^t f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds \right], \quad t \in [0,T].
\]

Let \( t = T \) in the above relation, since \( T \geq 0 \) was arbitrarily chosen, considering \( u(t) \leq (x(t) + k(t))^\frac{1}{p} \), we get (2.1). \( \square \)

**Theorem 2.1.** Let \( f_i, \varphi, \alpha, k \) be as in Lemma 2.1, \( i = 1,2,3,4 \). Assume in addition that \( g_j \in C(R_+ \times R_+ \times R_+) \) with \( (t,s) \mapsto \partial_t g_j(t,s), \partial_s g_j(t,s) \in C(R_+ \times R_+, R_+) \), \( j = 1,2 \), and \( p > 1 \) be a constant. If \( u \in C(R_+, R_+) \) satisfies

\[
u^p(t) \leq k(t) + \frac{p}{p-1} \int_0^t \left[ f_1(t,s)u(s) + g_1(t,s)u(s)\varphi(u(s)) \right] ds + \frac{p}{p-1} \int_0^t \left[ f_2(t,s)u(s) + g_2(t,s)u(s)\varphi(u(s)) \right] ds + \frac{p}{p-1} \int_0^t \left[ f_3(t,s)u(s) + g_3(t,s)u(s)\varphi(u(s)) \right] ds \quad \text{for } t \geq 0,
\]

then

\[
u(t) \leq \left\{ \Phi_p^{-\frac{1}{p}} \left[ \Phi_{p-\frac{1}{p}} \left( z(t) + B(t) + C(t) \right) \right] \right\} \frac{1}{p-1}, \quad t \in \Delta,
\]
where
\[
z(t) = (k(t))^{1 - \frac{1}{p}},
\]
\[
B(t) = (k(t))^{1 - \frac{1}{p}} + \int_0^\alpha f_1(t,s)ds + \int_0^t f_2(t,s)ds,
\]
\[
C(t) = \int_0^\alpha g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^\alpha f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds,
\]
\[
\Phi_{\frac{1}{p}}^{-1}\text{ is the inverse of } \Phi_{\frac{1}{p}}, \text{ and } t \in \Delta \text{ is chosen in such a way that } \Phi_{\frac{1}{p}}^{-1}(B(t)) \in \text{Dom}(\Phi_{\frac{1}{p}}).
\]

**Proof.** Let \( T \in \Delta, T \geq 0 \) be fixed and denote
\[
x(t) = \frac{p}{p-1} \left[ f_1(t,\alpha(t))u(\alpha(t))\alpha'(t) + g_1(t,\alpha(t))u(\alpha(t))\Phi(\alpha(t))\alpha'(t)\right.
\]
\[
+ \int_0^\alpha \left( \partial_t f_1(t,s)u(s) + \partial_s g_1(t,s)u(s)\Phi(u(s)) \right)ds
\]
\[
+ \frac{p}{p-1} \left[ f_2(t,t)u(t) + g_2(t,t)u(t)\Phi(u(t)) \right.
\]
\[
+ \int_0^t \left( \partial_t f_2(t,s)u(s) + \partial_s g_2(t,s)u(s)\Phi(u(s)) \right)ds
\]
\[
+ \frac{p}{p-1} \left[ f_3(t,t)\Phi(u(t)) + \int_0^\alpha \partial_t f_3(t,s)u(s)ds \right] \cdot \int_0^t f_4(t,s)\Phi(u(s))ds
\]
\[
+ \frac{p}{p-1} \left[ f_4(t,t)\Phi(u(t)) + \int_0^\alpha \partial_t f_4(t,s)\Phi(u(s))ds \right] \cdot \int_0^\alpha f_3(t,s)u(s)ds
\]
\[
\leq \frac{p}{p-1} \frac{d}{dt} \left[ \int_0^\alpha f_1(t,s)ds + \int_0^t f_2(t,s)ds \right] \cdot [k(T) + x(t)]^{\frac{1}{p}}
\]
\[
+ \frac{p}{p-1} \frac{d}{dt} \left[ \int_0^\alpha g_1(t,s)\Phi((k(T) + x(t))^{\frac{1}{p}})ds + \int_0^t g_2(t,s)\Phi((k(T) + x(t))^{\frac{1}{p}})ds
\]
\[
+ \int_0^\alpha f_3(t,s)ds \cdot \int_0^t f_4(t,s)\Phi((k(T) + x(t))^{\frac{1}{p}})ds \right] \cdot [k(T) + x(t)]^{\frac{1}{p}}, \quad t \in [0,T],
\]
then we get
\[
\frac{p-1}{p} \frac{x'(t)}{[k(T)+x(t)]^{\frac{1}{p}}} \leq \frac{d}{dt} \left[ \int_0^T f_1(t,s)ds + \int_0^t f_2(t,s)ds \right]
+ \frac{d}{dt} \left[ \int_0^T g_1(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \right]
+ \frac{d}{dt} \left[ \int_0^T g_2(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \right]
+ \frac{d}{dt} \left[ \int_0^T f_3(t,s)ds \cdot \int_0^t f_4(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds \right].
\]

Considering the integration on \([0,T]\) to obtain
\[
[k(T)+x(t)]^{1-\frac{1}{p}} \leq (k(T))^{1-\frac{1}{p}} + \int_0^T f_1(t,s)ds + \int_0^T f_2(t,s)ds
+ \int_0^T g_1(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds + \int_0^T g_2(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds
+ \int_0^T f_3(t,s)ds \cdot \int_0^T f_4(t,s)\varphi((k(T)+x(t))^{\frac{1}{p}})ds, \quad t \in [0,T].
\]

Let
\[
z(T) = (k(T))^{1-\frac{1}{p}},
B(t) = \int_0^T f_1(t,s)ds + \int_0^T f_2(t,s)ds,
C(t) = \int_0^T g_1(t,s)ds + \int_0^T g_2(t,s)ds + \int_0^T f_3(t,s)ds \cdot \int_0^T f_4(t,s)ds,
\]
then from Lemma 2.1 we can easily get
\[
[k(T)+x(t)]^{1-\frac{1}{p}} \leq \Phi^{-1}_{1-\frac{1}{p}} \left[ \Phi_{1-\frac{1}{p}}(z(T) + B(t)) + C(t) \right], \quad t \in [0,T].
\]

Let \(t = T\) in the above relation, since \(T \geq 0\) was arbitrarily chosen, considering \(u(t) \leq [k(t)+x(t)]^{\frac{1}{p}}\), we get (2.2). \(\square\)

**Corollary 2.1.** Let \(\varphi, \alpha, k, p\) be as in Lemma 2.1. Assume in addition that \(f_i, g_j, a_i, b_j \in C^1(R_+, R_+), i = 1, 2, 3, 4, j = 1, 2.\) If \(u \in C(R_+, R_+)\) satisfies
\[
u^p(t) \leq k(t) + \frac{p}{p-1} \int_0^T [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u(s)\varphi(u(s))]ds
+ \frac{p}{p-1} \int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u(s)\varphi(u(s))]ds
+ \frac{p}{p-1} \int_0^T a_3(t)f_3(s)u(s)ds \cdot \int_0^t a_4(t)f_4(s)\varphi(u(s))ds, \quad \text{for } t \geq 0,
\]
then
\[ u(t) \leq \left\{ \Phi_{1 - \frac{1}{p}}^{-1} \left[ \Phi_{1 - \frac{1}{p}}^{-1} (B(t)) + C(t) \right] \right\}^{\frac{1}{p-1}}, \quad t \in \Delta, \]

where
\[
B(t) = (k(t))^{1 - \frac{1}{p}} + \int_0^t a_1(t)f_1(s)ds + \int_0^t a_2(t)f_2(s)ds,
\]
\[
C(t) = \int_0^t b_1(t)g_1(s)ds + \int_0^t b_2(t)g_2(s)ds + \int_0^t a_3(t)f_3(s)ds \cdot \int_0^t a_4(t)f_4(s)ds,
\]
and \( \Phi_{1 - \frac{1}{p}}^{-1} \) is the inverse of \( \Phi_{1 - \frac{1}{p}} \), and \( t \in \Delta \) is chosen in such a way that \( \Phi_{1 - \frac{1}{p}}^{-1} (B(t)) + C(t) \in \text{Dom} \left( \Phi_{1 - \frac{1}{p}}^{-1} \right) \).

**Corollary 2.2.** Let \( f_i, g_j, \varphi, \alpha, k \) \((i = 1, 2, 3, 4, j = 1, 2)\) be as in Theorem 2.1. Assume in addition that \( p = 2 \) be a constant. If \( u \in C(R_+, R_+) \) satisfies
\[
u^2(t) \leq k(t) + 2 \int_0^t [f_1(t,s)u(s) + g_1(t,s)u(s)\varphi(u(s))] ds
\]
\[
+ 2 \int_0^t [f_2(t,s)u(s) + g_2(t,s)u(s)\varphi(u(s))] ds
\]
\[
+ 2 \int_0^t f_3(t,s)u(s)ds \cdot \int_0^t f_4(t,s)\varphi(u(s))ds, \quad \text{for } t \geq 0,
\]
then
\[ u(t) \leq \Phi_{\frac{1}{2}}^{-1} \left[ \Phi_{\frac{1}{2}} (B(t)) + C(t) \right], \quad t \in \Delta, \]

where
\[
B(t) = (k(t))^{\frac{1}{2}} + \int_0^t f_1(t,s)ds + \int_0^t f_2(t,s)ds,
\]
\[
C(t) = \int_0^t g_1(t,s)ds + \int_0^t g_2(t,s)ds + \int_0^t f_3(t,s)ds \cdot \int_0^t f_4(t,s)ds
\]
and \( \Phi_{\frac{1}{2}}^{-1} \) is the inverse of \( \Phi_{\frac{1}{2}} \), and \( t \in \Delta \) is chosen in such a way that \( \Phi_{\frac{1}{2}} (B(t)) + C(t) \in \text{Dom} \left( \Phi_{\frac{1}{2}}^{-1} \right) \).

**Corollary 2.3.** Let \( f_i, a_i, g_j, b_j, \varphi, \alpha, k \) \((i = 1, 2, 3, 4, j = 1, 2)\) be as in Corollary 2.1. Assume in addition that \( p = 2 \) be a constant. If \( u \in C(R_+, R_+) \) satisfies
\[
u^2(t) \leq k(t) + 2 \int_0^t [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u(s)\varphi(u(s))] ds
\]
\[
+ 2 \int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u(s)\varphi(u(s))] ds
\]
\[
+ 2 \int_0^t a_3(t)f_3(s)u(s)ds \cdot \int_0^t a_4(t)f_4(s)\varphi(u(s))ds, \quad \text{for } t \geq 0,
\]
then

\[ u(t) \leq \Phi_{\frac{1}{2}}^{-1} \left[ \Phi_{\frac{1}{2}}(B(t)) + C(t) \right], \quad t \in \Delta, \]

where

\[
B(t) = (k(t))^\frac{1}{p} + \int_0^\alpha f_1(t) u(s) + g_1(t, s) u^p(s) ds,
\]

\[
C(t) = \int_0^\alpha b_1(t) u(s) + g_2(t, s) u^p(s) ds + \int_0^\alpha a_3(t, s) u^p(s) \left( \int_0^t a_4(t, s) ds \right) ds,
\]

and \( \Phi_{\frac{1}{2}}^{-1} \) is the inverse of \( \Phi_{\frac{1}{2}} \), and \( t \in \Delta \) is chosen in such a way that \( \Phi_{\frac{1}{2}}(B(t)) + C(t) \in \text{Dom} \left( \Phi_{\frac{1}{2}}^{-1} \right) \).

\textbf{Theorem 2.2.} Let \( f_i, g_j \in C(R_+ \times R_+, R_+) \) with \( (t, s) \mapsto \partial_t f_i(t, s), \partial_s g_j(t, s) \in C(R_+ \times R_+, R_+) \), \( i = 1, 2, 3, 4 \), \( j = 1, 2 \), and \( p > 1 \) be a constant. Assume in addition that \( k \in C(R_+, R_+) \), \( \alpha \in C^1(R_+, R_+) \) are non-decreasing functions with \( \alpha(t) \leq t \) for \( t \geq 0 \), \( \varphi(t) > 0 \), and \( \int_1^\infty \frac{dt}{\varphi(t)} = \infty \). If \( u \in C(R_+, R_+) \) satisfies

\[
u^p(t) \leq k(t) + \frac{p}{p-1} \int_0^\alpha \left[ f_1(t, s) u(s) + g_1(t, s) u^p(s) \right] ds
\]

\[
+ \frac{p}{p-1} \int_0^\alpha \left[ f_2(t, s) u(s) + g_2(t, s) u^p(s) \right] ds
\]

\[
+ \frac{p}{p-1} \int_0^\alpha f_3(t, s) u(s) ds \cdot \int_0^t f_4(t, s) u^{p-1}(s) ds, \quad \text{for } t \geq 0,
\]

then

\[
u(t) \leq \left\{ (k(t))^{\frac{1}{p}} + \int_0^\alpha \left( e^{-C(s)} B(s) \right) ds \cdot e^{C(t)} \right\}^{\frac{1}{p-1}}, \quad t \in \Delta,
\]

where

\[
B(t) = \frac{d}{dt} \left( \int_0^\alpha f_1(t, s) ds + \int_0^t f_2(t, s) ds \right),
\]

\[
C(t) = \int_0^\alpha g_1(t, s) ds + \int_0^t g_2(t, s) ds + \int_0^\alpha f_3(t, s) ds \cdot \int_0^t f_4(t, s) ds,
\]

and \( \Phi_{\frac{1}{2}}^{-1} \) is the inverse of \( \Phi_{\frac{1}{2}}^{-\frac{1}{p}} \), and \( t \in \Delta \) is chosen in such a way that \( \Phi_{\frac{1}{2}}^{-\frac{1}{p}}(B(t)) \in \text{Dom} \left( \Phi_{\frac{1}{2}}^{-\frac{1}{p}} \right) \).
Proof. Let $T \in \Delta$, $T \geq 0$ be fixed and denote

$$x(t) = \frac{p}{p-1} \int_0^\alpha [f_1(t,s)u(s) + g_1(t,s)u^p(s)] ds$$

$$+ \frac{p}{p-1} \int_0^t [f_2(t,s)u(s) + g_2(t,s)u^p(s)] ds$$

$$+ \frac{p}{p-1} \int_0^\alpha f_3(t,s)u(s)ds \cdot \int_0^t f_4(t,s)u^{p-1}(s) ds,$$

then

$$u(t) \leq (k(t) + x(t))^{\frac{1}{p}}.$$

Our assumption on $f_i, g_j, \alpha, (i = 1, 2, 3, 4, j = 1, 2)$ imply that $x$ is non-decreasing on $R_+$. Hence, for $t \in [0,T]$, by calculations we have

$$x'(t) = \frac{p}{p-1} \left\{ [f_1(t,\alpha(t))u(\alpha(t))\alpha'(t) + g_1(t,\alpha(t))u^p(\alpha(t))\alpha'(t)

+ \int_0^{\alpha(t)} \partial_t f_1(t,s)u(s) + \partial_t g_1(t,s)u^p(s)] ds

+ [f_2(t,t)u(t) + g_2(t,t)u^p(t)] + \int_0^t [\partial_t f_2(t,s)u(s) + \partial_t g_2(t,s)u^p(s)] ds

+ \left[ f_3(t,\alpha(t))u(\alpha(t))\alpha'(t) + \int_0^{\alpha(t)} \partial_t f_3(t,s)u(s)ds \right] \cdot \int_0^t f_4(t,s)u^{p-1}(s) ds

+ \left[ f_4(t,t)u^{p-1}(t) + \int_0^t \partial_t f_4(t,s)u^{p-1}(s) ds \right] \cdot \int_0^{\alpha(t)} f_3(t,s)u(s) ds \right\}

\leq \frac{p}{p-1} \frac{d}{dt} \left[ \int_0^{\alpha(t)} f_1(t,s) ds + \int_0^t f_2(t,s) ds \right] \cdot [k(T) + x(t)]^{\frac{1}{p}}

+ \frac{p}{p-1} \frac{d}{dt} \left[ \int_0^{\alpha(t)} g_1(t,s) ds + \int_0^t g_2(t,s) ds \right.

+ \int_0^{\alpha(t)} f_3(t,s) ds \cdot \int_0^t f_4(t,s) ds \left] \cdot (k(T) + x(t))^{1-\frac{1}{p}} \right.$$

then we get

$$\frac{p-1}{p} \frac{x'(t)}{[k(T) + x(t)]^{\frac{1}{p}}} = \frac{d}{dt} \left[ \int_0^{\alpha(t)} g_1(t,s) ds + \int_0^t g_2(t,s) ds \right.

+ \int_0^{\alpha(t)} f_3(t,s) ds \cdot \int_0^t f_4(t,s) ds \left] \cdot (k(T) + x(t))^{1-\frac{1}{p}} \right.$$

$\leq \frac{d}{dt} \left[ \int_0^{\alpha(t)} f_1(t,s) ds + \int_0^t f_2(t,s) ds \right].$

Let

$$z(t) = [k(T) + x(t)]^{1-\frac{1}{p}},$$
\[ B(t) = \frac{d}{dt} \left( \int_0^t f_1(t,s) ds + \int_0^t f_2(t,s) ds \right), \quad (2.4) \]
\[ C(t) = \int_0^t g_1(t,s) ds + \int_0^t g_2(t,s) ds + \int_0^t f_3(t,s) ds \cdot \int_0^t f_4(t,s) ds, \]

then the above relation is equivalent to
\[ z'(t) - \frac{d}{dt} C(t) z(t) \leq B(t). \]

Multiplying the above inequality by \( e^{-C(t)} \) and considering the integration on \([0,t]\) to obtain
\[ z(t) \leq \left\{ (k(T))^{1 - \frac{1}{p}} + \int_0^t \left( e^{-C(s)} B(s) \right) ds \right\} e^{C(t)}, \quad \text{for all } t \in [0,T]. \quad (2.5) \]

Now, using (2.4), (2.5), and let \( t = T \), then
\[ k(T) + x(T) \leq \left\{ (k(T))^{1 - \frac{1}{p}} + \int_0^T \left( e^{-C(s)} B(s) \right) ds \cdot e^{C(T)} \right\}^{\frac{1}{p-1}} \]

since \( T \geq 0 \) was arbitrarily chosen, considering \( u(t) \leq [k(t) + x(t)]^{\frac{1}{p}} \), we get(2.3). \( \square \)

**Corollary 2.4.** Let \( f_i, g_j, a_i, b_j, \alpha, k, p \) be as in Corollary 2.1, \( i = 1, 2, 3, 4, j = 1, 2 \). If \( u \in C(R_+, R_+) \) satisfies
\[ u^p(t) \leq k(t) + \frac{p}{p-1} \int_0^t [a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u^p(s)] ds \]
\[ + \frac{p}{p-1} \int_0^t [a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u^p(s)] ds \]
\[ + \frac{p}{p-1} \int_0^t a_3(t)f_3(s)u(s) ds \cdot \int_0^t a_4(t)f_4(s)^p-1(s) ds, \quad \text{for } t \geq 0, \]

then
\[ u(t) \leq \left\{ (k(t))^{1 - \frac{1}{p}} + \int_0^t \left( e^{-C(s)} B(s) \right) ds \cdot e^{C(t)} \right\}^{\frac{1}{p-1}}, \quad t \in \Delta, \]

where
\[ B(t) = \frac{d}{dt} \left( \int_0^t a_1(t)f_1(s) ds + \int_0^t a_2(t)f_2(s) ds \right), \]
\[ C(t) = \int_0^t b_1(t)g_1(s) ds + \int_0^t b_2(t)g_2(s) ds + \int_0^t a_3(t)f_3(s) ds \cdot \int_0^t a_4(t)f_4(s) ds. \]
COROLLARY 2.5. Let \( f_i, g_j, \alpha, k \) be as in Theorem 2.2, \( i = 1,2,3,4, \ j = 1,2 \). Assume in addition that \( p = 2 \) be a constant. If \( u \in C(R_+, R_+) \) satisfies
\[
\begin{align*}
u^2(t) &\leq k(t) + 2 \int_0^{\alpha(t)} \left[ f_1(t,s)u(s) + g_1(t,s)u^2(s) \right] ds \\
&\quad + 2 \int_0^{t} \left[ f_2(t,s)u(s) + g_2(t,s)u^2(s) \right] ds \\
&\quad + 2 \int_0^{\alpha(t)} f_3(t,s)u(s)ds \cdot \int_0^{t} f_4(t,s)u(s)ds, \text{ for } t \geq 0,
\end{align*}
\]
then
\[
u(t) \leq (k(t))^{\frac{1}{2}} + \int_0^{t} \left( e^{-C(s)}B(s) \right) ds \cdot e^{C(t)}, \ t \geq 0,
\]
where
\[
\begin{align*}
B(t) &= \frac{d}{dt} \left( \int_0^{\alpha(t)} f_1(t,s)ds + \int_0^{t} f_2(t,s)ds \right), \\
C(t) &= \int_0^{\alpha(t)} g_1(t,s)ds + \int_0^{t} g_2(t,s)ds + \int_0^{\alpha(t)} f_3(t,s)ds \cdot \int_0^{t} f_4(t,s)ds.
\end{align*}
\]

COROLLARY 2.6. Let \( f_i, a_i, g_j, b_j, \alpha, k \), be as in Corollary 2.1, \( i = 1,2,3,4, \ j = 1,2 \). Assume in addition that \( p = 2 \) be a constant. If \( u \in C(R_+, R_+) \) satisfies
\[
\begin{align*}
u^2(t) &\leq k(t) + 2 \int_0^{\alpha(t)} \left[ a_1(t)f_1(s)u(s) + b_1(t)g_1(s)u^p(s) \right] ds \\
&\quad + 2 \int_0^{t} \left[ a_2(t)f_2(s)u(s) + b_2(t)g_2(s)u^p(s) \right] ds \\
&\quad + 2 \int_0^{\alpha(t)} a_3(t)f_3(s)u(s)ds \cdot \int_0^{t} a_4(t)f_4(s)u^p(s)ds, \text{ for } t \geq 0,
\end{align*}
\]
then
\[
u(t) \leq (k(t))^{\frac{1}{2}} + \int_0^{t} \left( e^{-C(s)}B(s) \right) ds \cdot e^{C(t)}, \ t \geq 0,
\]
where
\[
\begin{align*}
B(t) &= \frac{d}{dt} \left( \int_0^{\alpha(t)} a_1(t)f_1(s)ds + \int_0^{t} a_2(t)f_2(s)ds \right), \\
C(t) &= \int_0^{\alpha(t)} b_1(t)g_1(s)ds + \int_0^{t} b_2(t)g_2(s)ds + \int_0^{\alpha(t)} a_3(t)f_3(s)ds \cdot \int_0^{t} a_4(t)f_4(s)ds.
\end{align*}
\]

THEOREM 2.3. Let \( f_i, g_j \in C(R_+ \times R_+, R_+) \) with \((t,s) \mapsto \partial_t f_i(t,s), \partial_t g_j(t,s) \in C(R_+ \times R_+, R_+), i = 1,2,3,4, \ j = 1,2, \ p > q \geq 0 \) be constants. Assume in addition
that $\varphi, k \in C(R_+, R_+)$, $\alpha \in C^1(R_+, R_+)$ are non-decreasing functions with $\alpha(t) \leq t$ for $t \geq 0$, $\varphi(t) > 0$, and $\int_1^\infty \frac{dt}{\varphi(t)} = \infty$. If $u \in C(R_+, R_+)$ satisfies

\[ u^p(t) \leq k(t) + \frac{p}{p-q} \int_0^{\alpha(t)} \left[ f_1(t, s)u^q(s) + g_1(t, s)u^q(s)\varphi(u(s)) \right] ds \]

\[ + \frac{p}{p-q} \int_0^{\alpha(t)} \left[ f_2(t, s)u^q(s) + g_2(t, s)u^q(s)\varphi(u(s)) \right] ds \]

\[ + \frac{p}{p-q} \int_0^{\alpha(t)} f_3(t, s)u^q(s)ds \cdot \int_0^t f_4(t, s)\varphi(u(s))ds, \quad \text{for } t \geq 0, \quad (2.6) \]

then

\[ u(t) \leq \left\{ \Phi_{p-q}^{-1}\left[ \Phi_{p-q}(B(t)) + C(t) \right] \right\}^{\frac{1}{p-q}}, \quad t \in \Delta, \]

where

\[ B(t) = (k(t))^{\frac{q}{p}} + \int_0^{\alpha(t)} f_1(t, s)ds + \int_0^t f_2(t, s)ds, \quad (2.7) \]

\[ C(t) = \int_0^{\alpha(t)} g_1(t, s)ds + \int_0^t g_2(t, s)ds + \int_0^{\alpha(t)} f_3(t, s)ds \cdot \int_0^t f_4(t, s)ds, \quad (2.8) \]

and $\Phi_{p-q}^{-1}$ is the inverse of $\Phi_{p-q}$, and $t \in \Delta$ is chosen in such a way that $\Phi_{p-q}(B(t)) + C(t) \in \text{Dom} \left( \Phi_{p-q}^{-1} \right)$.

**Proof.** For any $r > 0$, define $\psi(r) = \varphi(r^{\frac{1}{2}})$, then our assumption on $\varphi$ imply that $\psi$ is non-decreasing on $R_+$, then (2.6) is equivalent to

\[ u^p(t) \leq k(t) + \frac{p}{p-q} \int_0^{\alpha(t)} \left[ f_1(t, s)u^q(s) + g_1(t, s)u^q(s)\psi(u^q(s)) \right] ds \]

\[ + \frac{p}{p-q} \int_0^{\alpha(t)} \left[ f_2(t, s)u^q(s) + g_2(t, s)u^q(s)\psi(u^q(s)) \right] ds \]

\[ + \frac{p}{p-q} \int_0^{\alpha(t)} f_3(t, s)u^q(s)ds \cdot \int_0^t f_4(t, s)\psi(u^q(s))ds. \]

Let $v(t) = u^q(t)$, then the above inequality is equivalent to

\[ v^\frac{p}{q}(t) \leq k(t) + \frac{p}{\frac{p}{q} - 1} \int_0^{\alpha(t)} \left[ f_1(t, s)v(s) + g_1(t, s)v(s)\psi(v(s)) \right] ds \]

\[ + \frac{p}{\frac{p}{q} - 1} \int_0^{\alpha(t)} \left[ f_2(t, s)v(s) + g_2(t, s)v(s)\psi(v(s)) \right] ds \]

\[ + \frac{p}{\frac{p}{q} - 1} \int_0^{\alpha(t)} f_3(t, s)v(s)ds \cdot \int_0^t f_4(t, s)\psi(v(s))ds. \]
Since \( \frac{p}{q} > 1 \), it follows from Theorem 2.1 that
\[
v(t) \leq \left\{ \Psi_{\frac{p}{q}-1} \left[ \Psi_{\frac{p}{q}}(B(t)) + C(t) \right] \right\}^{\frac{1}{\frac{p}{q}-1}}, \quad t \in [0, T],
\]
where \( B(t), C(t) \) is defined as (2.7), (2.8). Now it is elementary to check by the definition of \( \Psi \), then
\[
\Psi_{\frac{p}{q}-1}(r) = \Phi_{p-q}(r),
\]
thus we have
\[
v(t) \leq \left\{ \Phi_{p-q}^{-1} \left[ \Phi_{p-q}(B(t)) + C(t) \right] \right\}^{\frac{q}{p-q}}, \quad \text{for all } t \in [0, T],
\]
considering \( u(t) = \nu^\frac{1}{q}(t) \), we get
\[
u(t) \leq \left\{ \Phi_{p-q}^{-1} \left[ \Phi_{p-q}(B(t)) + C(t) \right] \right\}^{\frac{q}{p-q}}, \quad \text{for all } t \in [0, T]. \quad \Box
\]

COROLLARY 2.7. Let \( \alpha, k, p, q \) be as in Theorem 2.3. Assume in addition that \( f_i, g_j, a_i, b_j \in C^1(R_+, R_+), \ i = 1, 2, 3, 4, \ j = 1, 2. \) If \( u \in C(R_+, R_+) \) satisfies
\[
u^p(t) \leq k(t) + \frac{p}{p-q} \int_0^{\alpha(t)} \left[ a_1(t) f_1(s) u^q(s) + b_1(t) g_1(s) u^q(s) \phi(u(s)) \right] ds
\]
\[
+ \frac{p}{p-q} \int_0^t \left[ a_2(t) f_2(s) u^q(s) + b_2(t) g_2(s) u^q(s) \phi(u(s)) \right] ds
\]
\[
+ \frac{p}{p-q} \int_0^{\alpha(t)} a_3(t) f_3(s) u^q(s) ds \cdot \int_0^t a_4(t) f_4(s) \phi(u(s)) ds, \quad \text{for } t \geq 0,
\]
then
\[
u(t) \leq \left\{ \Phi_{p-q}^{-1} \left[ \Phi_{p-q}(B(t)) + C(t) \right] \right\}^{\frac{1}{\frac{p}{q}-1}}, \quad t \in \Delta,
\]
where
\[
B(t) = (k(t))^{\frac{p}{q} - 1} + \int_0^{\alpha(t)} a_1(t) f_1(s) ds + \int_0^t a_2(t) f_2(s) ds,
\]
\[
C(t) = \int_0^{\alpha(t)} b_1(t) g_1(s) ds + \int_0^t b_2(t) g_2(t) s ds + \int_0^{\alpha(t)} a_3(t) f_3(s) ds \cdot \int_0^t a_4(t) f_4(s) ds.
\]
and \( \Phi_{p-q}^{-1} \) is the inverse of \( \Phi_{p-q} \), and \( t \in \Delta \) is chosen in such a way that \( \Phi_{p-q}(B(t)) + C(t) \in \text{Dom} \left( \Phi_{p-q}^{-1} \right) \).

REMARK. Different choices of \( k, \alpha, a_i, b_j \) can give many different inequalities, \( i = 1, 2, 3, 4, \ j = 1, 2. \) For example, let \( a_1(t) = 1, \ a_2(t) = b_1(t) = b_2(t) = a_3(t) = 0, \ a(t) = t, \ k(t) = k^2, \ t \in [0, \infty), \) where \( k \geq 0 \) be a constant, our Corollary 2.6 reduces to Theorem A; or let \( a_1(t) = b_1(t) = 1, \ a_2(t) = b_2(t) = a_3(t) = 0, \ a(t) = t, \ k(t) = k^2, \ t \in [0, \infty), \) where \( k \geq 0 \) is a constant, our Corollary 2.3 reduces to Theorem B.
3. Examples

Now, we will show that our results are useful in studying the boundedness and stability of solutions to certain integro-differential equations with time delay. These applications are given as examples.

**Example 1.** Consider the nonlinear integro-differential equation with time delay

\[
\begin{aligned}
\left\{ \\
\begin{array}{l}
x'(t) = y - F(t,x) + \int_0^t H(s,x(s - \tau(s)))ds, \\
y'(t) = G(t,x(t - \tau(t))),
\end{array}
\right.
\end{aligned}
\]  

(3.1)

where \( F, H, G \in C(R_+ \times R_+, R_+) \), \( \tau \in C^1(R_+, R_+) \), and \( \tau(t) \leq t \) on \( R_+ \). If \( \alpha(t) = t - \tau(t) \) is an increasing diffeomorphism on \( R_+ \), and

\[
-xF(t,x) \leq a(t)|x|v(|x|),
\]

\[
G^2(t,x) \leq b(t)|x|v(|x|),
\]

\[
\left( \int_0^t H(s,x)ds \right)^2 \leq |x| \int_0^t c(s)|x|ds,
\]

for \( a(t), b(t), c(t) \in C(R_+, R_+) \) and some non-decreasing function \( v \in C(R_+, R_+) \) with the properties \( v(u) > 0 \) for \( u > 0 \) and \( \int_0^\infty \frac{ds}{v(s)} = \infty \), then all the solutions of (3.1) are bounded and global.

In fact, if \( (x(t), y(t)) \) is a solution of (3.1) defined on the maximal existence interval \( [0, T) \), let \( u(t) = \sqrt{x^2(t) + y^2(t)} \) and \( p(t) = \max\{1, a(t)\} \in C(R_+, R_+) \) for \( t \in [0, T) \). From (3.1) and our hypotheses on the functions \( F, H, G \) and \( v \), we obtain

\[
\frac{d}{dt} u^2(t) = 2xx' + 2yy' \\
= 2xy - 2xF(t,x) + 2yG(t,x(\alpha)) + 2x \int_0^t H(s,x(\alpha))ds \\
\leq x^2 + y^2 + 2a(t)|x|v(|x|) + y^2 + G^2(t,x(\alpha)) + x^2 + \left( \int_0^t H(s,x(\alpha))ds \right)^2 \\
\leq 2p(t)u^2 + 2p(t)uv(u) + b(t)|x(\alpha)|v(|x(\alpha)|) + |x(\alpha)| \int_0^t c(s)||x(\alpha)||ds,
\]

\( t \in [0, T) \).

With \( \varphi(u) := u + v(u) \), an integration on \([0, t] \), with \( t < T \) yields

\[
u^2(t) \leq u^2(0) + \int_0^t p(s)u(s)\varphi(u(s))ds + 2 \int_0^t b(s)|x(\alpha(s))|v(|x(\alpha(s))|)ds \\
+ \int_0^t (|x(\alpha(s))|) \left( \int_0^s c(r)||x(\alpha(r))||dr \right)ds \\
\leq u^2(0) + 2 \int_0^t p(s)u(s)\varphi(u(s))ds + \int_0^t b(s)u(\alpha(s))\varphi(u(\alpha(s)))ds \\
+ \int_0^t \varphi(u(s)) \left( \int_0^s c(r)u(\alpha(r))dr \right)ds
\]
after performing the change of variable \( r = \alpha(s) \) and some intermediate steps, where \( \alpha^{-1} \) is the inverse of the diffeomorphism \( \alpha \). Our hypotheses on \( v \) guarantee that \( \int_1^\infty \frac{dr}{\phi(r)} dr = \infty \) (see [8]). Therefore, \( \Phi(r) = \int_1^r \frac{ds}{\phi(s)}, r > 0 \), then from Corollary 2.3, we deduce that

\[
\Phi(t) \leq \Phi^{-1}_\frac{1}{2} \left[ \Phi_\frac{1}{2}(\Phi(0)) + \int_0^t p(s) ds + \int_0^\alpha \frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u(s) ds \right] + t \int_0^\alpha \frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u(s) ds,
\]

\[
= \Phi^{-1}_\frac{1}{2} \left[ \Phi_\frac{1}{2}(\Phi(0)) + \int_0^t p(s) ds + \int_0^t b(s) ds + t \int_0^t c(s) ds \right], \quad t \in [0, T).
\]

This proves that \( u(t) \) is bounded on \( [0, T] \) if \( T < \infty \), and all solutions of (3.1) are global. \( T \in \Delta \) is chosen in such a way that \( \Phi_\frac{1}{2}(\Phi(0)) + \int_0^T p(s) ds + \int_0^T b(s) ds + t \int_0^T c(s) ds \in \text{Dom} \left( \Phi^{-1}_\frac{1}{2} \right) \), \( \Phi^{-1}_\frac{1}{2} \) is the inverse of \( \Phi_\frac{1}{2} \).

**Example 2.** Consider the nonlinear integro-differential equation with time delay

\[
\begin{align*}
x'(t) &= y + \int_0^t H(s,x(s-\tau(s)))ds, \\
y'(t) &= -F(t,y) + G(t,x(t-\tau(t))),
\end{align*}
\]  

(3.2)

where \( F, H, G \in C(R_+ \times R_+ \times R_+) \), \( \tau \in C_1(R_+ \times R_+) \), and \( \tau(t) \leq t \) on \( R_+ \). If \( \alpha(t) = t - \tau(t) \) is an increasing diffeomorphism on \( R_+ \), and

\[
\begin{align*}
-yF(t,y) &\leq a(t)|y|v(|y|), \\
G^2(t,x) &\leq b(t)|x|v(|x|), \\
\left( \int_0^t H(s,x)ds \right)^2 &\leq |x| \int_0^t c(s)|x|ds, \quad (t,y),(t,x) \in (R_+ \times R_+),
\end{align*}
\]

for \( a(t), b(t), c(t) \in C(R_+, R_+) \) and some non-decreasing function \( v \in C(R_+, R_+) \) with the properties \( v(u) > 0 \) for \( u > 0 \) and \( \int_1^\infty \frac{ds}{v(s)} = \infty \), then all the solutions of (3.2) are bounded and global.

In fact, if \( (x(t),y(t)) \) is a solution of (3.2) defined on the maximal existence interval \( [0, T) \), let \( u(t) = \sqrt{x^2(t) + y^2(t)} \) and \( p(t) = \max \{1,a(t)\} \in C(R_+, R_+) \) for \( t \in [0, T) \). From (3.2) and our hypotheses on the functions \( F, H, G \) and \( v \), we obtain

\[
\frac{d}{dt} u^2(t) = 2xx' + 2yy'
\]

\[
= 2xy - 2yF(t,y) + 2yG(t,x(\alpha)) + 2x \int_0^t H(s,x(\alpha))ds
\]
\[ \leq x^2 + y^2 + 2a(t)|y|v(|y|) + y^2 + C^2(t, x(\alpha)) + x^2 + \left( \int_0^t H(s, x(\alpha))ds \right)^2 \]
\[ \leq 2p(t)u^2 + 2p(t)uv(u) + b(t)|x(\alpha)|v(|x(\alpha)|) \]
\[ + |x(\alpha)| \int_0^t c(s)|x(\alpha)|ds, \quad t \in [0, T). \]

With \( \varphi(u) := u + v(u) \), an integration on \([0, t]\), with \( t < T \) yields
\[ u^2(t) \leq u^2(0) + 2 \int_0^t p(s)u(s)\varphi(u(s))ds + \int_0^t b(s) |x(\alpha(s))|v(|x(\alpha(s))|)ds \]
\[ + 2 \int_0^t |x(\alpha)| \left( \int_0^s c(r)|x(\alpha)|dr \right)ds \]
\[ \leq u^2(0) + 2 \int_0^t p(s)u(s)\varphi(u(s))ds + \int_0^t b(s)u(\alpha(s))\varphi(u(\alpha(s)))ds \]
\[ + \int_0^t \varphi(u(s)) \left[ \int_0^x c(r)u(\alpha(r))dr \right]ds \]
\[ \leq u^2(0) + 2 \int_0^t p(s)u(s)\varphi(u(s))ds + 2 \int_0^t \frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u(s)\varphi(u(s))ds \]
\[ + 2 \int_0^t \varphi(u(s))ds \int_0^t \frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u(s)ds, \]

after performing the change of variables \( r = \alpha(s) \) and some intermediate steps, where \( \alpha^{-1} \) is the inverse of the diffeomorphism \( \alpha \). Our hypotheses on \( v \) guarantee that \( \int_1^\infty \frac{dr}{\varphi(r)} dr = \infty \) (see [8]). Therefore, \( \Phi(r) = \int_0^r \frac{ds}{\varphi(s)} \), \( r > 0 \), then from Corollary 2.3, we deduce that
\[ u(t) \leq \Phi^{-1}_{\frac{1}{2}} \left[ \Phi_{\frac{1}{2}}(u(0)) + \int_0^t p(s)ds + \int_0^t \frac{b(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} ds \right] \]
\[ + t \int_0^t \frac{c(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} u(s)ds \]
\[ = \Phi^{-1}_{\frac{1}{2}} \left[ \Phi_{\frac{1}{2}}(u(0)) + \int_0^t p(s)ds + \int_0^t b(s)ds + t\int_0^t c(s)ds \right], \quad t \in [0, T). \]

This proves that \( u(t) \) is bounded on \([0, T)\) if \( T < \infty \), and all solutions of (3.2) are global. \( T \in \Delta \) is chosen in such a way that \( \Phi_{\frac{1}{2}}(u(0)) + \int_0^t p(s)ds + \int_0^t b(s)ds + t\int_0^t c(s)ds \in \text{Dom} \left( \Phi^{-1}_{\frac{1}{2}} \right) \), \( \Phi^{-1}_{\frac{1}{2}} \) is the inverse of \( \Phi_{\frac{1}{2}} \).
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(Received July 9, 2012)

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