Journal of Mathematical Inequalities
Volume 7, Number 3 (2013), 513–516 doi:10.7153/jmi-07-45

NORM INEQUALITY OF $AP+BQ$ FOR SELFADJOINT PROJECTIONS P AND Q WITH $PQ=0$

TAKAHIKO NAKAZI

(*Communicated by W. D. Evans*)

Abstract. Let A,B,P and Q be bounded operators on a Hilbert space $\mathscr{L}$ which P and Q are selfadjoint projections with $PQ=0$. We study the norm of $AP+BQ$ using a Hankel type operator : $H_{B^*A}=QB^*A \mid P\mathscr{L}$.

Let $\mathscr{L}$ be a Hilbert space and let $\mathscr{H}$ and $\mathscr{K}$ denote closed subspaces of $\mathscr{L}$ such that $\mathscr{H}$ is orthogonal to $\mathscr{K}$. P and Q denote orthogonal projections with $PQ=0$. We suppose $\mathscr{H}=P\mathscr{L}$ and $\mathscr{K}=Q\mathscr{L}$. For a bounded linear operator X on $\mathscr{L}$, we define a Hankel type operator H_X from $\mathscr{H}$ to $\mathscr{K}$: $H_X f=Q(Xf)$ $(f\in H)$.

In a very special case, Yamamoto and the author [3] have established a theorem about the norm of $AP+BQ$. Let m be the normalized Lebesgue measure on the unit circle T. For $p=2,\infty$, $L^p(T)$ denotes the usual Lebesgue space on T and $H^p(T)$ denotes the usual Hardy space on T. Let $\mathscr{L}=L^2(T), \mathscr{H}=H^2(T)$ and $\mathscr{K}=L^2(T)\ominus H^2(T)$. Suppose $A=M_\alpha$ and $B=M_\beta$ are multiplication operators on $L^2(T)$ where α and β are in $L^\infty(T)$. Then the following formula is known [3].

$$\|M_\alpha P+M_\beta Q\| = \inf_{g\in H^\infty}\left\| \frac{|\alpha|^2+|\beta|^2}{2}+\sqrt{\|\alpha\bar{\beta}+g\|_\infty^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty .$$

In the proof, a lifting theorem of Cotlar and Sadosky was essential (see [1]). Later the author [4] tried to generalize it when A,B,P and Q have a lifting property. In this paper, we do not use such a lifting theorem. In the formula above, it is well known that $\|H_{B^*A}\|=\|\alpha\bar{\beta}+H^\infty\|=\inf\{\|\alpha\bar{\beta}+g\|_\infty : g\in H^\infty\}$ by a theorem of Nehari (see [5]). In this special case, we usually write $H_{B^*A}=H_{\bar{\beta}\alpha}$.

LEMMA. *For any bounded A and B,*

$$\|AP+BQ\|^2 = \sup_{\|f\|=\|g\|=1}\left\{\frac{\|Af\|^2+\|Bg\|^2}{2}+\sqrt{|\langle B^*Af,g\rangle|^2+\left(\frac{\|Af\|^2-\|Bg\|^2}{2}\right)^2} : f\in\mathscr{H}, g\in\mathscr{K}\right\}$$

Mathematics subject classification (2010): 47B35.
Keywords and phrases: Singular integral operator, Hankel operator, norm, norm inequality.
This research is partially supposed by Grant-in Aid Research No. 20540148.

Proof. Put $\gamma = \|AP+BQ\|$, then $\|Af+Bg\|^2 \leqslant \gamma^2\|f+g\|^2$ where $f \in \mathscr{H}$ and $g \in \mathscr{K}$. For any $f \in \mathscr{H}$ and $g \in \mathscr{K}$

$$\langle(\gamma^2 - A^*A)f, f\rangle + \langle(\gamma^2 - B^*B)g, g\rangle - 2\mathrm{Re}\langle(B^*A-\gamma^2)f, g\rangle \geqslant 0$$

and hence as $g=0$, $\langle(\gamma^2 - A^*A)f, f\rangle \geqslant 0$ and as $f=0$, $\langle(\gamma^2 - B^*B)g, g\rangle \geqslant 0$. Hence

$$\langle(\gamma^2 - A^*A)f, f\rangle\langle(\gamma^2 - B^*B)g, g\rangle \geqslant |\langle B^*Af, g\rangle|^2$$

for any $f \in \mathscr{H}$ and $g \in \mathscr{K}$. Therefore

$$\gamma^4\|f\|^2\|g\|^2 - \gamma^2(\|Af\|^2\|g\|^2 + \|f\|^2\|Bg\|^2) + \|Af\|^2\|Bg\|^2 - |\langle B^*Af, g\rangle|^2 \geqslant 0$$

for any $f \in \mathscr{H}$ and $g \in \mathscr{K}$. If $\|f\| = \|g\| = 1$, then

$$\gamma^4 - \gamma^2(\|Af\|^2 + \|Bg\|^2) + \|Af\|^2\|Bg\|^2 - |\langle B^*Af, g\rangle| \geqslant 0.$$

Thus, either

$$\gamma^2 \leqslant \frac{\|Af\|^2 + \|Bg\|^2}{2} - \left\{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2 - \|Bg\|^2}{2}\right)^2\right\}^{1/2}$$

or

$$\gamma^2 \geqslant \frac{\|Af\|^2 + \|Bg\|^2}{2} + \left\{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2 - \|Bg\|^2}{2}\right)^2\right\}^{1/2}.$$

The first inequality above is not valid because $\gamma^2 \geqslant \max(\|Af\|^2, \|Bg\|^2)$. Hence

$$\gamma^2 \geqslant \sup_{\|f\|=\|g\|=1}\left\{\frac{\|Af\|^2+\|Bg\|^2}{2} + \sqrt{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2-\|Bg\|^2}{2}\right)^2} : f \in \mathscr{H}, g \in \mathscr{K}\right\}.$$

Since the argument are reversible, the converse inequality is also valid. □

When A and B are unitary operators, by Lemma we can prove $\|AP+BQ\|^2 = 1 + \|H_{B^*A}\|$. In fact, we can show the following theorem.

THEOREM 1. *Suppose α and β are complex numbers. If $A^*A = |\alpha|^2 I$ and $B^*B = |\beta|^2 I$ then*

$$\|AP+BQ\|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\|H_{B^*A}\|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

Proof. It is clear by Lemma. □

From the C^*-identity and the triangle inequality, it follows that $\|AP+BQ\|^2 \leqslant \|AP\|^2 + \|BQ\|^2 + \|QB^*AP\| + \|PA^*BQ\|$. The following theorem gives a better inequality.

THEOREM 2. *For any bounded operators A and B,*

$$\max(\|AP\|^2,\|BQ\|^2) \leqslant \|AP+BQ\|^2 \leqslant \max(\|AP\|^2,\|BQ\|^2)+\|H_{B^*A}\|$$

Proof. At first we show the first inequality assuming $\|AP\| \geqslant \|BQ\|$. By Lemma,

$$\begin{aligned}\|AP+BQ\|^2 &\geqslant \sup_{\|g\|=1}\sup_{\|f\|=1}\left\{\frac{\|Af\|^2+\|Bg\|^2}{2}+\left|\frac{\|Af\|^2-\|Bg\|^2}{2}\right| : f\in\mathscr{H}, g\in\mathscr{K}\right\}\\ &= \sup_{\|g\|=1}\left\{\frac{\|AP\|^2+\|Bg\|^2}{2}+\left|\frac{\|AP\|^2-\|Bg\|^2}{2}\right| : g\in\mathscr{K}\right\}\\ &= \|AP\|^2\end{aligned}$$

because $\|AP\|^2 \geqslant \|Bg\|^2$. When $\|AP\| \leqslant \|BQ\|$, the same argument implies $\|AP+BQ\|^2 \geqslant \|BQ\|^2$. This shows the first inequality.

We show the second inequality. We may assume $\|AP\| \geqslant \|BQ\|$. By Lemma and the inequality : $\sqrt{x+y} \leqslant \sqrt{x}+\sqrt{y}$,

$$\begin{aligned}\|AP+BQ\|^2 &\leqslant \sup_{g\in\mathscr{K},\|g\|=1}\left\{\frac{\|AP\|^2+\|Bg\|^2}{2}+\|H_{B^*A}\|+\left|\frac{\|AP\|^2-\|Bg\|^2}{2}\right|\right\}\\ &= \|AP\|^2+\|H_{B^*A}\|\end{aligned}$$

because $\|AP\|^2 \geqslant \|Bg\|^2$. When $\|AP\| \leqslant \|BQ\|$, the same argument implies $\|AP+BQ\|^2 \leqslant \|BQ\|^2+\|H_{B^*A}\|$. This shows the second inequality. $\square$

REMARK. Let $Q=I-P$, that is, $\mathscr{L}=\mathscr{H}\oplus\mathscr{K}$. Let $\mathscr{B}$ denote a von Newmann algebra on $\mathscr{L}$ which contains I and $\mathscr{A}$ denote a weakly closed subalgebra of $\mathscr{B}$ which has $\mathscr{H}$ as an invariant subspace. Suppose A and B are in $\mathscr{B}$ and D is in $\mathscr{A}$.

(1) Lemma shows that $\|AP+BQ\|^2 = \sup_{\|f\|=\|g\|=1}\{(\|Af\|^2+\|Bg\|^2)/2+\sqrt{|\langle(B^*A+D)f,g\rangle|^2+(\|Af\|^2-\|Bg\|^2)^2/4} : f\in\mathscr{H}, g\in\mathscr{K}\}$.

(2) If $(\mathscr{B},\mathscr{M},P)$ has a lifting property (see [2]), then it is known [4, Proposition 1] that $\|AP+BQ\|^2 = \inf_{D\in\mathscr{A}}\sup_{\|F\|=\|G\|=1}\{(\|AF\|^2+\|BG\|^2)/2+\sqrt{|\langle(B^*A+D)F,G\rangle|^2+(\|AF|^2-\|BG\|^2)^2/4} : F,G\in\mathscr{L}\}$. This is stronger than Lemma.

(3) When $(\mathscr{B},\mathscr{A},P)$ has a lifting property, $\|H_{B^*A}\| = \|B^*A+\mathscr{A}\|$ but in general the inequality may hold, that is, $\|H_{B^*A}\| \leqslant \|B^*A+\mathscr{A}\|$.

(4) Theorem 1 should be compared with [4, Corollary 1]. It is clearly general by (3).

(5) Theorem 2 should be compared with [4, Theorem 2]. These two theorems are similar but different even if with (3). The first point is that we do not assume $\max(\|A\|,\|B\|) \leqslant \|S_{A,B}\|$. The second point is that the old one is better in the lower estimate and the new one is better in the upper estimate.

REFERENCES

[1] R. AROCENA, M. COTLAR AND C. SADOSKY, *Weighted inequalities in L^2 and lifting properties*, Adv. in Math. Suppl. Studies 7A, Academic Press, New York, 1981, pp. 258–269.

[2] T. NAKAZI, *A lifting theorem and analytic operator algebras*, Proc. A. M. S., 104 (1988), 1081–1085.

[3] T. NAKAZI AND T. YAMAMOTO, *Norms of some singular integral operators*, J. Operator Th. 40 (1998), 187–207.

[4] T. NAKAZI, *Norm inequalities for some singular integral operators*, Math. Inequality&Appl., 3 (2000), 411–421.

[5] S. C. POWER, *Hankel Operators On Hilbert Space*, Research Notes in Mathematics 64, Pitman Advanced Publishing Program, Boston-London-Melbourne.

(Received July 16, 2012)

Takahiko Nakazi
Hokusei Gakuen University
2-3-1, Ohyachi-Nishi, Atsubetsu-ku
Sapporo 004-8631, Japan
e-mail: z00547@hokusei.ac.jp