Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 517–522
doi:10.7153/jmi-07-46

GENERALIZATION ON KANTOROVICH INEQUALITY

MASATOSHI FUJII, HONGLIANG ZUO AND NAN CHENG

(*Communicated by J. I. Fujii*)

Abstract. In this paper, we provide a new form of upper bound for the converse of Jensen's inequality. Thereby, known estimations of the difference and ratio in Jensen's inequality are essentially improved. As an application, we also obtain an improvement of Kantorovich inequality.

1. Introduction

Throughout this paper, A, B are selfadjoint operators on a Hilbert space H, and $m_1 \leqslant A \leqslant M_1$, $m_2 \leqslant B \leqslant M_2$, $C(A,x) = ((M_1 - A)(A - m_1)x,x)$.

If f is a real valued continuous convex function, then the well-known Jensen's inequality asserts that

$$f((Ax,x)) \leqslant (f(A)x,x). \tag{1.1}$$

for every unit vector $x \in H$. In particular, if $f(t) = \frac{1}{t}$ (resp. t^2), then we have $(Ax,x)^{-1} \leqslant (A^{-1}x,x)$ (resp. $(Ax,x)^2 \leqslant (A^2x,x)$).

As a complementary inequality to Jensen's inequality, the Kantorovich inequality estimates the upper bound of the ratio in Jensen's inequality: if A is a positive operator on Hilbert space H, then

$$(Ax,x)(A^{-1}x,x) \leqslant \frac{(M_1+m_1)^2}{4M_1m_1}. \tag{1.2}$$

Many authors have investigated on extensions of the Kantorovich one, such as Zhibing Liu, Kanmin Wang, Chengfeng Xu [11], Furuta [7, 8] and Ky Fan [1]. Among others, we pay our attentions to the long research series of Mond-Pečarić method [9]. The authors established the method by which complementary inequalities to Jensen's type inequalities and extensions of the Kantorovich type one are obtained. Fujii et al. [4] gave the recent developments of Mond -Pečarić method in operator inequalities.

Mathematics subject classification (2010): 20D10, 20D20.
Keywords and phrases: Converse of Jensen's inequality, Kantorovich inequaliy, covariance and variance.

This research was partially supported by the National Natural Science Foundation of China (10471085). The corresponding author Hongliang Zuo was supported by the Basic Science and Technological Frontier Project of Henan(132300410261) and partially supported by NSF(10401085).

We look over development for operator inequalities including Seo's studies [3, 5, 6, 10, 12, 13] up to inequalities of covariance-variance of operators introduced by Fujii, Furuta, Nakamoto and Takahasi [2] as follows:

$$(A^2x,x)-(Ax,x)^2 \leqslant \frac{(M_1-m_1)^2}{4} \tag{1.3}$$

$$|(ABx,x)-(Ax,x)(Bx,x)| \leqslant \frac{(M_1-m_1)(M_2-m_2)}{4} \tag{1.4}$$

We observe that the so-called (noncommutative) covariance-variance inequality gives a unified method to prove certain operator inequalities including the celebrated Kantorovich inequality, Bernstein's inequality and so on. Though their inequalities are of different kinds, they have common ingredients such as the estimations of the difference and the ratio in Jensen's inequality. Obviously, the converse of Jensen's inequality is important.

In this paper, we improve the inequalities (1.3) and (1.4) to obtain the more accurate estimations via $C(A,x)$ as follows:

$$0 \leqslant (A^{k+1}x,x)-(Ax,x)^{k+1} \leqslant \frac{1}{4}(M_1-m_1)^2\sum_{p=1}^{k}(k-p+1)m_1^{p-1}M_1^{k-p} \\ -\sum_{p=1}^{k}M_1^{k-p}\sqrt{C(A^p,x)C(A,x)} \tag{1.5}$$

$$|(ABx,x)-(Ax,x)(Bx,x)| \leqslant \frac{(M_1-m_1)(M_2-m_2)}{4}-\sqrt{C(A,x)C(B,x)} \tag{1.6}$$

by which we extend Kantorovich inequalities as follows:

$$(Ax,x)(Bx,x)-(A\sharp Bx,x)^2 \leqslant \frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4M_2m_2}(Bx,x)^2 \tag{1.7}$$

2. The estimations of variance and covariance

The following basic lemma is essentially known as in [9], but our expression is a little bit different from those in [9]. For the sake of convenience, we give it a slim proof.

LEMMA 2.1. *Let A be a selfadjoint operator on Hilbert space with $m_1 \leqslant A \leqslant M_1$. Then, for $\|x\|=1$*

$$(A^2x,x)-(Ax,x)^2 \leqslant \frac{(M_1-m_1)^2}{4}-C(A,x). \tag{2.1}$$

Proof. We first note that $(M_1 - t)(t - m_1) \leqslant \frac{(M_1 - m_1)^2}{4}$ for all real numbers t. Hence it follows that

$$\begin{aligned}
&(A^2x,x) - (Ax,x)^2 \\
&= (M_1 - (Ax,x))((Ax,x) - m_1) - ((M_1 - A)(A - m_1)x,x) \\
&\leqslant \frac{(M_1 - m_1)^2}{4} - C(A,x).
\end{aligned}$$

COROLLARY 2.2. *Let A be a positive operator on Hilbert space with $0 \leqslant m_1 \leqslant A \leqslant M_1$. Then, for $\|x\| = 1$, we have*

$$(Ax,x) - (A^{\frac{1}{2}}x,x)^2 \leqslant \frac{(\sqrt{M_1} - \sqrt{m_1})^2}{4} - C(A^{\frac{1}{2}},x), \tag{2.2}$$

$$(A^{-1}x,x) - (A^{-\frac{1}{2}}x,x)^2 \leqslant \frac{(\sqrt{M_1} - \sqrt{m_1})^2}{4m_1M_1} - C(A^{-\frac{1}{2}},x). \tag{2.3}$$

THEOREM 2.3. *Let A,B be positive operators on Hilbert space with $0 \leqslant m_1 \leqslant A \leqslant M_1$ and $0 \leqslant m_2 \leqslant B \leqslant M_2$. Then, for $\|x\| = 1$, we have*

$$|(ABx,x) - (Ax,x)(Bx,x)| \leqslant \frac{(M_1 - m_1)(M_2 - m_2)}{4} - \sqrt{C(A,x)C(B,x)}. \tag{2.4}$$

□

Proof. First of all, since

$$(ABx,x) - (Ax,x)(Bx,x) = ((A - (Ax,x))(B - (Bx,x))x,x),$$

we have

$$|(ABx,x) - (Ax,x)(Bx,x)| \leqslant \|(A - (Ax,x))x\|\|(B - (Bx,x))x\|.$$

Morwover, it follows from Lemma 2.1 that

$$\|(A - (Ax,x))x\|^2 = (A^2x,x) - (Ax,x)^2 \leqslant \frac{(M_1 - m_1)^2}{4} - C(A,x).$$

Therefore it implies that

$$\begin{aligned}
&|(ABx,x) - (Ax,x)(Bx,x)|^2 \\
&\leqslant \left[\frac{(M_1 - m_1)^2}{4} - C(A,x)\right]\left[\frac{(M_2 - m_2)^2}{4} - C(B,x)\right] \\
&\leqslant \left(\frac{(M_1 - m_1)(M_2 - m_2)}{4} - \sqrt{C(A,x)C(B,x)}\right)^2
\end{aligned}$$

from $(a^2 - b^2)(c^2 - d^2) \leqslant (ac - bd)^2$ for real numbers a,b,c,d and $C(A,x) \geqslant 0$, $C(B,x) \geqslant 0$. □

THEOREM 2.4. *Let A be a selfadjoint operator on Hilbert space with $m_1 \leqslant A \leqslant M_1$, then, for $\|x\| = 1$ and all natural numbers k*

$$0 \leqslant (A^{k+1}x,x) - (Ax,x)^{k+1} \leqslant \frac{1}{4}(M_1 - m_1)^2 \sum_{p=1}^{k} (k-p+1) m_1^{p-1} M_1^{k-p} - \sum_{p=1}^{k} M_1^{k-p} \sqrt{C(A^p,x)C(A,x)} \tag{3.6}$$

Proof. For $k = 1$ it is shown by Lemma 2.1, that is, the following holds:

$$(A^2x,x) - (Ax,x)^2 \leqslant \frac{(M_1 - m_1)^2}{4} - C(A,x). \tag{3.7}$$

Assume (3.6) holds for some k, that is,

$$0 \leqslant (A^k x,x) - (Ax,x)^k \leqslant \frac{1}{4}(M_1 - m_1)^2 \sum_{p=1}^{k-1} (k-p) m_1^{p-1} M_1^{k-p-1} - \sum_{p=1}^{k-1} M_1^{k-p-1} \sqrt{C(A^p,x)C(A,x)} \tag{3.8}$$

then we prove (3.6) for $k+1$ by (3.8) and applying Theorem 3.2 to $B = A^k$ as follows:

$$\begin{aligned} &(A^{k+1}x,x) - (Ax,x)^{k+1} \\ &= (A^{k+1}x,x) - (Ax,x)(A^k x,x) + (Ax,x)[(A^k x,x) - (Ax,x)^k] \\ &\leqslant \frac{1}{4}(M_1 - m_1)(M_1^k - m_1^k) - \sqrt{C(A^k,x)C(A,x)} \\ &+ M_1[\frac{1}{4}(M_1 - m_1)^2 \sum_{p=1}^{k-1} (k-p) m_1^{p-1} M_1^{k-p-1} - \sum_{p=1}^{k-1} M_1^{k-p-1} \sqrt{C(A^p,x)C(A,x)}] \\ &= \frac{1}{4}(M_1 - m_1)^2 \sum_{p=1}^{k} (k-p+1) m_1^{p-1} M_1^{k-p} - \sum_{p=1}^{k} M_1^{k-p} \sqrt{C(A^p,x)C(A,x)}. \quad \square \end{aligned}$$

3. The extensions of Kantorovich inequality

Substituting B by A^{-1} in Theorem 2.3, we have the following improvement of Kantorovich inequality.

COROLLARY 3.1. *Let A be an operator on Hilbert space with $0 < m \leqslant A \leqslant M$, then, for $\|x\| = 1$*

$$(Ax,x)(A^{-1}x,x) - 1 \leqslant \frac{(M-m)^2}{4Mm} - \sqrt{C(A,x)C(A^{-1},x)}. \tag{3.1}$$

THEOREM 3.2. *Let A, B be operators on Hilbert space with $0<m_1\leqslant A\leqslant M_1$, $0<m_2\leqslant B\leqslant M_2$, then, for $\|x\|=1$*

$$(Ax,x)(Bx,x)-(A\sharp Bx,x)^2\leqslant\frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4M_2m_2}(Bx,x)^2. \tag{3.2}$$

Proof. Lemma 2.1 says that, for $x\neq 0$,

$$\frac{(A^2x,x)}{\|x\|^2}-\frac{(Ax,x)^2}{\|x\|^4}\leqslant\frac{(M_1-m_1)^2}{4}-\frac{C(A,x)}{\|x\|^2}$$

if $0<m_1\leqslant A\leqslant M_1$. Replace A by $(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}$ and x by $B^{\frac{1}{2}}x$ in the above inequality, then

$$\begin{aligned}&\frac{(B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}}x,x)}{(Bx,x)}-\frac{(B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}x,x)^2}{(Bx,x)^2}\\ \leqslant\;&\frac{(\sqrt{\frac{M_1}{m_2}}-\sqrt{\frac{m_1}{M_2}})^2}{4}-\frac{C((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}},B^{\frac{1}{2}}x)}{(Bx,x)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(Ax,x)(Bx,x)-(A\sharp Bx,x)^2\\ \leqslant\;&\frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4M_2m_2}(Bx,x)^2-C((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}},B^{\frac{1}{2}}x)(Bx,x)\\ \leqslant\;&\frac{(\sqrt{M_1M_2}-\sqrt{m_1m_2})^2}{4M_2m_2}(Bx,x)^2.\quad\square\end{aligned}$$

From the proof of Theorem 3.2 we can directly obtain the other form of the generalized Kantorovich inequality.

COROLLARY 3.3. *With the assumptions in Corollary 3.1,*

$$(Ax,x)(A^{-1}x,x)-1\leqslant\frac{(M_1-m_1)^2}{4}(A^{-1}x,x)^2-C(A,A^{-\frac{1}{2}}x)(A^{-1}x,x). \tag{3.3}$$

Acknowledgement. The authors would like to express their thanks to the referees for their valuable comments and suggestions, which helped to improve the paper.

REFERENCES

[1] KY FAN, *Some matrix inequalities*, Abh. Math. Sem. Univ. Hamburg **29** (1966), 185–196.

[2] M. FUJII, T. FURUTA, R. NAKAMOTO AND S. E. TAKAHASHI, *Operator inequalities and covariance in noncommutative probability*, Math. Japon. **46** (1996), 317–320.

[3] M. FUJII, S. IZUMINO, R. NAKAMOTO AND Y. SEO, *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J. **8** (1997), 117–122.
[4] M. FUJII, J. MIĆIĆ, J. E. PEČARIĆ AND Y. SEO, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Element, Zegreb, 2012, In Print.
[5] M. FUJII, R. NAKAMOTO AND Y. SEO, *Covariance in Bernstein's inequality for operators*, Nihonkai Math. J. **8** (1997), 1–6.
[6] J. I. FUJII AND Y. SEO, *Determinant for positive operators*, Sci. Math. **1** (1998), 153–156.
[7] T. FURUTA, *Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 38–41.
[8] T. FURUTA, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl. **2** (1998), 137–148.
[9] T. FURUTA, J. MIĆIĆ, J. E. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities **1**, Element, Zegreb, 2005.
[10] S. IZUMINO AND Y. SEO, *Ozeki's inequality and noncommuative covariance*, Nihonkai Math. J. **8** (1997), 55–58.
[11] Z. LIU, K. WANG, AND C. XU, *A note on Kantorovich inequality for Hermite matrices*, J. Inequal Appl. (2011), Artical ID 245767, 1–6.
[12] Y. SEO, *Operator convexity and its applications to operator inequalities*, Doctoral dissertation: Niigata University, 1999.
[13] Y. SEO, *Variance in noncommuative probability*, Math. Japon. **46**, 3 (1997), 439–444.

(Received July 29, 2012)

Masatoshi Fujii
Department of Mathematics
Osaka Kyoiku University
Asahigaoka, Kashiwara
Osaka 582-8582, Japan
e-mail: mfujii@cc.osaka-kyoiku.ac.jp

Hongliang Zuo
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007
China
e-mail: zuodke@yahoo.com

Nan Cheng
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007
China
e-mail: sghkanting@163.com