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SOME NESBITT TYPE INEQUALITIES WITH APPLICATIONS FOR THE ZETA FUNCTIONS

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Abstract. In this paper, we show some new generalizations of Nesbitt's inequality. As applications of the generalizations, we obtain the minimum value related to the Riemann zeta function and the Hurwitz zeta function.

1. Introduction

The following inequality is known in the literature as Nesbitt's inequality[3]: if x,y and z are positive real numbers, then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geqslant \frac{3}{2}, \tag{1.1}$$

with equality if and only if the numbers x,y and z are equal.

Recently there are some papers about Nesbitt's inequality, see [2, 4]. In [2], the authors prove that: If $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha \in (-\infty,0] \cup [1,\infty)$, $x_k > 0$, $p_k \in [0,1]$, $k \in \{1,2,\cdots,n\}$ such that $\sum\limits_{k=1}^{n} p_k = 1$, then

$$\frac{\sum\limits_{k=1}^{n}(p_k x_k)^{\alpha}}{\sum\limits_{k=1}^{n} p_k(x_1+x_2+\cdots+x_{k-1}+x_{k+1}+\cdots+x_n)} \leqslant \sum_{k=1}^{n} \frac{p_k x_k^{\alpha}}{x_1+x_2+\cdots+x_{k-1}+x_{k+1}+\cdots+x_n}, \tag{1.2}$$

where $\mathbb{N} = \{1,2,\ldots\}$.

The special case of (1.2) with $\alpha = 1$, $n = 3$ and $p_1 = p_2 = p_3 = \frac{1}{3}$, result in classical Nesbitt's inequality (1.1). In this paper, we give some new generalizations of Nesbitt's inequality. As applications, we obtain some new inequalities and the minimum value related to the Riemann zeta function and the Hurwitz zeta function.

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2. Main results

In order to prove the main results, we need the Jensen's inequality, which can be stated as: Let I a real interval, $\varphi : I \longrightarrow R$ a convex function, then for any $n \in \mathbb{N}$, $x_1, x_2, \cdots, x_n \in I$, and positive weights $p_1, p_2, \cdots, p_n$, such that $\sum_{k=1}^{n} p_k = 1$, the following inequality holds

$$\varphi\left(\sum_{k=1}^{n} p_k x_k\right) \leqslant \sum_{k=1}^{n} p_k \varphi(x_k). \tag{2.1}$$

The main result of this paper is the following inequality:

THEOREM 2.1. *Let $\alpha \geqslant 1$, $\beta \geqslant 0$ and $0 < x_k < M$, $p_k \in [0,1]$, $k \in \{1,2,\cdots,n\}$, such that $\sum_{k=1}^{n} p_k = 1$, then*

$$\frac{(\sum_{k=1}^{n} p_k x_k)^{\alpha}}{(M - \sum_{k=1}^{n} p_k x_k)^{\beta}} \leqslant \sum_{k=1}^{n} p_k \left\{ \frac{x_k^{\alpha}}{(M - x_k)^{\beta}} \right\}. \tag{2.2}$$

Proof. Let

$$f(x) = \frac{x^{\alpha}}{(M-x)^{\beta}}, \quad 0 < x < M.$$

It is easy to obtain

$$f''(x) = \frac{\alpha(\alpha-1)x^{\alpha-2}}{(M-x)^{\beta}} + \frac{2\alpha\beta x^{\alpha-1}}{(M-x)^{\beta+1}} + \frac{\beta(\beta+1)x^{\alpha}}{(M-x)^{\beta+2}}. \tag{2.3}$$

Under the conditions $\alpha \geqslant 1$, $\beta \geqslant 0$ and $0 < x_k < M$, it is obvious that $f''(x) \geqslant 0$ for $x \in (0,M)$. So, f is convex in $(0,M)$. Using Jensen's inequality gives the result (2.2). $\square$

Inequality (2.2) has many applications. In fact, if we take different p_k, α, β and M in (2.2), we can derive some other inequalities. First, we point out that Nesbitt's inequality is a special case of (2.2).

COROLLARY 2.2. *Let x_i be positive real numbers with $i \in \{1,2,\cdots,n\}$, then for $\alpha \geqslant 1$*

$$\sum_{i=1}^{n} \left(\frac{x_i}{\sum_{k=1}^{n} x_k - x_i} \right)^{\alpha} \geqslant \frac{n}{(n-1)^{\alpha}}, \tag{2.4}$$

Proof. Let $p_k = \frac{1}{n}$, $\alpha = \beta$ and $M = \sum_{i=1}^{n} x_i$ in (2.2), then

$$\frac{(\sum_{k=1}^{n} p_k x_k)^\alpha}{(M - \sum_{k=1}^{n} p_k x_k)^\beta} = \frac{1}{(n-1)^\alpha}, \tag{2.5}$$

and

$$\sum_{k=1}^{n} p_k \left\{ \frac{x_k^\alpha}{(M - x_k)^\beta} \right\} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i}{\sum_{k=1}^{n} x_k - x_i} \right)^\alpha. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.2) gives the inequality we seek. □

REMARK 2.3. For $M = x_1 + x_2 + \cdots + x_n = \sum_{k=1}^{n} p_k (x_1 + x_2 + \cdots + x_n)$ and $\beta = 1$ in (2.2), we obtain (1.2).

REMARK 2.4. Letting $n = 3$ in (2.4) gives: if x, y and z are positive real numbers, $\alpha \geqslant 1$, then

$$\left(\frac{x}{y+z}\right)^\alpha + \left(\frac{y}{z+x}\right)^\alpha + \left(\frac{z}{x+y}\right)^\alpha \geqslant \frac{3}{2^\alpha}. \tag{2.7}$$

Nesbitt's inequality (1.1) is a special case of (2.7) for $\alpha = 1$.

3. Some applications of the inequality for the zeta functions

Now, we show some applications of the inequality (2.2) for the Riemann zeta function and the Hurwitz zeta function. First, we get the following inequality, which will be used to obtain the minimum value related to the Riemann zeta function and the Hurwitz zeta function.

THEOREM 3.1. *Let x_i and y_j be positive real numbers, such that $x_i < y_j$, where $i \in \{1, 2, \cdots, n\}$ and $j \in \{1, 2, \cdots, m\}$, then for $\alpha \geqslant 1$, $\beta \geqslant 0$*

$$n^{\alpha-\beta-1} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{x_i^\alpha}{(y_j - x_i)^\beta} \geqslant (\sum_{i=1}^{n} x_i)^\alpha \sum_{j=1}^{m} \frac{1}{(n y_j - \sum_{i=1}^{n} x_i)^\beta}. \tag{3.1}$$

Proof. Let $p_k = \frac{1}{n}$ in (2.2). Consequently, we have

$$\frac{(\sum_{k=1}^{n} p_k x_k)^\alpha}{(M - \sum_{k=1}^{n} p_k x_k)^\beta} = \frac{(\frac{1}{n} \sum_{i=1}^{n} x_i)^\alpha}{(M - \frac{1}{n} \sum_{i=1}^{n} x_i)^\beta}, \tag{3.2}$$

and

$$\sum_{k=1}^{n} p_k \left\{ \frac{x_k^\alpha}{(M-x_k)^\beta} \right\} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^\alpha}{(M-x_i)^\beta}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (2.2) gives

$$\frac{1}{n} \sum_{i=1}^{n} \frac{x_i^\alpha}{(M-x_i)^\beta} \geqslant \frac{(\frac{1}{n}\sum\limits_{i=1}^{n} x_i)^\alpha}{(M-\frac{1}{n}\sum\limits_{i=1}^{n} x_i)^\beta}. \tag{3.4}$$

First, let $M = y_j$ in (3.4) and then take the summation with respect to j from 1 to m, we obtain

$$\frac{1}{n} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{x_i^\alpha}{(y_j-x_i)^\beta} \geqslant \sum_{j=1}^{m} \frac{(\frac{1}{n}\sum\limits_{i=1}^{n} x_i)^\alpha}{(y_j-\frac{1}{n}\sum\limits_{i=1}^{n} x_i)^\beta}, \tag{3.5}$$

which can be rewritten as (3.1). □

Now, we use the inequality (3.1) to obtain the minimum value related to the Riemann zeta function and the Hurwitz zeta function [1]. The Riemann zeta function is given by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{3.6}$$

This function as a function of a real argument was introduced and studied by Leonhard Euler in the first half of the eighteenth century without using complex analysis, which was not available at that time. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. A related function, the Hurwitz zeta function, is given by

$$\zeta(s,u) = \sum_{k=0}^{\infty} \frac{1}{(k+u)^s}. \tag{3.7}$$

The Riemann zeta function and the Hurwitz zeta function have the following relationship

$$\zeta(s,1) = \zeta(s). \tag{3.8}$$

THEOREM 3.2. *Let* $X = (x_1, x_2, \cdots, x_n)$, $V = \{X : x_i > 0, \sum\limits_{i=1}^{n} x_i = 1\}$, *then for* $s > 1$

$$\min_{X \in V} \sum_{i=1}^{n} x_i \zeta(s, 1 + \frac{1}{n} - x_i) = \zeta(s) \tag{3.9}$$

Proof. For any positive integer, suppose $y_j = j + \frac{1}{n}$, $j \in \{1, 2, \cdots, m\}$. It is easy to see $y_j > 1 \geqslant x_i$. Letting $\alpha = 1$, $\beta = s > 1$, $(x_1, x_2, \cdots, x_n) \in V$ and $y_j = j + \frac{1}{n}$,

$j \in \{1,2,\cdots,m\}$ in (3.1) gives

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{x_i}{(j+\frac{1}{n}-x_i)^s} \geqslant \sum_{j=1}^{m}\frac{1}{j^s}. \tag{3.10}$$

Let $m \to \infty$ in (3.10), one obtains

$$\sum_{i=1}^{n} x_i \zeta(s, 1+\frac{1}{n}-x_i) \geqslant \zeta(s). \tag{3.11}$$

On the other hand, if $x_i = \frac{1}{n}$ for $i \in \{1,2,\cdots,n\}$, then

$$\sum_{i=1}^{n} x_i \zeta(s, 1+\frac{1}{n}-x_i) = \frac{1}{n}\sum_{i=1}^{n}\zeta(s,1) = \zeta(s). \tag{3.12}$$

So, we obtain (3.9). □

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