Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 529–534 doi:10.7153/jmi-07-48

# SOME INEQUALITIES FOR THE SPECTRAL RADIUS OF THE HADAMARD PRODUCT OF TWO NONNEGATIVE MATRICES

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(*Communicated by A. Guessab*)

*Abstract.* In this paper, we propose some sharper upper bounds for the spectral radius of the Hadamard product of two nonnegative matrices. The results involve the directed graph of the Hadamard product of associated matrices.

## 1. Introduction

For any two $n\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, the Hadamard product of $A$ and $B$ is defined by $A\circ B=(a_{ij}b_{ij})$. A matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is called a nonnegative matrix if $a_{ij}\geqslant 0$. A matrix $A\in\mathbb{R}^{n\times n}$ is called a nonsingular $M$-matrix [1] if there exist $P\geqslant 0$ and $\alpha>0$ such that

$$A=\alpha I-P \quad \text{and} \quad \alpha>\rho(P),$$

where $\rho(P)$ is the spectral radius of the nonnegative matrix $P$, $I$ is the $n\times n$ identity matrix. Denote by $\mathscr{M}_n$ the set of all $n\times n$ nonsingular $M$-matrices. The matrices in $\mathscr{M}_n^{-1}:=\{A^{-1}:A\in\mathscr{M}_n\}$ are called inverse $M$-matrices. Let

$$\tau(A)=\min\{\operatorname{Re}\lambda:\lambda\in\sigma(A)\},$$

and $\sigma(A)$ denotes the spectrum of $A$. It is known that $\tau(A)=\frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A\in\mathscr{M}_n$, and the corresponding eigenvector is nonnegative [5, p. 129–130]. The set $\{1,\cdots,n\}$ is denoted by $N$, where $n$ is any positive integer. The $i$th row sum of matrix $A$ is denoted by $r_i(A)$.

A matrix $A$ is irreducible if there does not exist a permutation matrix $P$ such that

$$PAP^T=\begin{bmatrix}A_{1,1} & A_{1,2}\\ 0 & A_{2,2}\end{bmatrix},$$

where $A_{1,1}$ and $A_{2,2}$ are square matrices.

Denote the set of all simple circuits in the digraph $\Gamma_A$ of $A$ by $\Psi(A)$. A circuit of length $k$ in $\Gamma_A$ is an ordered sequence $\gamma=(i_1,\cdots,i_k,i_{k+1})$, where $i_1,\cdots,i_k\in N$ are all

*Mathematics subject classification* (2010): 15A42, 15A18.
*Keywords and phrases*: The Hadamard product, the spectral radius, nonnegative matrix, inverse $M$-matrix, directed graph.


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distinct, $i_{k+1} = i_1$. The set $\{i_1, \cdots, i_k\}$ is called the support of $\gamma$ and is denoted by $\bar{\gamma}$. The length of the circuit $\gamma$ is denoted by $|\gamma|$.

Recently, many contributions have been seen about the bounds of the eigenvalue of the Hadamard product of matrices in [2, 3, 6, 8, 9]. In this paper, our purpose is to propose some sharper upper bounds for the spectral radius of the Hadamard product of two nonnegative matrices.

## 2. Main results

LEMMA 1. [4, p. 507] *Let $A \geqslant 0$ be an irreducible $n \times n$ matrix. Then*
*1) $A$ has a positive real eigenvalue equal to its spectral radius;*
*2) to $\rho(A)$ there corresponds an eigenvector $x > 0$.*

LEMMA 2. [2, 3] *If $B = (b_{ij}) \in \mathscr{M}_n^{-1}$ and $B^{-1} = (\beta_{ij}) \in \mathscr{M}_n$, then*

$$b_{ij} \leqslant \frac{(\beta_{jj} - \tau(B^{-1}))v_j b_{ii}}{\beta_{jj} v_i}, \tag{1}$$

*where $v = (v_1, \ldots, v_n)^T > 0$ is the right Perron eigenvector of $B^T$.*

LEMMA 3. [8] *Let $A, B \in \mathbb{R}^{n\times n}$. If $E, F$ are diagonal matrices of order $n$, then*

$$E(A \circ B)F = (EAF) \circ B = (EA) \circ (BF) = (AF) \circ (EB) = A \circ (EBF).$$

LEMMA 4. [7] *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and let $\Psi(A) \neq \emptyset$. Then for any diagonal matrix $D$ with positive diagonal entries, we have*

$$\min_{\gamma \in \Psi(A)} \left[ \prod_{i \in \bar{\gamma}} r_i(D^{-1}AD) \right]^{\frac{1}{|\gamma|}} \leqslant \rho(A) \leqslant \max_{\gamma \in \Psi(A)} \left[ \prod_{i \in \bar{\gamma}} r_i(D^{-1}AD) \right]^{\frac{1}{|\gamma|}}.$$

THEOREM 1. [3] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and $B \in \mathscr{M}_n^{-1}$.*
*1). If $A$ is nilpotent, i.e., $\rho(A) = 0$, then $\rho(A \circ B) = 0$.*
*2). If $A$ is not nilpotent, then*

$$\rho(A \circ B) \leqslant \frac{\rho(A)}{\rho(B)} \max_{i \in N} \left[ \left( \frac{a_{ii}}{\rho(A)} + \beta_{ii}\rho(B) - 1 \right) \frac{b_{ii}}{\beta_{ii}} \right] \leqslant \rho(A) \max_{i \in N} b_{ii}.$$

THEOREM 2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and $B \in \mathscr{M}_n^{-1}$. If $A$ is not nilpotent, then*

$$\rho(A \circ B) \leqslant \frac{\rho(A)}{\rho(B)} \max_{\gamma \in \Psi(A \circ B)} \left[ \prod_{i \in \bar{\gamma}} \left( \frac{a_{ii}}{\rho(A)} + \beta_{ii}\rho(B) - 1 \right) \frac{b_{ii}}{\beta_{ii}} \right]^{\frac{1}{|\gamma|}}.$$

*Proof.* Since $B^{-1} \in \mathscr{M}_n$, we have $b_{ij} \geqslant 0$ and [1, p. 159]

$$\beta_{ii} > \tau(B^{-1}) > 0, \quad \forall i \in N.$$

Since the Hadamard product, and taking inverse are continuous functions, we may assume that $B^{-1} = \alpha I - P \in \mathscr{M}_n$, where $P \geqslant 0$ with $\alpha > \rho(P)$, are irreducible (thus $B \geqslant 0$ is also irreducible).

According to Lemma 1, let $v$ and $u$ be the right Perron eigenvectors of $B^T$ and $A$ respectively, *i.e.*, $v = (v_1, \cdots, v_n)^T$, $u = (u_1, \cdots, u_n)^T \in \mathbb{R}^n$ are positive vectors such that

$$(B^{-1})^T v = \tau(B^{-1})v, \qquad Au = \rho(A)u.$$

Hence, we have

$$a_{ii} + \sum_{j \neq i} \frac{a_{ij}u_j}{u_i} = \rho(A), \quad \forall i \in N. \tag{2}$$

Now define a diagonal matrix $Z = \operatorname{diag}(z_1, \cdots, z_n)$, where

$$z_i = \frac{u_i \beta_{ii}}{v_i(\beta_{ii} - \tau(B^{-1}))} > 0, \quad \forall i \in N. \tag{3}$$

By Lemma 2 and (3), we get

$$b_{ij}z_j \leqslant \frac{(\beta_{jj} - \tau(B^{-1}))v_j b_{ii}}{\beta_{jj}v_i} \cdot \frac{u_j \beta_{jj}}{v_j(\beta_{jj} - \tau(B^{-1}))} = \frac{b_{ii}u_j}{v_i}. \tag{4}$$

By Lemma 3, we have $\rho(A \circ B) = \rho(Z^{-1}(A \circ B)Z) = \rho(A \circ (Z^{-1}BZ))$. Let $\hat{B} = (\hat{b}_{ij}) = Z^{-1}BZ$. According to (2) and (4), we obtain

$$\begin{aligned} r_i[Z^{-1}(A \circ B)Z] &= r_i(A \circ \hat{B}) \\ &= a_{ii}b_{ii} + \sum_{j \neq i} a_{ij}\hat{b}_{ij} \\ &= a_{ii}b_{ii} + \sum_{j \neq i} a_{ij}b_{ij}\frac{z_j}{z_i} \\ &\leqslant a_{ii}b_{ii} + \sum_{j \neq i} a_{ij}\frac{b_{ii}u_j}{v_i} \cdot \frac{v_i(\beta_{ii} - \tau(B^{-1}))}{u_i\beta_{ii}} \\ &= a_{ii}b_{ii} + \frac{b_{ii}}{\beta_{ii}}(\beta_{ii} - \tau(B^{-1}))\sum_{j \neq i}\frac{a_{ij}u_j}{u_i} \\ &= a_{ii}b_{ii} + \frac{b_{ii}}{\beta_{ii}}(\beta_{ii} - \tau(B^{-1}))(\rho(A) - a_{ii}) \\ &= \rho(A)\tau(B^{-1})\left(\frac{a_{ii}}{\rho(A)} + \frac{\beta_{ii}}{\tau(B^{-1})} - 1\right)\frac{b_{ii}}{\beta_{ii}}. \end{aligned} \tag{5}$$

By Lemma 4 and (5), we get

$$\rho(A \circ B) \leqslant \frac{\rho(A)}{\rho(B)} \max_{\gamma \in \Psi(A \circ B)} \left[\prod_{i \in \overline{\gamma}} \left(\frac{a_{ii}}{\rho(A)} + \beta_{ii}\rho(B) - 1\right)\frac{b_{ii}}{\beta_{ii}}\right]^{\frac{1}{|\gamma|}}.$$

If $A\circ B$ is reducible, let $T=(t_{ij})$ be the permutation matrix such that $t_{12}=t_{23}=\cdots=t_{n-1,n}=t_{n,1}=1$ and the remaining $t_{ij}=0$. Then there exists a positive real number $\varepsilon$ such that $A+\varepsilon T$ is an irreducible nonnegative matrix and $(B^{-1}-\varepsilon T)^{-1}$ is an irreducible inverse $M$-matrix. Apply the irreducible case on them and then use continuity argument (the spectral radius $\rho(\cdot)$ and inverse taking are continuous functions) and to complete the proof. □

REMARK 1. Since

$$\begin{aligned}\rho(A\circ B) &\leqslant \frac{\rho(A)}{\rho(B)}\max_{\gamma\in\Psi(A\circ B)}\left[\prod_{i\in\overline{\gamma}}\left(\frac{a_{ii}}{\rho(A)}+\beta_{ii}\rho(B)-1\right)\frac{b_{ii}}{\beta_{ii}}\right]^{\frac{1}{|\gamma|}}\\ &\leqslant \frac{\rho(A)}{\rho(B)}\max_{\gamma\in\Psi(A\circ B)}\left[\left(\max_{i\in N}\{(\frac{a_{ii}}{\rho(A)}+\beta_{ii}\rho(B)-1)\frac{b_{ii}}{\beta_{ii}}\}\right)^{|\gamma|}\right]^{\frac{1}{|\gamma|}}\\ &= \frac{\rho(A)}{\rho(B)}\max_{i\in N}\left[\left(\frac{a_{ii}}{\rho(A)}+\beta_{ii}\rho(B)-1\right)\frac{b_{ii}}{\beta_{ii}}\right],\end{aligned}$$

we know that the bound of Theorem 2 is sharper than the one of Theorem 1.

THEOREM 3. [10] *If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$ are nonnegative, then*

$$\rho(A\circ B)\leqslant\max_{i\in N}\{a_{ii}b_{ii}+\alpha_i\rho(B)-\alpha_i b_{ii}\},\tag{6}$$

*where* $\alpha_i=\max_{k\neq i}\{a_{ik}\}$, $\forall i\in N$.

THEOREM 4. [10] *If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$ are nonnegative, then*

$$\rho(A\circ B)\leqslant\max_{i\in N}\{a_{ii}b_{ii}+\beta_i\rho(A)-\beta_i a_{ii}\},\tag{7}$$

*where* $\beta_i=\max_{k\neq i}\{b_{ik}\}$, $\forall i\in N$.

THEOREM 5. *If $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$ are nonnegative, then*

$$\rho(A\circ B)\leqslant\max_{\gamma\in\Psi(A\circ B)}\left[\prod_{i\in\overline{\gamma}}\left(a_{ii}b_{ii}+\alpha_i\rho(B)-\alpha_i b_{ii}\right)\right]^{\frac{1}{|\gamma|}}.\tag{8}$$

*where* $\alpha_i=\max_{k\neq i}\{a_{ik}\}$, $\forall i\in N$.

*Proof.* It is easy to know that (8) holds with equality for $n=1$. In the following, we consider $n\geqslant 2$.

Firstly, we may assume that $A\circ B$ is irreducible, then $A$ and $B$ are irreducible. According to Lemma 1, we may assume that $v=(v_1,\ldots,v_n)^T>0$ be the right Perron eigenvector of $B$. Then we get

$$v_i b_{ii}+\sum_{j\neq i}b_{ij}v_j=\rho(B)v_j,\quad\forall i\in N,$$

or equivalently,

$$\sum_{j\neq i} b_{ij}v_j = [\rho(B) - b_{ii}]v_i, \quad \forall i \in N.$$

Denote $\alpha_i = \max_{k\neq i}\{a_{ik}\}$, $\forall i \in N$. Since $A$ is an irreducible nonnegative matrix, $\alpha_i > 0$, $\forall i \in N$. Define a positive diagonal matrix $Z = \mathrm{diag}(z_1,\dots,z_n)$, where

$$z_i = \frac{v_i}{\alpha_i} > 0, \quad \forall i \in N.$$

By Lemma 3, we have $\rho(A\circ B) = \rho(Z^{-1}(A\circ B)Z) = \rho(A\circ(Z^{-1}BZ))$. Let $\hat{B} = Z^{-1}BZ$. So, we have

$$\begin{aligned} r_i[Z^{-1}(A\circ B)Z] &= r_i(A\circ\hat{B}) = a_{ii}b_{ii} + \sum_{j\neq i} a_{ij}b_{ij}\frac{z_j}{z_i} \\ &= a_{ii}b_{ii} + \sum_{j\neq i} b_{ij}a_{ij}\frac{v_j}{\alpha_j}\frac{\alpha_i}{v_i} \leqslant a_{ii}b_{ii} + \sum_{j\neq i} b_{ij}v_j\frac{\alpha_i}{v_i} \\ &= a_{ii}b_{ii} + \alpha_i\rho(B) - \alpha_i b_{ii}. \end{aligned}$$

By Lemma 4, we get the desired result

$$\rho(A\circ B) \leqslant \max_{\gamma\in\Psi(A\circ B)} \left[\prod_{i\in\bar{\gamma}}\left(a_{ii}b_{ii} + \alpha_i\rho(B) - \alpha_i b_{ii}\right)\right]^{\frac{1}{|\gamma|}}.$$

If $A\circ B$ is reducible, let $T = (t_{ij})$ be the permutation matrix such that $t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n,1} = 1$ and the remaining $t_{ij} = 0$. Then there exists a positive real number $\varepsilon$ such that $A + \varepsilon T$ and $B + \varepsilon T$ are irreducible nonnegative matrices. Apply the irreducible case on them and then use continuity argument and to complete the proof. $\square$

Since the Hadamard product is commutative, we obtain the following result.

THEOREM 6. *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ are nonnegative, then*

$$\rho(A\circ B) \leqslant \max_{\gamma\in\Psi(A\circ B)} \left[\prod_{i\in\bar{\gamma}}\left(a_{ii}b_{ii} + \beta_i\rho(A) - \beta_i a_{ii}\right)\right]^{\frac{1}{|\gamma|}}.$$

*where $\beta_i = \max_{k\neq i}\{b_{ik}\}$, $\forall i \in N$.*

According to Theorem 5 and Theorem 6, we have the following corollary.

COROLLARY 1. *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ are nonnegative, then*

$$\begin{aligned} \rho(A\circ B) \leqslant \ \min\Bigg\{ &\max\nolimits_{\gamma\in\Psi(A\circ B)} \left[\prod\nolimits_{i\in\bar{\gamma}}\left(a_{ii}b_{ii} + \alpha_i\rho(B) - \alpha_i b_{ii}\right)\right]^{\frac{1}{|\gamma|}}, \\ &\max\nolimits_{\gamma\in\Psi(A\circ B)} \left[\prod\nolimits_{i\in\bar{\gamma}}\left(a_{ii}b_{ii} + \beta_i\rho(A) - \beta_i a_{ii}\right)\right]^{\frac{1}{|\gamma|}}\Bigg\}. \end{aligned}$$

*where $\alpha_i = \max_{k\neq i}\{a_{ik}\}$ and $\beta_i = \max_{k\neq i}\{b_{ik}\}$, $\forall i \in N$.*

REMARK 2. Since

$$\begin{aligned}\rho(A\circ B) &\leqslant \max_{\gamma\in\Psi(A\circ B)}\left[\prod_{i\in\overline{\gamma}}\left(a_{ii}b_{ii}+\alpha_i\rho(B)-\alpha_i b_{ii}\right)\right]^{\frac{1}{|\gamma|}}\\ &\leqslant \max_{\gamma\in\Psi(A\circ B)}\left[\left(\max_{i\in N}\{a_{ii}b_{ii}+\alpha_i\rho(B)-\alpha_i b_{ii}\}\right)^{|\gamma|}\right]^{\frac{1}{|\gamma|}}\\ &= \max_{i\in N}\{a_{ii}b_{ii}+\alpha_i\rho(B)-\alpha_i b_{ii}\},\end{aligned}$$

we know that the bound of Theorem 5 is sharper than the one of Theorem 3. Similarly, we also have that the bound of Theorem 6 is sharper than the one of Theorem 4.

## 3. Conclusions

In this paper, some inequalities for the spectral radius of the Hadamard product of two nonnegative matrices are given. Furthermore, we prove that the results of this paper are sharper than the ones of [3] and [10].

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(Received August 15, 2012)

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