NEW RESULTS RELATED TO THE CONVEXITY OF THE BERNARDI INTEGRAL OPERATOR

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Abstract. In this paper we prove the convexity of the image of a close-to-convex function by the Bernardi integral operator given by

\[ L_γ(f)(z) = F(z) = \frac{γ+1}{z^γ} \int_0^z f(t)t^{γ-1}dt, \quad z \in U. \]  

(1)

This result extends the result obtained by N. Pascu in [9], where it has been shown that the Bernardi operator transforms a close-to-convex function into a close-to-convex function under certain conditions.

1. Introduction and preliminaries

Let \( U \) be the unit disc of the complex plane:

\[ U = \{ z \in \mathbb{C} : |z| < 1 \}. \]

Let \( \mathcal{H}(U) \) be the space of holomorphic functions in \( U \). Also, let

\[ A_n = \{ f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \} \]

with \( A_1 = A \) and

\[ S = \{ f \in A : f \text{ is univalent in } U \}. \]

Let

\[ K = \left\{ f \in A, \ \text{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}, \]

denote the class of normalized convex functions in \( U \),

\[ S^* = \left\{ f \in A, \ \text{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\} \]

denote the class of starlike functions in \( U \), and

\[ C = \left\{ f \in A : \ \exists \ \varphi \in K, \ \text{Re} \frac{f'(z)}{\varphi'(z)} > 0, \ z \in U \right\} \]

denote the class of close-to-convex functions.

In order to prove our original results, we use the following lemmas:

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**Lemma 1.** [3, 4, 6, Theorem 2.3.i, p. 35] Let \( \psi : \mathbb{C}^2 \times U \to \mathbb{C} \), satisfy the condition

\[
\Re \psi(is,t;z) \leq 0, \quad z \in U,
\]

for \( s,t \in \mathbb{R}, \ t \leq -\frac{n}{2}(1+s^2) \).

If \( p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots \) satisfies

\[
\Re [p(z),zp'(z);z] > 0
\]

then

\[
\Re p(z) > 0, \quad z \in U.
\]

More general forms of this lemma can be found in [6].

**Lemma 2.** [7, Theorem 4.6.3, p. 84] The function \( f \in A \), with \( f'(z) \neq 0 \), \( z \in U \), is close-to-convex if and only if

\[
\int_{\theta_1}^{\theta_2} \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] d\theta > -\pi, \quad z = re^{i\theta},
\]

for all \( \theta_1, \theta_2 \) with \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \) and all \( r \in (0,1) \).

If \( L_\gamma : A \to A \) is the integral operator defined by \( L_\gamma[f] = F \), where \( F \) is given by

\[
L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt
\]

and \( \Re \gamma \geq 0, \ z \in U \), then it is well known that

(i) \( L_\gamma[S^*] \subset S^* \),

(ii) \( L_\gamma[K] \subset K \),

(iii) \( L_\gamma[C] \subset C \).

These results are obtained in [2] and [9].

**2. Main results**

We determine conditions such that, for a function \( f \in A_n \), the image under the Bernardi integral operator is convex.

**Theorem 1.** Let \( f \in A_n, \ \gamma \geq 1, n \geq 1 \) and

\[
L_\gamma(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt, \quad z \in U. \tag{2}
\]

If

\[
\Re \left[ \frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2\gamma}, \quad z \in U, \tag{3}
\]
then the function $F$ given by (2) is convex.

**Proof.** Let $f \in A_n$, $f(z) = z + a_{n+1}z^{n+1} + \ldots$, $z \in U$. Then, from (2), we have:

$$L_\gamma(f)(z) = F(z) = \frac{\gamma + 1}{z} \int_0^z (t + a_{n+1}t^{n+1} + \ldots) t^{\gamma-1} dt$$

$$= \frac{\gamma + 1}{z} \left( \frac{z^{n+1}}{n+1} + a_{n+1} \frac{z^{n+1}}{n+1} + \ldots \right) = z + b_{n+1}z^{n+1} + \ldots,$$

hence $F \in A_n$.

According to Lemma 2 we obtain

$$\int_{\theta_1}^{\theta_2} \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] d\theta \geq \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma} d\theta = -\frac{1}{2\gamma} \int_{\theta_1}^{\theta_2} d\theta$$

$$= -\frac{1}{2\gamma} (\theta_2 - \theta_1) = -\frac{2\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \gamma \geq 1.$$  

From (5) we have $f \in C$, hence it is univalent. If $f \in C$, then from (iii) we have $L_\gamma[f] = F \in C$, hence $F$ is univalent.

From (2), we have

$$z^{\gamma} F(z) = \left( 1 + \frac{\gamma}{n} \right) \int_0^z f(t) t^{\gamma-1} dt, \quad z \in U.$$  

By differentiating (6), we obtain

$$\frac{\gamma}{n} z^{\gamma-1} F(z) + z^{\gamma} F'(z) = \left( \frac{\gamma}{n} + 1 \right) f(z) \cdot z^{\gamma-1}, \quad z \in U,$$

and by a simple calculation, we have

$$\frac{\gamma}{n} F(z) + z F'(z) = \left( \frac{\gamma}{n} + 1 \right) f(z), \quad z \in U.$$  

By differentiating (8) and by a simple calculation, we obtain

$$\frac{\gamma}{n} F'(z) + F'(z) \left[ 1 + \frac{zf''(z)}{F'(z)} \right] = \left( \frac{\gamma}{n} + 1 \right) f'(z), \quad z \in U.$$  

Let

$$1 + \frac{zf''(z)}{F'(z)} = p(z), \quad z \in U, \quad p(0) = 1, \quad p(z) = 1 + p_nz^n + \ldots,$$

then (9) is equivalent to

$$F'(z) \left[ \frac{\gamma}{n} + p(z) \right] = \left( \frac{\gamma}{n} + 1 \right) f'(z), \quad z \in U.$$  

(11)
Since $F'(z) \neq 0$, $p(z) + \gamma \neq 0$, $f \in C$, we have $f'(z) \neq 0$, $z \in U$, and by differentiating (11), we obtain

$$1 + \frac{zF''(z)}{F'(z)} + \frac{zp'(z)}{p(z) + \gamma} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \quad (12)$$

Using (10), we have

$$p(z) + \frac{zp'(z)}{p(z) + \gamma} = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in U. \quad (13)$$

Using (3), we obtain

$$\operatorname{Re} \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} \right] > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1, \quad (14)$$

which is equivalent to

$$\operatorname{Re} \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma} \right] > 0, \quad z \in U, \quad \gamma \geq 1. \quad (15)$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z),zp'(z);z) = p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1. \quad (16)$$

Then (15) is equivalent to

$$\operatorname{Re} \psi(p(z),zp'(z);z) > 0, \quad z \in U. \quad (17)$$

In order to prove Theorem 1, we use Lemma 1. For that we calculate

$$\operatorname{Re} \psi(is,t;z) = \operatorname{Re} \left[ is + \frac{t}{is + \gamma} + \frac{1}{2\gamma} \right] = \operatorname{Re} \left[ is + \frac{1}{2\gamma} + \frac{t \left( \frac{\gamma}{n} - is \right)}{\gamma^2/n^2 + s^2} \right]$$

$$= \frac{1}{2\gamma} + \frac{\gamma^2}{n^2 + s^2} \leq \frac{1}{2\gamma} - \frac{\gamma^2 (1 + s^2)}{2n^2} \leq \frac{1}{2\gamma} - \frac{\gamma(1 + s^2)}{2 \left( \gamma^2/n^2 + s^2 \right)}$$

$$= \frac{\gamma^2/n^2 + s^2 - \gamma^2 - \gamma^2 s^2}{2\gamma \left( \gamma^2/n^2 + s^2 \right)} = \frac{\gamma^2 \left( 1/n^2 - 1 \right) + s^2 (1 - \gamma^2)}{2\gamma \left( \gamma^2/n^2 + s^2 \right)} \leq 0,$$
since \( n \geq 1 \) and \( \gamma \geq 1 \).

Now, using Lemma 1 we get that \( \text{Re} \ p(z) > 0, \ z \in U \), i.e.

\[
\text{Re} \ \frac{zF''(z)}{F'(z)} + 1 > 0, \quad z \in U,
\]

hence \( F \in K \). \( \square \)

**Remark 1.** For \( n = 1 \) we obtain the results from [8].

We determine conditions such that, for a function \( f \in H[1, 1] \), the image under the Bernardi integral operator is convex.

**Theorem 2.** Let \( f \in H[1, 1], \ \gamma \geq 1, \) and

\[
L_\gamma[f](z) = F(z) = \frac{\gamma}{z^{\gamma}} \int_0^z f(t)t^{\gamma-1}dt, \quad z \in U. \tag{18}
\]

If

\[
\text{Re} \left[ \frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2\gamma}, \quad z \in U, \tag{19}
\]

then the function \( F \) given by (18) is convex.

**Proof.** Let \( f \in H[1, 1], \ f(z) = 1 + a_1z + a_2z^2 + \ldots, \ z \in U \). Then, from (18), we have

\[
L_\gamma[f](z) = F(z) = \frac{\gamma}{z^{\gamma}} \int_0^z (1 + a_1t + a_2t^2 + \ldots)t^{\gamma-1}dt \tag{20}
\]

\[
= \frac{\gamma}{z^{\gamma}} \left[ \frac{z^{\gamma}}{\gamma} + \frac{a_1z^{\gamma+1}}{\gamma+1} + \frac{a_2z^{\gamma+2}}{\gamma+2} + \ldots \right] = 1 + b_1z + b_2z + \ldots,
\]

hence \( f \in H[1, 1] \).

According to Lemma 2 we obtain

\[
\int_{\theta_1}^{\theta_2} \text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] d\theta \geq \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma}d\theta = -\frac{1}{2\gamma} \int_{\theta_1}^{\theta_2} d\theta \tag{21}
\]

\[
= -\frac{1}{2\gamma} (\theta_2 - \theta_1) = -\frac{\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \gamma \geq 1.
\]

From (21) we have \( f \in C \), hence it is univalent. If \( f \in C \), then from (iii) we have \( L_\gamma[f] = F \in C \), hence \( F \) is univalent, \( F'(z) \neq 0, \ z \in U \).

From (18), we have

\[
z^\gamma F(z) = \gamma \int_0^z f(t)t^{\gamma-1}dt, \quad z \in U. \tag{22}
\]

By differentiating (22) and by a simple calculation, we obtain

\[
\gamma F'(z) + F'(z) \left[ 1 + \frac{zf''(z)}{f'(z)} \right] = \gamma f'(z), \quad z \in U. \tag{23}
\]
Let
\[ 1 + \frac{zF''(z)}{F'(z)} = p(z), \quad z \in U, \quad p(0) = 1, \quad p(z) = 1 + p_1 z + p_2 z^2 + \ldots. \tag{24} \]

Then (23) is equivalent to
\[ F'(z)[p(z) + \gamma] = \gamma f'(z), \quad z \in U. \tag{25} \]

Since \( F'(z) \neq 0 \), \( p(z) + \gamma \neq 0 \), \( f \in C \), we have \( f'(z) \neq 0 \), \( z \in U \), and by differentiating (25), we obtain
\[ 1 + \frac{zF''(z)}{F'(z)} + \frac{zp'(z)}{p(z) + \gamma} = \frac{zf''(z)}{f'(z)} + 1, \quad z \in U. \tag{26} \]

Using (24), we have
\[ p(z) + \frac{zp'(z)}{p(z) + \gamma} = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in U. \tag{27} \]

Using (19), we obtain
\[ \Re \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} \right] > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1, \tag{28} \]

which is equivalent to
\[ \Re \left[ p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma} \right] > 0, \quad z \in U, \quad \gamma \geq 1. \tag{29} \]

Let \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \),
\[ \psi(p(z), zp'(z); z) = p(z) + \frac{zp'(z)}{p(z) + \gamma} + \frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geq 1. \tag{30} \]

Then (29) is equivalent to
\[ \Re \psi(p(z), zp'(z); z) > 0, \quad z \in U. \tag{31} \]

In order to prove Theorem 2, we use Lemma 1. For that we calculate
\[ \Re \psi(is, t; z) = \Re \left[ is + \frac{1}{2\gamma} + \frac{t}{is + \gamma} \right] = \Re \left[ is + \frac{1}{2\gamma} + \frac{t(\gamma - is)}{\gamma^2 + s^2} \right] = \frac{1}{2\gamma} + \frac{t\gamma}{\gamma^2 + s^2} \leq \frac{1}{2\gamma} - \frac{\gamma(1+s^2)}{2(\gamma^2+s^2)} = \frac{\gamma^2 + s^2 - \gamma^2 - \gamma^2 s^2}{2\gamma(\gamma^2 + s^2)} = \frac{s^2(1-\gamma^2)}{2\gamma(\gamma^2 + s^2)} \leq 0, \]

since \( \gamma \geq 1, n \geq 1 \).

Now, using Lemma 1, we get that \( \Re p(z) > 0, z \in U \), i.e.
\[ \Re \frac{zF''(z)}{F'(z)} + 1 > 0, \quad z \in U, \]

hence \( F \in K \). □
REFERENCES


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