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SIMPLE PROOFS OF THE CUSA–HUYGENS–TYPE AND BECKER–STARK–TYPE INEQUALITIES

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Abstract. In this paper, we respectively give some simple proofs of the Cusa-Huygens- and Becker-Stark-type inequalities presented by Chen and Cheung in [9].

1. Introduction

For $0 < x < \pi/2$, the Cusa-Huygens inequality (see [1–5]) and Becker-Stark inequality (see [6–8]) are known as

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}, \tag{1}$$

and

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \tag{2}$$

respectively.

In recent paper [9], Chen and Cheung sharpen the two inequalities above and obtain the following results.

THEOREM 1.1. *For* $0 < x < \pi/2$,

$$\left(\frac{2+\cos x}{3}\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\vartheta} \tag{3}$$

with the best constants $\theta = \ln(\pi/2)/\ln(3/2) = 1.11373998\cdots$, *and* $\vartheta = 1$.

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THEOREM 1.2. *For* $0 < x < \pi/2$,

$$\left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\alpha} < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2 - 4x^2}\right)^{\beta} \tag{4}$$

with the best constants $\alpha = \pi^2/12 = 0.822467033\cdots$, *and* $\beta = 1$.

Using a monotone form of l'Hospital's rule and the power series expansions of sine and cosine functions in [9], the authors verify the double inequality (3) in a tortuous process. Meanwhile, in [9], in order to prove (4) the authors employ a method for proving inequalities by computer which can be found in [19] by Malešević. In this paper, we give two shorter and lucid proofs of the inequalities (3) and (4) just using the differentiation.

2. Lemmas

LEMMA 2.1. ([10–15]) *Let* $f, g : [a,b] \longrightarrow R$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g' \neq 0$ *on* (a,b). *If* f'/g' *is increasing (or decreasing)on* (a,b), *then the functions* $(f(x) - f(b))/(g(x) - g(b))$ *and* $(f(x) - f(a))/(g(x) - g(a))$ *are also increasing (or decreasing) on* (a,b).

LEMMA 2.2. ([16], [17], [3]) *Let* $|x| < \pi$, *and* B_{2n} *be the even-indexed Bernoulli numbers. Then*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} |B_{2n}| x^{2n} \tag{5}$$

holds.

LEMMA 2.3. ([18]) *For all integers* $n \geqslant 1$, *let* B_{2n} *be the even-indexed Bernoulli numbers. Then*

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\alpha - 2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\beta - 2n}} \tag{6}$$

with the best constants $\alpha = 0$ *and* $\beta = 2 + (\ln(1 - 6/\pi^2))/\ln 2 \approx 0.6491\cdots$.

3. New Proof of Theorem 1.1

Let

$$f(x) = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln\left(\frac{2+\cos x}{3}\right)},$$

we compute

$$f'(x) = \frac{g(x)}{\left[\ln\left(\frac{2+\cos x}{3}\right)\right]^2},$$

where

$$g(x) = \frac{x\cos x - \sin x}{x \sin x} \ln \frac{2 + \cosh x}{3} + \frac{\sin x}{2 + \cos x} \ln \frac{\sin x}{x}.$$

In the following part, we shall prove that $g(x) > 0$. Firstly, since

$$\ln\frac{2+\cos x}{3} > \ln\frac{\sin x}{x},$$

we obtain

$$g(x) > \left(\frac{x\cos x-\sin x}{x\sin x}+\frac{\sin x}{2+\cos x}\right)\cdot\ln\frac{\sin x}{x}.$$

In view of that $\ln((\sin x)/x) < 0$ for $0 < x < \pi/2$, the proof of $g(x) > 0$ is complete if we prove the following inequality

$$\frac{x\cos x-\sin x}{x\sin x}+\frac{\sin x}{2+\cos x} < 0.$$

We can simplify the above inequality as follows:

$$\begin{aligned}\frac{x\cos x-\sin x}{x\sin x}+\frac{\sin x}{2+\cos x} &= \frac{x\sin^2 x+(x\cos x-\sin x)(2+\cos x)}{x\sin x(2+\cos x)}\\ &= \frac{h(x)}{x\sin x(2+\cos x)},\end{aligned}$$

where $h(x) = x+2x\cos x-2\sin x-\sin x\cos x$.

Since $h'(x) = 1-2x\sin x-\cos 2x = 2(\sin x)^2-2x\sin x = 2\sin x(\sin x-x) < 0$, and $h(0) = 0$, we have $h(x) < 0$, which indicates that $g(x) > 0$ and $f'(x) > 0$. So $f(x)$ is increasing for $0 < x < \pi/2$.

At the same time, $f(0^+) = 1$ and

$$f\left(\frac{\pi}{2}\right) = \frac{\ln(\frac{\pi}{2})}{\ln(\frac{3}{2})},$$

so 1 and $\ln(\pi/2)/\ln(3/2)$ are the best constants in (3).

The proof of Theorem 1.1 is complete. □

4. New Proof of Theorem 1.2

Let

$$k(x) = \frac{\ln\frac{\tan x}{x}}{\ln\frac{\pi^2}{\pi^2-4x^2}} = \frac{f_1(x)}{f_2(x)},$$

we compute

$$\frac{f_1'(x)}{f_2'(x)} = \frac{(\pi^2-4x^2)(2x-\sin(2x))}{8x^2\sin(2x)} = \frac{(\pi^2-4x^2)}{8x^2}\left(\frac{2x}{\sin(2x)}-1\right) \equiv p(x).$$

By the Lemma 2.2 we obtain

$$
\begin{aligned}
p(x) =& \frac{(\pi^2-4x^2)}{8x^2}\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n}\\
=& \frac{1}{8}(\pi^2-4x^2)\sum_{n=1}^{\infty}\frac{(2^{2n}-2)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}\\
=& \frac{1}{8}\left[\pi^2\sum_{n=1}^{\infty}\frac{(2^{2n}-2)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}-4\sum_{n=1}^{\infty}\frac{(2^{2n}-2)2^{2n}}{(2n)!}|B_{2n}|x^{2n}\right]\\
=& \frac{1}{8}\left[\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}\left(\frac{\pi^2(2^{2n+2}-2)2^{2n+2}}{(2n+2)!}|B_{2n+2}|-\frac{4(2^{2n}-2)2^{2n}}{(2n)!}|B_{2n}|\right)x^{2n}\right]\\
=& \frac{1}{8}\left[\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}a_nx^{2n}\right],
\end{aligned}
$$

where

$$
a_n = \frac{\pi^2(2^{2n+2}-2)2^{2n+2}}{(2n+2)!}|B_{2n+2}|-\frac{4(2^{2n}-2)2^{2n}}{(2n)!}|B_{2n}|.
$$

From the Lemma 2.3, we compute

$$
\begin{aligned}
a_n >& \frac{2^{2n+3}}{\pi^{2n}}\left(\frac{2^{2n+2}-2}{2^{2n+2}-1}-\frac{2^{2n}-2}{2^{2n}-2^{\beta}}\right)\\
=& \frac{2^{2n+3}}{\pi^{2n}}\cdot\frac{(2-2^{\beta})2^{2n+2}-2^{2n}}{(2^{2n+2}-1)(2^{2n}-2^{\beta})}\\
=& \frac{2^{4n+3}}{\pi^{2n}}\cdot\frac{4(2-2^{\beta})-1}{(2^{2n+2}-1)(2^{2n}-2^{\beta})}.
\end{aligned}
$$

Since $4(2-2^{\beta})-1\approx 0.7268\cdots>0$, we get that $a_n>0$ for $n=1,2,\cdots$. This indicates that $p(x)$ is increasing for $0<x<\pi/2$. Then the function $k(x)$ is increasing on $(0,\pi/2)$ by Lemma 2.1.

At the same time, $k(0^+)=1$ and $k((\pi/2)^-)=\pi^2/12$, so 1 and $\pi^2/12$ are the best constants in (4).

The proof of Theorem 1.2 is complete. □

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