Journal of Mathematical Inequalities
Volume 7, Number 4 (2013), 569–575 doi:10.7153/jmi-07-53

INEQUALITY CHAINS RELATED TO TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS AND INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

CHAO-PING CHEN AND JÓZSEF SÁNDOR

(*Communicated by N. Elezović*)

Abstract. We present inequality chains related to trigonometric and hyperbolic and inverse trigonometric and hyperbolic functions.

1. Introduction

Neuman and Sándor [6] have proved that

$$\left(\frac{x}{\arctan x}\right)^{1/2} < \frac{x}{\operatorname{arcsinh} x} < \left(\frac{1+(x/\arctan x)^2}{2}\right)^{1/2} \tag{1.1}$$

for $x \neq 0$, by the theory of means. Inequality (1.1) appears in another form (with mean theory notation) as the third inequality of relation (3.6) of Corollary 3.2 of [6]. More precisely, the following inequality holds

$$AT \leqslant M^2 \leqslant \frac{A^2+T^2}{2}, \tag{1.2}$$

where

$$M = \frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}, \quad T = \frac{a-b}{2\arctan\frac{a-b}{a+b}} \quad \text{and} \quad A = \frac{a+b}{2}.$$

Let $x = \frac{a-b}{a+b}$ in (1.2), then one can get (1.1).

It is clear that $\left(\frac{x}{\arctan x}\right)^{1/2}$ and $\left(\frac{1+(x/\arctan x)^2}{2}\right)^{1/2}$ are the geometric and root-mean square means of 1 and $x/\arctan x$, respectively. The first aim of this paper is to establish Theorem 1.1, which shows that the second inequality in (1.1) can be separated by the arithmetic mean of 1 and $x/\arctan x$.

Mathematics subject classification (2010): 26D05, 26D07.
Keywords and phrases: Inequalities, trigonometric functions, hyperbolic functions, inverse trigonometric functions, inverse hyperbolic functions.

THEOREM 1.1. *For all* $x \neq 0$,

$$\frac{x}{\operatorname{arcsinh} x} < \frac{1 + (x/\arctan x)}{2} < \left(\frac{1 + (x/\arctan x)^2}{2}\right)^{1/2}. \tag{1.3}$$

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \tag{1.4}$$

for $0 < |x| < \frac{\pi}{2}$, and

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} \tag{1.5}$$

for $x \neq 0$. The left-hand side inequality (1.4) first appeared in [4, p. 238], while the right-hand side inequality (1.4) is due to Cusa and Huygens (see [9] for more details regarding this result). The first inequality in (1.5) was established by Lazarević [2] (see, e.g., [4, p. 238]), while the second inequality in (1.5) appears in [7].

The inequalities (1.4) and (1.5) have been grouped into the following inequality chain:

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{3}{2 + \cos x} < \frac{2 + \cosh x}{3} < \frac{x}{\sin x} < \frac{1}{(\cos x)^{1/3}} \tag{1.6}$$

for $0 < |x| < \pi/2$ in [8] (See pp. 23–24 for proofs, as well as pp. 16–17 for the history of such relations). Very recently, Chen and Sándor [1, Remark 2.1] showed that the first inequalities in (1.4) and (1.5) can be separated. More precisely, the authors proved that

$$(\cos x)^{1/3} < \left(\frac{1 + 2\cos x}{3}\right)^{1/2} < \frac{\sin x}{x}, \quad 0 < |x| < \frac{\pi}{2} \tag{1.7}$$

and

$$(\cosh x)^{1/3} < \left(\frac{1 + 2\cosh x}{3}\right)^{1/2} < \frac{\sinh x}{x}, \quad x \neq 0. \tag{1.8}$$

The second aim of this paper is to establish Theorem 1.2, which shows that if we restrict $0 < |x| < 1$, then we have longer inequality chain:

$$\begin{aligned}\left(\frac{x}{\arctan x}\right)^{1/2} &< \frac{x}{\operatorname{arcsinh} x} < (\cosh x)^{1/3} < \left(\frac{1 + 2\cosh x}{3}\right)^{1/2} < \frac{\sinh x}{x} \\ &< \frac{3}{2 + \cos x} < \frac{2 + \cosh x}{3} < \frac{x}{\sin x} < \left(\frac{3}{1 + 2\cos x}\right)^{1/2} < \frac{1}{(\cos x)^{1/3}} \\ &< \frac{\arcsin x}{x} < \left(\frac{\operatorname{arctanh} x}{x}\right)^{1/2}.\end{aligned} \tag{1.9}$$

THEOREM 1.2. (i) *The following inequalities hold:*

$$\left(\frac{x}{\operatorname{arctan} x}\right)^{1/2} < \frac{x}{\operatorname{arcsinh} x} < (\cosh x)^{1/3}. \tag{1.10}$$

The first inequality holds for $x \neq 0$*, while the second inequality is valid provided* $0 < |x| < 1$*. The exponents* $1/2$ *and* $1/3$ *are the best possible.*

(ii) *For* $0 < |x| < 1$*,*

$$\frac{1}{(\cos x)^{1/3}} < \frac{\arcsin x}{x} < \left(\frac{\operatorname{arctanh} x}{x}\right)^{1/2}. \tag{1.11}$$

The exponents $1/3$ *and* $1/2$ *are the best possible.*

The left side of inequality (1.10), and the right side of inequality (1.11) do appear in another notation (see [6, Corollary 3.2]) as a particular result. By putting in the left side of inequality (1.11), $x = \sin t, t \in (0, \pi/2)$, we get the inequality

$$\frac{\sin t}{t} < \left(\cos(\sin t)\right)^{1/3}, \quad 0 < t < \frac{\pi}{2}.$$

This is a counterpart to

$$\frac{\sin t}{t} > (\cos t)^{1/3}, \quad 0 < t < \frac{\pi}{2}.$$

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The first inequality in (1.3) can be re-written as

$$\arctan x < \frac{x \operatorname{arcsinh} x}{2x - \operatorname{arcsinh} x}, \qquad x \neq 0.$$

Consider the function $A(x)$ for $x > 0$ defined by

$$A(x) = \frac{x \operatorname{arcsinh} x}{2x - \operatorname{arcsinh} x} - \arctan x.$$

Differentiation yields

$$(2x - \operatorname{arcsinh} x)^2 (1 + x^2) A'(x) = B(x),$$

where

$$B(x) = 2x^2 \sqrt{1 + x^2} - (\operatorname{arcsinh} x)^2 (1 + x^2) - (2x - \operatorname{arcsinh} x)^2.$$

By an elementary change of variable

$$x = \sinh t\,, \quad t > 0\,,$$

we obtain

$$
\begin{aligned}
B(x) = C(t) :&= 2\sinh^2 t \cosh t - t^2 \cosh^2 t - (2\sinh t - t)^2 \\
&= \frac{1}{2}\cosh(3t) - \left(\frac{1}{2}t^2 + 2\right)\cosh(2t) + 4t\sinh t - \frac{1}{2}\cosh t - \frac{3}{2}t^2 + 2 \\
&= \sum_{n=3}^{\infty} \frac{2\cdot 9^n - (2n^2 - n + 8)\cdot 4^n + 2(16n-1)}{4\cdot(2n)!} t^{2n} > 0, \quad t > 0.
\end{aligned}
$$

Hence, we have $B(x) > 0$ and $A'(x) > 0$ for $x > 0$. Therefore, the function $A(x)$ is strictly increasing for $x > 0$, and we have

$$
A(x) = \frac{x \operatorname{arcsinh} x}{2x - \operatorname{arcsinh} x} - \arctan x > \lim_{x\to 0^+} A(x) = 0, \qquad x > 0.
$$

Hence, the first inequality in (1.3) holds for $x \neq 0$.

It is well-known that for $a, b > 0$ and $a \neq b$, the function

$$
M(r) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0; \\ \sqrt{ab}, & r = 0 \end{cases}
$$

is strictly increasing for $r \in (-\infty, \infty)$. This implies that the second inequality in (1.3) holds for $x \neq 0$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. The first inequality in (1.10) is obtained by considering the function $f(x)$ defined for $x > 0$ by

$$
f(x) = x \arctan x - (\operatorname{arcsinh} x)^2.
$$

Elementary calculations show that

$$
\begin{aligned}
f'(x) &= \arctan x + \frac{x}{1+x^2} - \frac{2\operatorname{arcsinh} x}{\sqrt{1+x^2}}, \\
\frac{(1+x^2)^{3/2}}{2x} f''(x) &= -\frac{x}{\sqrt{1+x^2}} + \operatorname{arcsinh} x \triangleq g(x), \\
g'(x) &= \frac{x^2}{(1+x^2)^{3/2}} > 0 \qquad (x > 0).
\end{aligned}
$$

Hence, for $x > 0$,

$$
g(x) > g(0) = 0 \Longrightarrow f''(x) > 0 \Longrightarrow f'(x) > f'(0) = 0 \Longrightarrow f(x) > f(0) = 0.
$$

Therefore, the first inequality in (1.10) is valid for $x \neq 0$.

Let

$$
h(x) = (\operatorname{arcsinh} x)^3 \cosh x - x^3, \quad 0 < x < 1.
$$

Motivated by the investigations in [3], we now show $h(x) > 0$ for $0 < x < 1$. Consider the function $H(x)$ by

$$H(x) = \begin{cases} \dfrac{h(x)}{x^7}, & 0 < x \leqslant 1, \\ \mu, & x = 0, \end{cases}$$

where μ is constant determined with limit:

$$\mu = \lim_{x \to 0} \frac{h(x)}{x^7} = \frac{1}{10}.$$

Using Maple we determine Taylor approximation for the function $H(x)$ by the polynomial of the tenth order:

$$P(x) = \frac{1}{10} - \frac{149}{1890}x^2 + \frac{2431}{37800}x^4 - \frac{3683}{69300}x^6 + \frac{152069191}{3405402000}x^8 - \frac{129780571}{3405402000}x^{10},$$

which has a bound of absolute error

$$\varepsilon = 3\ln(1+\sqrt{2}) \cdot \cosh(1) - \frac{117926293}{113513400} = 0.01762256\ldots$$

for values $x \in [0,1]$. It is true that

$$\begin{aligned} &H(x) - (P(x) - \varepsilon) \geqslant 0, \\ &P(x) - \varepsilon = \left(\frac{1}{10} - \varepsilon - \frac{149}{1890}x^2\right) + x^4\left(\frac{2431}{37800} - \frac{3683}{69300}x^2\right) \\ &\qquad\qquad + x^8\left(\frac{152069191}{3405402000} - \frac{129780571}{3405402000}x^2\right) > 0 \end{aligned}$$

for $x \in [0,1]$. Hence, for $x \in [0,1]$ it is true that $H(x) > 0$ and therefore $h(x) > 0$ for $x \in (0,1)$. Therefore, the second inequality in (1.10) is valid for $0 < |x| < 1$.

Write (1.10) as

$$\frac{1}{2} < \frac{\ln(x/\operatorname{arcsinh} x)}{\ln(x/\arctan x)} \quad \text{and} \quad \frac{\ln(x/\operatorname{arcsinh} x)}{\ln(\cosh x)} < \frac{1}{3}.$$

Elementary calculations show that

$$\lim_{x \to 0} \frac{\ln(x/\operatorname{arcsinh} x)}{\ln(x/\arctan x)} = \frac{1}{2} \quad \text{and} \quad \lim_{x \to 0} \frac{\ln(x/\operatorname{arcsinh} x)}{\ln(\cosh x)} = \frac{1}{3}.$$

Hence, the exponents $1/2$ and $1/3$ in (1.10) are the best possible.

Direct computation would yield

$$\begin{aligned} (\arcsin x)^3 &= \left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}\right)^3 \\ &= x^3 + \frac{1}{2}x^5 + \frac{37}{120}x^7 + \frac{3229}{15120}x^9 + \frac{10679}{67200}x^{11} + \ldots \\ &> x^3 + \frac{1}{2}x^5 + \frac{37}{120}x^7 + \frac{3229}{15120}x^9, \quad 0 < x < 1. \end{aligned}$$

Hence, we have for $0 < x < 1$,

$$(\arcsin x)^3 \cos x - x^3 > \left(x^3 + \frac{1}{2}x^5 + \frac{37}{120}x^7 + \frac{3229}{15120}x^9\right)\left(1 - \frac{1}{2}x^2\right) - x^3$$
$$= x^7\left(\frac{7}{120} + \frac{449}{7560}x^2 - \frac{3229}{30240}x^4\right) > 0.$$

Therefore, the first inequality in (1.11) is valid for $0 < |x| < 1$.

The second inequality in (1.11) is obtained by considering the function $I(x)$ defined for $0 < x < 1$ by

$$I(x) = x \operatorname{arctanh} x - (\arcsin x)^2.$$

Elementary calculations show that

$$I'(x) = \operatorname{arctanh} x + \frac{x}{1-x^2} - \frac{2\arcsin x}{\sqrt{1-x^2}},$$
$$\frac{(1-x^2)^{3/2}}{2x} I''(x) = \frac{x}{\sqrt{1-x^2}} - \arcsin x \triangleq G(x),$$
$$G'(x) = \frac{x^2}{(1-x^2)^{3/2}} > 0, \qquad 0 < x < 1.$$

Hence, for $x > 0$,

$$G(x) > G(0) = 0 \Longrightarrow I''(x) > 0 \Longrightarrow I'(x) > I'(0) = 0 \Longrightarrow I(x) > I(0) = 0.$$

Therefore, the second inequality in (1.11) is valid for $0 < |x| < 1$.

Write (1.11) as

$$\frac{1}{3} < \frac{\ln(\arcsin x/x)}{\ln(\sec x)} \quad \text{and} \quad \frac{\ln(\arcsin x/x)}{\ln(\operatorname{arctanh} x/x)} < \frac{1}{2}.$$

Elementary calculations show that

$$\lim_{x\to 0} \frac{\ln(\arcsin x/x)}{\ln(\sec x)} = \frac{1}{3} \quad \text{and} \quad \lim_{x\to 0} \frac{\ln(\arcsin x/x)}{\ln(\operatorname{arctanh} x/x)} = \frac{1}{2}.$$

Hence, the exponents $1/3$ and $1/2$ in (1.11) are the best possible. The proof of Theorem 1.2 is complete. □

REMARK. The first inequality in (1.10) and the second inequality in (1.11) have been obtained earlier (see Theorem 3.1) in [5].

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(Received March 31, 2012)

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454003
Henan Province, China
e-mail: chenchaoping@sohu.com

József Sándor
Babeş-Bolyai University
Department of Mathematics
Str. Kogălniceanu nr. 1
400084 Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro