OPTIMAL INEQUALITIES BETWEEN NEUMAN–SÁNDOR, CENTROIDAL AND HARMONIC MEANS

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Abstract. In this paper, we answer the question: what are the greatest values $\alpha_1$, $\alpha_2$ and the least values $\beta_1$, $\beta_2$, such that the inequalities

$$\alpha_1 T(a, b) + (1 - \alpha_1)H(a, b) < R(a, b) < \beta_1 T(a, b) + (1 - \beta_1)H(a, b)$$

and

$$T^{\alpha_2}(a, b)H^{1-\alpha_2}(a, b) < R(a, b) < T^{\beta_2}(a, b)H^{1-\beta_2}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$? Here, $R(a, b)$, $T(a, b)$ and $H(a, b)$ denote the Neuman-Sándor, centroidal and harmonic means of two positive numbers $a$ and $b$, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $R(a, b)$ is introduced by Neuman and Sándor in [1, 2] as follows:

$$R(a, b) = \frac{a - b}{2\text{arcsinh}(\frac{a-b}{a+b})}. \quad (1.1)$$

The main properties and inequalities for the Neuman-Sándor mean $R(a, b)$ can be found in [1, 2].

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities can be found in [3–32].

Let $T(a, b) = \frac{2}{3} \frac{a^2 + ab + b^2}{a+b}$, $A(a, b) = \frac{a+b}{2}$, $I(a, b) = \frac{1}{e}(\frac{a^a}{b^b})^{\frac{1}{a-b}}$, $P(a, b) = \frac{a-b}{2\text{arcsinh}(\frac{a-b}{a+b})},$

$L(a, b) = \frac{a-b}{\log a - \log b}$, $G(a, b) = \sqrt{ab}$ and $H(a, b) = \frac{2ab}{a+b}$ be the centroidal, arithmetic, seiffert, logarithmic, geometric and harmonic means of two positive numbers $a$ and $b$ with $a \neq b$, respectively. Then

$$H(a, b) < G(a, b) < L(a, b) < P(a, b) < I(a, b) < A(a, b) < R(a, b) < T(a, b). \quad (1.2)$$


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It’s natural to ask: what are the greatest values $\alpha_1, \alpha_2$ and the least values $\beta_1, \beta_2$, such that

$$\alpha_1 T(a, b) + (1 - \alpha_1)H(a, b) < R(a, b) < \beta_1 T(a, b) + (1 - \beta_1)H(a, b)$$

and

$$T^{\alpha_2}(a, b)H^{1 - \alpha_2}(a, b) < R(a, b) < T^{\beta_2}(a, b)H^{1 - \beta_2}(a, b)$$

hold for all $a, b > 0$ with $a \neq b$? The main purpose of this article is to answer this question.

2. Lemmas

In order to establish our results we need one lemma, which we present in this section.

**Lemma 1.** Let $p \in \left\{ \frac{3}{4\log(1+\sqrt{2})}, \frac{7}{8} \right\}$ and $g(t) = 3(-2p^2 + 9p - 6)t^6 + (20p^2 - 30p + 9)t^5 + (-26p^2 + 21p)t^4 + 6(4p^2 - 6p + 3)t^3 + (-26p^2 + 21p)t^2 + (20p^2 - 30p + 9)t - 6p^2 + 27p - 18$, the following statements hold:

1. If $p = \frac{3}{4\log(1+\sqrt{2})} = 0.8509\ldots$, then there exists $\lambda > 1$, such that $g(t) < 0$ for $t \in (1, \lambda)$ and $g(t) > 0$ for $t \in (\lambda, +\infty)$;
2. If $p = \frac{7}{8}$, then $g(t) > 0$ for $t \in (1, +\infty)$.

**Proof.** Let $p \in \left\{ \frac{3}{4\log(1+\sqrt{2})}, \frac{7}{8} \right\}$, then by elementary computations we have

$$g(1) = 0,$$

$$g'(t) = 18(-2p^2 + 9p - 6)t^5 + 5(20p^2 - 30p + 9)t^4 + 4(-26p^2 + 21p)t^3 + 18(4p^2 - 6p + 3)t^2 + 2(-26p^2 + 21p)t + 20p^2 - 30p + 9,$$

$$g'(1) = 0,$$

$$g''(t) = 90(-2p^2 + 9p - 6)t^4 + 20(20p^2 - 30p + 9)t^3 + 12(-26p^2 + 21p)t^2 + 36(4p^2 - 6p + 3)t - 52p^2 + 42p,$$

$$g''(1) = 288 \left( p - \frac{7}{8} \right),$$

$$g'''(t) = 360(-2p^2 + 9p - 6)t^3 + 60(20p^2 - 30p + 9)t^2 + 24(-26p^2 + 21p)t + 36(4p^2 - 6p + 3),$$

$$g'''(1) = 1728 \left( p - \frac{7}{8} \right).$$
\[ g^{(4)}(t) = 1080(-2p^2 + 9p - 6)t^2 + 120(20p^2 - 30p + 9)t + 24(-26p^2 + 21p), \quad (2.8) \]
\[ g^{(4)}(1) = -384p^2 + 6624p - 5400, \quad (2.9) \]
\[ g^{(5)}(t) = 2160(-2p^2 + 9p - 6)t + 120(20p^2 - 30p + 9) \quad (2.10) \]

and
\[ g^{(5)}(1) = -1920p^2 + 15840p - 11880 > 0. \quad (2.11) \]

(1) If \( p = \frac{3}{4\log(1 + \sqrt{2})} \approx 0.8509 \ldots \), then \(-2p^2 + 9p - 6 > 0\), and (2.2), (2.4)–(2.10) together with the expression of \( g(t) \) lead to
\[ \lim_{t \to +\infty} g(t) = +\infty, \quad (2.12) \]
\[ \lim_{t \to +\infty} g'(t) = +\infty, \quad (2.13) \]
\[ g''(1) < 0, \quad \lim_{t \to +\infty} g''(t) = +\infty, \quad (2.14) \]
\[ g'''(1) < 0, \quad \lim_{t \to +\infty} g'''(t) = +\infty, \quad (2.15) \]
\[ g^{(4)}(1) < 0, \quad \lim_{t \to +\infty} g^{(4)}(t) = +\infty \quad (2.16) \]

and \( g^{(5)}(t) \) is strictly increasing in \([1, +\infty)\).

The monotonicity of \( g^{(5)}(t) \) and (2.11) imply that \( g^{(4)}(t) \) is strictly increasing in \([1, +\infty)\).

It follows from the monotonicity of \( g^{(4)}(t) \) and (2.16) that there exists \( \lambda_1 > 1 \), such that \( g'''(t) \) is strictly decreasing in \([1, \lambda_1] \) and strictly increasing in \([\lambda_1, +\infty)\).

From the piecewise monotonicity of \( g'''(t) \) and (2.15) we clearly see that there exists \( \lambda_2 > \lambda_1 > 1 \), such that \( g''(t) \) is strictly decreasing in \([1, \lambda_2] \) and strictly increasing in \([\lambda_2, +\infty)\).

It follows from (2.14) and the piecewise monotonicity of \( g''(t) \) that there exists \( \lambda_3 > \lambda_2 > 1 \), such that \( g'(t) \) is strictly decreasing in \([1, \lambda_3] \) and strictly increasing in \([\lambda_3, +\infty)\).

From the piecewise monotonicity of \( g'(t) \) and (2.3) together with (2.13) we conclusion that there exists \( \lambda_4 > \lambda_3 > 1 \), such that \( g(t) \) is strictly decreasing in \([1, \lambda_4] \) and strictly increasing in \([\lambda_4, +\infty)\).

Therefore, Lemma 2.1 (1) follows from the piecewise monotonicity of \( g(t) \) and (2.1) together with (2.12).

(2) If \( p = \frac{7}{8} \), then \(-2p^2 + 9p - 6 > 0\), and (2.5), (2.7), (2.9) and (2.10) lead to
\[ g''(1) = 0, \quad (2.17) \]
\[ g'''(1) = 0, \quad (2.18) \]
\[ g^{(4)}(1) > 0 \quad (2.19) \]

and \( g^{(5)}(t) \) is strictly increasing in \([1, +\infty)\).

It follows from the monotonicity of \( g^{(5)}(t) \) and (2.11) that \( g^{(4)}(t) \) is strictly increasing in \([1, +\infty)\).

Hence, \( g(t) > 0 \) follows from the monotonicity of \( g^{(4)}(t) \) and (2.17)–(2.19), (2.3) together with (2.1).
3. Main Results

Theorem 3.1. The double inequality

\[ \alpha_1 T(a, b) + (1 - \alpha_1) H(a, b) < R(a, b) < \beta_1 T(a, b) + (1 - \beta_1) H(a, b) \]  \hspace{1cm} (3.1)

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_1 \leq \frac{3}{4 \log(1 + \sqrt{2})} = 0.8509 \ldots \) and \( \beta_1 \geq \frac{7}{8} \).

Proof. Firstly, we prove that

\[ R(a, b) > \alpha_1 T(a, b) + (1 - \alpha_1) H(a, b) \]  \hspace{1cm} (3.2)

and

\[ R(a, b) < \frac{7}{8} T(a, b) + \frac{1}{8} H(a, b) \]  \hspace{1cm} (3.3)

hold for all \( a, b > 0 \) with \( a \neq b \). Here \( \alpha_1 = \frac{3}{4 \log(1 + \sqrt{2})} = 0.8509 \ldots \)

Without loss of generality, we assume that \( t = \frac{a}{b} > 1 \) and \( p \in \{ \frac{3}{4 \log(1 + \sqrt{2})}, \frac{7}{8} \} \), then (1.1) leads to

\[ pT(a, b) + (1 - p)H(a, b) - R(a, b) = \frac{2p a^2 + ab + b^2}{3} \frac{a - b}{a + b} + (1 - p) \frac{2ab}{a + b} - \frac{1}{2 \arcsinh\left(\frac{a - b}{a + b}\right)} \]

\[ = b \left[ \frac{2pt^2 + (6 - 4p)t + 2p}{3(t + 1)} - \frac{t - 1}{2 \arcsinh\left(\frac{t - 1}{t + 1}\right)} \right] . \hspace{1cm} (3.4) \]

Let

\[ f(t) = \log \left[ \frac{2pt^2 + (6 - 4p)t + 2p}{3(t + 1)} \right] - \log \left[ \frac{t - 1}{2 \arcsinh\left(\frac{t - 1}{t + 1}\right)} \right] . \hspace{1cm} (3.5) \]

Then simple computations yield

\[ \lim_{t \to 1^+} f(t) = 0 \]  \hspace{1cm} (3.6)

and

\[ f'(t) = \frac{[3 - 2p]t^2 + 4pt + 3 - 2p]f_1(t)}{[pt^2 + (3 - 2p)t + t^2 - 1] \arcsinh\left(\frac{t - 1}{t + 1}\right)} , \hspace{1cm} (3.7) \]

where

\[ f_1(t) = \frac{2(t - 1)[pt^2 + (3 - 2p)t + p]}{[(3 - 2p)t^2 + 4pt + 3 - 2p] \sqrt{2t^2 + 2}} - \arcsinh\left(\frac{t - 1}{t + 1}\right) , \hspace{1cm} (3.8) \]

\[ f_1(1) = 0 \]  \hspace{1cm} (3.9)

and

\[ f_1'(t) = \frac{4g(t)}{[(3 - 2p)t^2 + 4pt + 3 - 2p] ^2 (2t^2 + 2)^{3/2}(t + 1)} , \hspace{1cm} (3.10) \]
Here, \( g(t) \) is defined as in Lemma 2.1.

1. If \( p = \frac{3}{4 \log(1 + \sqrt{2})} = 0.8509 \ldots \), then (3.5) and (3.8) yield

\[
\lim_{t \to +\infty} f(t) = 0
\]

and

\[
\lim_{t \to +\infty} f_1(t) > 0.
\]

From Lemma 2.1(1) and (3.10) we know that there exists \( \lambda > 1 \), such that \( f_1(t) \)
is strictly decreasing in \([1, \lambda]\) and strictly increasing in \([\lambda, +\infty)\).

It follows from the piecewise monotonicity of \( f_1(t) \) and (3.9) together with (3.12) that there exists \( \xi > \lambda > 1 \), such that \( f(t) \) is strictly decreasing in \([1, \xi]\) and strictly increasing in \([\xi, +\infty)\).

Therefore, inequality (3.2) follows from (3.6), (3.11) and the piecewise monotonicity of \( f(t) \) together with (3.4)–(3.5).

2. If \( p = \frac{7}{8} \), then Lemma 2.1(2) and (3.10) lead to \( f_1(t) \) is strictly increasing in

\([1, +\infty)\).

Hence, inequality (3.3) follows from (3.6), (3.7), (3.9) and the monotonicity of \( f_1(t) \) together with (3.4)–(3.5).

Secondly, we prove that the parameters \( \alpha_1 = \frac{3}{4 \log(1 + \sqrt{2})} = 0.8509 \ldots \) and \( \beta_1 = \frac{7}{8} \) are the best possible parameters such that inequality (3.1) holds for all \( a, b > 0 \) with \( a \neq b \).

For any \( \varepsilon > 0 \), \( \alpha_1 = \frac{3}{4 \log(1 + \sqrt{2})} = 0.8509 \ldots \) and \( x > 0 \), (1.1) leads to

\[
\lim_{x \to +\infty} \frac{(\alpha_1 + \varepsilon)T(x, 1) + (1 - \alpha_1 - \varepsilon)H(x, 1)}{R(x, 1)} = \frac{4}{3}(\alpha_1 + \varepsilon)\log(1 + \sqrt{2}) > 1.
\]

Inequality (3.13) implies that for any \( \varepsilon > 0 \) there exists \( X_1 = X_1(\varepsilon) > 1 \), such that

\( (\alpha_1 + \varepsilon)T(x, 1) + (1 - \alpha_1 - \varepsilon)H(x, 1) > R(x, 1) \) for \( x \in (X_1, +\infty) \).

For any \( \varepsilon > 0 \) and \( x > 0 (x \to 0) \), making use of Taylor expression one has

\[
R(1 + x, 1) - \left[ \left( \frac{7}{8} - \varepsilon \right) T(1 + x, 1) + \left( \frac{1}{8} + \varepsilon \right) H(1 + x, 1) \right]
= \frac{x}{2 \text{arcsinh}(\frac{1}{\sqrt{x+1}})} - \left[ \left( \frac{7}{8} - \varepsilon \right) \frac{1+x+x^2/3}{1+x/2} + \left( \frac{1}{8} + \varepsilon \right) \frac{1+x}{1+x/2} \right]
= \left[ 1 + \frac{1}{2} x + \frac{1}{24} x^2 + o(x^2) \right] - \left[ \left( \frac{7}{8} - \varepsilon \right) \left( 1 + \frac{1}{2} x + \frac{1}{12} x^2 + o(x^2) \right) \right.
\left. + \left( \frac{1}{8} + \varepsilon \right) \left( 1 + \frac{1}{2} x - \frac{1}{4} x^2 + o(x^2) \right) \right]
= \frac{\varepsilon}{3} x^2 + o(x^2).
\]

Equation (3.14) implies that for any \( \varepsilon > 0 \) there exists \( \delta_1 = \delta_1(\varepsilon) > 0 \), such that

\( R(1 + x, 1) > (\frac{7}{8} - \varepsilon)T(1 + x, 1) + (\frac{1}{8} + \varepsilon)H(1 + x, 1) \) for \( x \in (0, \delta_1) \).
The double inequality
\[ T^{\alpha_2}(a, b)H^{1-\alpha_2}(a, b) < R(a, b) < T^{\beta_2}(a, b)H^{1-\beta_2}(a, b) \] (3.15)
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_2 \leq \frac{7}{8}\) and \(\beta_2 \geq 1\).

**Proof.** Firstly, we prove that
\[ R(a, b) > T^{7/8}(a, b)H^{1/8}(a, b) \] (3.16)
and
\[ R(a, b) < T(a, b) \] (3.17)
hold for all \(a, b > 0\) with \(a \neq b\).

From (1.2) we clearly see that (3.17) is true. So, we only need to prove inequality (3.16). Without loss of generality, we assume that \(t = \frac{a}{b} > 1\), then (1.1) leads to
\[
R(a, b) - T^{7/8}(a, b)H^{1/8}(a, b) = b \left[ \frac{t - 1}{2\arcsinh\left(\frac{t-1}{t+1}\right)} - \left( \frac{2t^2 + t + 1}{3t + 1} \right)^{7/8} \cdot \left( \frac{2t}{t+1} \right)^{1/8} \right].
\] (3.18)
Let
\[ h(t) = \log \left[ \frac{t - 1}{2\arcsinh\left(\frac{t-1}{t+1}\right)} \right] - \log \left[ \left( \frac{2t^2 + t + 1}{3t + 1} \right)^{7/8} \cdot \left( \frac{2t}{t+1} \right)^{1/8} \right], \] (3.19)
then by simple computations we get
\[ \lim_{t \to 1^+} h(t) = 0 \] (3.20)
and
\[ h'(t) = \frac{(r^4 + 8r^3 + 30r^2 + 8r + 1)h_1(t)}{8t(t+1)(r^3 - 1)\arcsinh\left(\frac{t-1}{t+1}\right)}, \] (3.21)
where
\[ h_1(t) = \arcsinh\left(\frac{t - 1}{t+1}\right) - \frac{16t(r^3 - 1)}{(r^4 + 8r^3 + 30r^2 + 8r + 1)\sqrt{2r^2 + 2}}, \] (3.22)
h_1(1) = 0 (3.23)
and
\[ h'_1(t) = \frac{J(t)}{(r^4 + 8r^3 + 30r^2 + 8r + 1)^2(t+1)(2r^2 + 2)^3/2}. \] (3.24)
Here,
\[ J(x) = 8(t - 1)(r^2 + 1)(17r^9 + 113r^8 + 302r^7 + 646r^6 + 880r^5 + 184r^4 + 114r^3 + 90r^2 - 33t - 9) > 0 \] (3.25)
for \( t > 1 \).

Therefore, \( h(t) > 0 \) follows from (3.25), (3.24), (3.23), (3.21) and (3.20). Thus inequality (3.16) follows from (3.18) and (3.19).

Secondly, we prove that the parameters \( \alpha_2 = 7/8 \) and \( \beta_2 = 1 \) are the best possible parameters such that inequality (3.15) holds for all \( a, b > 0 \) with \( a \neq b \).

For any \( \varepsilon > 0 \) and \( x > 0(x \to 0) \), from (1.1) and Taylor expression we have

\[
T^{7/8+\varepsilon}(1 + x, 1)H^{1/8-\varepsilon}(1 + x, 1) - R(1 + x, 1) = \frac{2}{3} \cdot \frac{x^2 + 3x + 3}{x + 2} x^{7/8+\varepsilon} \cdot \left( \frac{2x + 2}{x + 2} \right)^{1/8-\varepsilon} - \frac{x}{2\arcsinh(x)}
\]

\[
= \left( 1 + \frac{1}{2} x + \frac{1}{12} x^2 + o(x^2) \right)^{7/8+\varepsilon} \cdot \left( 1 + \frac{1}{2} x - \frac{1}{4} x^2 + o(x^2) \right)^{1/8-\varepsilon}
\]

\[
= \frac{\varepsilon}{3} x^2 + o(x^2).
\]  

Equation (3.26) implies that for any \( \varepsilon > 0 \) there exists \( \delta_2 = \delta_2(\varepsilon) > 0 \), such that \( T^{7/8+\varepsilon}(1 + x, 1)H^{1/8-\varepsilon}(1 + x, 1) > R(1 + x, 1) \) for \( x \in (0, \delta_2) \).

For any \( \varepsilon > 0 \) and \( x > 0 \), one has

\[
\lim_{x \to +\infty} \frac{R(x, 1)}{T^{1-\varepsilon}(x, 1)H^{\varepsilon}(x, 1)} = \lim_{x \to +\infty} \frac{x^\varepsilon}{\frac{4}{3} \log(1 + \sqrt{2}) \cdot 3^\varepsilon} > 1.
\]  

Inequality (3.27) implies that for any \( \varepsilon > 0 \) there exists \( X_2 = X_2(\varepsilon) > 1 \), such that \( R(x, 1) > T^{1-\varepsilon}(x, 1)H^{\varepsilon}(x, 1) \) for \( x \in (X_2, +\infty) \).

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