Journal of
Mathematical
Inequalities
Volume 7, Number 4 (2013), 593–600 doi:10.7153/jmi-07-56

OPTIMAL INEQUALITIES BETWEEN NEUMAN–SÁNDOR, CENTROIDAL AND HARMONIC MEANS

WEIFENG XIA AND YUMING CHU

(*Communicated by E. Neuman*)

Abstract. In this paper, we answer the question: what are the greatest values α_1, α_2 and the least values β_1, β_2, such that the inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < R(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b)$$

and

$$T^{\alpha_2}(a,b)H^{1-\alpha_2}(a,b) < R(a,b) < T^{\beta_2}(a,b)H^{1-\beta_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$? Here, $R(a,b)$, $T(a,b)$ and $H(a,b)$ denote the Neuman-Sándor, centroidal and harmonic means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a \neq b$, the Neuman-Sándor mean $R(a,b)$ is introduced by Neuman and Sándor in [1, 2] as follows:

$$R(a,b) = \frac{a-b}{2\operatorname{arcsinh}(\frac{a-b}{a+b})}. \tag{1.1}$$

The main properties and inequalities for the Neuman-Sándor mean $R(a,b)$ can be found in [1, 2].

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities can be found in [3–32].

Let $T(a,b) = \frac{2}{3}\frac{a^2+ab+b^2}{a+b}$, $A(a,b) = \frac{a+b}{2}$, $I(a,b) = \frac{1}{e}(\frac{a^a}{b^b})^{\frac{1}{a-b}}$, $P(a,b) = \frac{a-b}{2\arcsin(\frac{a-b}{a+b})}$, $L(a,b) = \frac{a-b}{\log a - \log b}$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = \frac{2ab}{a+b}$ be the centroidal, arithmetic, identic, seiffert, logarithmic, geometric and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) < R(a,b) < T(a,b). \tag{1.2}$$

Mathematics subject classification (2010): 26D15, 26E60.
Keywords and phrases: Neuman-Sándor mean, centroidal mean, harmonic mean, inequalities.
This work was supported by the Natural Science Foundation of China (No. 11071069), Science Foundation of Zhejiang Province (No. Y13A010028) and Innovation Team Foundation of the Department of Education of Zhejinag Province (No. T200924).

It's natural to ask: what are the greatest values α_1, α_2 and the least values β_1, β_2, such that

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < R(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b)$$

and

$$T^{\alpha_2}(a,b)H^{1-\alpha_2}(a,b) < R(a,b) < T^{\beta_2}(a,b)H^{1-\beta_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$? The main purpose of this article is to answer this question.

2. Lemmas

In order to establish our results we need one lemma, which we present in this section.

LEMMA 1. *Let* $p \in \{\frac{3}{4\log(1+\sqrt{2})}, \frac{7}{8}\}$ *and* $g(t) = 3(-2p^2+9p-6)t^6 + (20p^2 - 30p+9)t^5 + (-26p^2+21p)t^4 + 6(4p^2-6p+3)t^3 + (-26p^2+21p)t^2 + (20p^2-30p+9)t - 6p^2 + 27p - 18$*, the following statements hold:*

(1) If $p = \frac{3}{4\log(1+\sqrt{2})} = 0.8509\ldots$*, then there exists* $\lambda > 1$*, such that* $g(t) < 0$ *for* $t \in (1,\lambda)$ *and* $g(t) > 0$ *for* $t \in (\lambda, +\infty)$*;*

(2) If $p = \frac{7}{8}$*, then* $g(t) > 0$ *for* $t \in (1,+\infty)$*.*

Proof. Let $p \in \{\frac{3}{4\log(1+\sqrt{2})}, \frac{7}{8}\}$, then by elementary computations we have

$$g(1) = 0, \tag{2.1}$$

$$\begin{aligned} g'(t) =& 18(-2p^2+9p-6)t^5 + 5(20p^2-30p+9)t^4 + 4(-26p^2+21p)t^3 \\ &+ 18(4p^2-6p+3)t^2 + 2(-26p^2+21p)t + 20p^2 - 30p + 9, \end{aligned} \tag{2.2}$$

$$g'(1) = 0, \tag{2.3}$$

$$\begin{aligned} g''(t) =& 90(-2p^2+9p-6)t^4 + 20(20p^2-30p+9)t^3 + 12(-26p^2+21p)t^2 \\ &+ 36(4p^2-6p+3)t - 52p^2 + 42p, \end{aligned} \tag{2.4}$$

$$g''(1) = 288\left(p - \frac{7}{8}\right), \tag{2.5}$$

$$\begin{aligned} g'''(t) =& 360(-2p^2+9p-6)t^3 + 60(20p^2-30p+9)t^2 + 24(-26p^2+21p)t \\ &+ 36(4p^2-6p+3), \end{aligned} \tag{2.6}$$

$$g'''(1) = 1728\left(p - \frac{7}{8}\right), \tag{2.7}$$

$$g^{(4)}(t) = 1080(-2p^2+9p-6)t^2+120(20p^2-30p+9)t+24(-26p^2+21p), \quad (2.8)$$

$$g^{(4)}(1) = -384p^2+6624p-5400, \quad (2.9)$$

$$g^{(5)}(t) = 2160(-2p^2+9p-6)t+120(20p^2-30p+9) \quad (2.10)$$

and

$$g^{(5)}(1) = -1920p^2+15840p-11880 > 0. \quad (2.11)$$

(1) If $p = \frac{3}{4\log(1+\sqrt{2})} = 0.8509\ldots$, then $-2p^2+9p-6>0$, and (2.2), (2.4)–(2.10) together with the expression of $g(t)$ lead to

$$\lim_{t\to+\infty} g(t) = +\infty, \quad (2.12)$$

$$\lim_{t\to+\infty} g'(t) = +\infty, \quad (2.13)$$

$$g''(1) < 0, \qquad \lim_{t\to+\infty} g''(t) = +\infty, \quad (2.14)$$

$$g'''(1) < 0, \qquad \lim_{t\to+\infty} g'''(t) = +\infty, \quad (2.15)$$

$$g^{(4)}(1) < 0, \qquad \lim_{t\to+\infty} g^{(4)}(t) = +\infty \quad (2.16)$$

and $g^{(5)}(t)$ is strictly increasing in $[1,+\infty)$.

The monotonicity of $g^{(5)}(t)$ and (2.11) imply that $g^{(4)}(t)$ is strictly increasing in $[1,+\infty)$.

It follows from the monotonicity of $g^{(4)}(t)$ and (2.16) that there exists $\lambda_1 > 1$, such that $g'''(t)$ is strictly decreasing in $[1,\lambda_1]$ and strictly increasing in $[\lambda_1,+\infty)$.

From the piecewise monotonicity of $g'''(t)$ and (2.15) we clearly see that there exists $\lambda_2 > \lambda_1 > 1$, such that $g''(t)$ is strictly decreasing in $[1,\lambda_2]$ and strictly increasing in $[\lambda_2,+\infty)$.

It follows from (2.14) and the piecewise monotonicity of $g''(t)$ that there exists $\lambda_3 > \lambda_2 > 1$, such that $g'(t)$ is strictly decreasing in $[1,\lambda_3]$ and strictly increasing in $[\lambda_3,+\infty)$.

From the piecewise monotonicity of $g'(t)$ and (2.3) together with (2.13) we conclusion that there exists $\lambda_4 > \lambda_3 > 1$, such that $g(t)$ is strictly decreasing in $[1,\lambda_4]$ and strictly increasing in $[\lambda_4,+\infty)$.

Therefore, Lemma 2.1 (1) follows from the piecewise monotonicity of $g(t)$ and (2.1) together with (2.12).

(2) If $p = \frac{7}{8}$, then $-2p^2+9p-6>0$, and (2.5), (2.7), (2.9) and (2.10) lead to

$$g''(1) = 0, \quad (2.17)$$

$$g'''(1) = 0, \quad (2.18)$$

$$g^{(4)}(1) > 0 \quad (2.19)$$

and $g^{(5)}(t)$ is strictly increasing in $[1,+\infty)$.

It follows from the monotonicity of $g^{(5)}(t)$ and (2.11) that $g^{(4)}(t)$ is strictly increasing in $[1,+\infty)$.

Hence, $g(t) > 0$ follows from the monotonicity of $g^{(4)}(t)$ and (2.17)–(2.19), (2.3) together with (2.1).

3. Main Results

THEOREM 3.1. *The double inequality*

$$\alpha_1 T(a,b)+(1-\alpha_1)H(a,b) < R(a,b) < \beta_1 T(a,b)+(1-\beta_1)H(a,b) \tag{3.1}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1 \leqslant \frac{3}{4\log(1+\sqrt{2})} = 0.8509\ldots$ *and* $\beta_1 \geqslant \frac{7}{8}$.

Proof. Firstly, we prove that

$$R(a,b) > \alpha_1 T(a,b)+(1-\alpha_1)H(a,b) \tag{3.2}$$

and

$$R(a,b) < \frac{7}{8}T(a,b)+\frac{1}{8}H(a,b) \tag{3.3}$$

hold for all $a,b>0$ with $a\neq b$. Here $\alpha_1 = \frac{3}{4\log(1+\sqrt{2})} = 0.8509\ldots$

Without loss of generality, we assume that $t=\frac{a}{b}>1$ and $p\in\{\frac{3}{4\log(1+\sqrt{2})},\frac{7}{8}\}$, then (1.1) leads to

$$\begin{aligned}
&pT(a,b)+(1-p)H(a,b)-R(a,b)\\
&=\frac{2p}{3}\frac{a^2+ab+b^2}{a+b}+(1-p)\frac{2ab}{a+b}-\frac{a-b}{2\operatorname{arcsinh}(\frac{a-b}{a+b})}\\
&=b\left[\frac{2pt^2+(6-4p)t+2p}{3(t+1)}-\frac{t-1}{2\operatorname{arcsinh}(\frac{t-1}{t+1})}\right].
\end{aligned} \tag{3.4}$$

Let

$$f(t)=\log\left[\frac{2pt^2+(6-4p)t+2p}{3(t+1)}\right]-\log\left[\frac{t-1}{2\operatorname{arcsinh}(\frac{t-1}{t+1})}\right]. \tag{3.5}$$

Then simple computations yield

$$\lim_{t\to 1^+} f(t)=0 \tag{3.6}$$

and

$$f'(t)=\frac{[(3-2p)t^2+4pt+3-2p]f_1(t)}{[pt^2+(3-2p)t+p](t^2-1)\operatorname{arcsinh}(\frac{t-1}{t+1})}, \tag{3.7}$$

where

$$f_1(t)=\frac{2(t-1)[pt^2+(3-2p)t+p]}{[(3-2p)t^2+4pt+3-2p]\sqrt{2t^2+2}}-\operatorname{arcsinh}\left(\frac{t-1}{t+1}\right), \tag{3.8}$$

$$f_1(1)=0 \tag{3.9}$$

and

$$f_1'(t)=\frac{4g(t)}{[(3-2p)t^2+4pt+3-2p]^2(2t^2+2)^{3/2}(t+1)}. \tag{3.10}$$

Here, $g(t)$ is defined as in Lemma 2.1.

(1) If $p=\frac{3}{4\log(1+\sqrt{2})}=0.8509\ldots$, then (3.5) and (3.8) yield

$$\lim_{t\to+\infty} f(t)=0 \tag{3.11}$$

and

$$\lim_{t\to+\infty} f_1(t)>0. \tag{3.12}$$

From Lemma 2.1(1) and (3.10) we know that there exists $\lambda>1$, such that $f_1(t)$ is strictly decreasing in $[1,\lambda]$ and strictly increasing in $[\lambda,+\infty)$.

It follows from the piecewise monotonicity of $f_1(t)$ and (3.9) together with (3.12) that there exists $\xi>\lambda>1$, such that $f(t)$ is strictly decreasing in $[1,\xi]$ and strictly increasing in $[\xi,+\infty)$.

Therefore, inequality (3.2) follows from (3.6), (3.11) and the piecewise monotonicity of $f(t)$ together with (3.4)–(3.5).

(2) If $p=\frac{7}{8}$, then Lemma 2.1(2) and (3.10) lead to $f_1(t)$ is strictly increasing in $[1,+\infty)$.

Hence, inequality (3.3) follows from (3.6), (3.7), (3.9) and the monotonicity of $f_1(t)$ together with (3.4)–(3.5).

Secondly, we prove that the parameters $\alpha_1=\frac{3}{4\log(1+\sqrt{2})}=0.8509\ldots$ and $\beta_1=\frac{7}{8}$ are the best possible parameters such that inequality (3.1) holds for all $a,b>0$ with $a\neq b$.

For any $\varepsilon>0$, $\alpha_1=\frac{3}{4\log(1+\sqrt{2})}=0.8509\ldots$ and $x>0$, (1.1) leads to

$$\begin{aligned}&\lim_{x\to+\infty}\frac{(\alpha_1+\varepsilon)T(x,1)+(1-\alpha_1-\varepsilon)H(x,1)}{R(x,1)}\\&=\frac{4}{3}(\alpha_1+\varepsilon)\log(1+\sqrt{2})>1.\end{aligned} \tag{3.13}$$

Inequality (3.13) implies that for any $\varepsilon>0$ there exists $X_1=X_1(\varepsilon)>1$, such that $(\alpha_1+\varepsilon)T(x,1)+(1-\alpha_1-\varepsilon)H(x,1)>R(x,1)$ for $x\in(X_1,+\infty)$.

For any $\varepsilon>0$ and $x>0(x\to 0)$, making use of Taylor expression one has

$$\begin{aligned}&R(1+x,1)-\left[\left(\frac{7}{8}-\varepsilon\right)T(1+x,1)+\left(\frac{1}{8}+\varepsilon\right)H(1+x,1)\right]\\&=\frac{x}{2\operatorname{arcsinh}(\frac{x}{x+2})}-\left[\left(\frac{7}{8}-\varepsilon\right)\frac{1+x+x^2/3}{1+x/2}+\left(\frac{1}{8}+\varepsilon\right)\frac{1+x}{1+x/2}\right]\\&=\left[1+\frac{1}{2}x+\frac{1}{24}x^2+o(x^2)\right]-\left[\left(\frac{7}{8}-\varepsilon\right)\left(1+\frac{1}{2}x+\frac{1}{12}x^2+o(x^2)\right)\right.\\&\quad\left.+\left(\frac{1}{8}+\varepsilon\right)\left(1+\frac{1}{2}x-\frac{1}{4}x^2+o(x^2)\right)\right]\\&=\frac{\varepsilon}{3}x^2+o(x^2).\end{aligned} \tag{3.14}$$

Equation (3.14) implies that for any $\varepsilon>0$ there exists $\delta_1=\delta_1(\varepsilon)>0$, such that $R(1+x,1)>(\frac{7}{8}-\varepsilon)T(1+x,1)+(\frac{1}{8}+\varepsilon)H(1+x,1)$ for $x\in(0,\delta_1)$.

THEOREM 3.2. *The double inequality*

$$T^{\alpha_2}(a,b)H^{1-\alpha_2}(a,b) < R(a,b) < T^{\beta_2}(a,b)H^{1-\beta_2}(a,b) \tag{3.15}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_2 \leqslant \frac{7}{8}$ *and* $\beta_2 \geqslant 1$.

Proof. Firstly, we prove that

$$R(a,b) > T^{7/8}(a,b)H^{1/8}(a,b) \tag{3.16}$$

and

$$R(a,b) < T(a,b) \tag{3.17}$$

hold for all $a,b>0$ with $a \neq b$.

From (1.2) we clearly see that (3.17) is true. So, we only need to prove inequality (3.16). Without loss of generality, we assume that $t = \frac{a}{b} > 1$, then (1.1) leads to

$$\begin{aligned} &R(a,b) - T^{7/8}(a,b)H^{1/8}(a,b) \\ &= b\left[\frac{t-1}{2\operatorname{arcsinh}(\frac{t-1}{t+1})} - \left(\frac{2}{3}\frac{t^2+t+1}{t+1}\right)^{7/8} \cdot \left(\frac{2t}{t+1}\right)^{1/8}\right]. \end{aligned} \tag{3.18}$$

Let

$$h(t) = \log\left[\frac{t-1}{2\operatorname{arcsinh}(\frac{t-1}{t+1})}\right] - \log\left[\left(\frac{2}{3}\frac{t^2+t+1}{t+1}\right)^{7/8} \cdot \left(\frac{2t}{t+1}\right)^{1/8}\right], \tag{3.19}$$

then by simple computations we get

$$\lim_{t \to 1^+} h(t) = 0 \tag{3.20}$$

and

$$h'(t) = \frac{(t^4+8t^3+30t^2+8t+1)h_1(t)}{8t(t+1)(t^3-1)\operatorname{arcsinh}(\frac{t-1}{t+1})}, \tag{3.21}$$

where

$$h_1(t) = \operatorname{arcsinh}\left(\frac{t-1}{t+1}\right) - \frac{16t(t^3-1)}{(t^4+8t^3+30t^2+8t+1)\sqrt{2t^2+2}}, \tag{3.22}$$

$$h_1(1) = 0 \tag{3.23}$$

and

$$h_1'(t) = \frac{J(t)}{(t^4+8t^3+30t^2+8t+1)^2(t+1)(2t^2+2)^{3/2}}. \tag{3.24}$$

Here,

$$\begin{aligned} J(x) =& 8(t-1)(t^2+1)(17t^9+113t^8+302t^7+646t^6+880t^5 \\ &+184t^4+114t^3+90t^2-33t-9) > 0 \end{aligned} \tag{3.25}$$

for $t > 1$.

Therefore, $h(t) > 0$ follows from (3.25), (3.24), (3.23), (3.21) and (3.20). Thus inequality (3.16) follows from (3.18) and (3.19).

Secondly, we prove that the parameters $\alpha_2 = 7/8$ and $\beta_2 = 1$ are the best possible parameters such that inequality (3.15) holds for all $a, b > 0$ with $a \neq b$.

For any $\varepsilon > 0$ and $x > 0 (x \to 0)$, from (1.1) and Taylor expression we have

$$\begin{aligned}
&T^{7/8+\varepsilon}(1+x,1)H^{1/8-\varepsilon}(1+x,1) - R(1+x,1) \\
=&\left(\frac{2}{3}\cdot\frac{x^2+3x+3}{x+2}\right)^{7/8+\varepsilon}\cdot\left(\frac{2x+2}{x+2}\right)^{1/8-\varepsilon} - \frac{x}{2\operatorname{arcsinh}(\frac{x}{x+2})} \\
=&\left(1+\frac{1}{2}x+\frac{1}{12}x^2+o(x^2)\right)^{7/8+\varepsilon}\cdot\left(1+\frac{1}{2}x-\frac{1}{4}x^2+o(x^2)\right)^{1/8-\varepsilon} \\
&-\left(1+\frac{1}{2}x+\frac{1}{24}x^2+o(x^2)\right) \\
=&\frac{\varepsilon}{3}x^2+o(x^2).
\end{aligned} \tag{3.26}$$

Equation (3.26) implies that for any $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$, such that $T^{\frac{7}{8}+\varepsilon}(1+x,1)H^{\frac{1}{8}-\varepsilon}(1+x,1) > R(1+x,1)$ for $x \in (0, \delta_2)$.

For any $\varepsilon > 0$ and $x > 0$, one has

$$\lim_{x\to+\infty}\frac{R(x,1)}{T^{1-\varepsilon}(x,1)H^{\varepsilon}(x,1)} = \lim_{x\to+\infty}\frac{x^{\varepsilon}}{\frac{4}{3}\log(1+\sqrt{2})\cdot 3^{\varepsilon}} > 1. \tag{3.27}$$

Inequality (3.27) implies that for any $\varepsilon > 0$ there exists $X_2 = X_2(\varepsilon) > 1$, such that $R(x,1) > T^{1-\varepsilon}(x,1)H^{\varepsilon}(x,1)$ for $x \in (X_2, +\infty)$.

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(Received October 31, 2012)

Weifeng Xia
School of Teacher Education
Huzhou Teachers College
Huzhou 313000, Zhejiang, China,
e-mail: xwf212@163.com, xwf212@hutc.zj.cn

Yuming Chu
School of Mathematics and Computation Sciences
Hunan City University
Yiyang 413000, China
e-mail: chuyuming@hutc.zj.cn