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REFINEMENTS OF A TWO–SIDED INEQUALITY FOR TRIGONOMETRIC FUNCTIONS

ZHEN-HANG YANG

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Abstract. In this paper, we prove that for $x \in (0,\pi/2)$

$$(\cos p_1 x)^{1/(3p_1^2)} < \frac{\sin x}{x} < (\cos p_0 x)^{1/(3p_0^2)} < \cdots < e^{-x^2/6} < \frac{2+\cos x}{3}$$

with the best constants $p_1 = 0.45346...$ and $p_0 = 1/\sqrt{5}$, and the function $p \mapsto (\cos px)^{1/(3p^2)}$ is decreasing on $(0,1)$. Our results greatly refine Adamović-Mitrinović's and Cusa's inequality. As applications, some precise estimations for certain special functions and constants are presented.

1. Introduction

In the recent past, the following two-side inequality

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3} \quad \left(0 < x < \frac{\pi}{2}\right) \tag{1.1}$$

has attracted the attention of many scholars, where the left inequality was obtained by Adamović and Mitrinović (see [12, 2, p. 238]), while the right one is due to Cusa and Huygens (see, e.g., [7]) and it is now known as *Cusa's inequality* [5], [13], [15], [20], [25].

A nice refinement of the inequalities (1.1) appeared in [12, 3.4.6]. For convenience, we record it as follows.

THEOREM A. *For* $x \in (0,\pi/2)$,

$$\cos px \leqslant \frac{\sin x}{x} \leqslant \cos qx \tag{1.2}$$

with the best possible constants

$$p = \frac{1}{\sqrt{3}} \quad \textit{and} \quad q = \frac{2}{\pi}\arccos\frac{2}{\pi}.$$

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Moreover, we have

$$\cos x \leqslant \frac{\cos x}{1-x^2/3} \leqslant (\cos x)^{1/3} \leqslant \cos\frac{x}{\sqrt{3}} \leqslant \frac{\sin x}{x} \leqslant \cos qx \leqslant \cos\frac{x}{2} \leqslant 1. \tag{1.3}$$

Recently, Klén et al. [9, Theorem 2.4] showed that the function $p \mapsto (\cos px)^{1/p}$ is decreasing on $(0,1)$ and improved Cusa's inequality (the right one in (1.1)), which is stated as follows.

THEOREM B. *For* $x \in \left(-\sqrt{27/5}, \sqrt{27/5}\right)$

$$\cos^2\frac{x}{2} \leqslant \frac{\sin x}{x} \leqslant \cos^3\frac{x}{3} \leqslant \frac{2+\cos x}{3}. \tag{1.4}$$

An improvement for the first inequality in (1.4) is due to Neuman [15]:

$$\cos^{4/3}\frac{x}{2} = \left(\frac{1+\cos x}{2}\right)^{2/3} < \frac{\sin x}{x},\ x \in \left(0, \tfrac{\pi}{2}\right). \tag{1.5}$$

Lv et al. [10] showed that for $x \in (0, \pi/2)$ inequalities

$$\left(\cos\frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \left(\cos\frac{x}{2}\right)^{\theta} \tag{1.6}$$

hold, where $\theta = 2(\ln\pi - \ln 2)/\ln 2 = 1.3030...$ and $4/3$ are the best possible constants.

Very recently, by using inequalities involving Schwab-Borchardt mean, Neuman obtained an inportant and interesting refinement of Adamović-Mitrinović's one (the left one of (1.1)) in [14, Theorem 1] (also see [16]), that is, the following chain of inequalities

$$(\cos x)^{1/3} < \left(\cos x\frac{\sin x}{x}\right)^{1/4} < \left(\frac{\sin x}{\operatorname{arctanh}\sin x}\right)^{1/2} < \left(\frac{\cos x + (\sin x)/x}{2}\right)^{1/2}$$
$$< \left(\frac{1+2\cos x}{3}\right)^{1/2} < \left(\frac{1+\cos x}{2}\right)^{2/3} < \frac{\sin x}{x} \tag{1.7}$$

holds for $x \in (0, \pi/2)$.

Other results involving inequalities (1.1) can be found in [5], [8], [13], [15], [16], [18], [20], [24], [25], and related references therein.

The aim of this paper is to give sharp bounds

$$U_p(x) = (\cos px)^{1/(3p^2)} \text{ if } p \in (0,1] \text{ and } U_0(x) = \lim_{p\to 0^+} U_p(x) = e^{-x^2/6} \tag{1.8}$$

$(x \in (0, \pi/2))$ for $(\sin x)/x$ to refine two-side inequality (1.1), that is, for $x \in (0, \pi/2)$, to determine the best $p, q \in [0,1)$ such that

$$(\cos x)^{1/3} < U_p(x) < \frac{\sin x}{x} < U_q(x) < \frac{2+\cos x}{3} \tag{1.9}$$

hold.

The paper is organized as follows. Some useful lemmas are given in Section 2. In Section 3, the sharp bounds $U_p(x)$ for $(\sin x)/x$ and their relative error estimations are established. In the last section, some precise estimations for certain special functions and constants are presented.

2. Lemmas

LEMMA 1. ([21], [1]) *Let $f,g:[a,b]\to\mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g'\neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then so are the functions*

$$x\mapsto \frac{f(x)-f(b)}{g(x)-g(b)} \quad \textit{and} \quad x\mapsto \frac{f(x)-f(a)}{g(x)-g(a)}.$$

LEMMA 2. ([2]) *Let a_n and b_n $(n=0,1,2,\ldots)$ be real numbers and let the power series $A(t)=\sum_{n=1}^{\infty}a_nt^n$ and $B(t)=\sum_{n=1}^{\infty}b_nt^n$ be convergent for $|t|<R$. If $b_n>0$ for $n=0,1,2,\ldots$, and a_n/b_n is strictly increasing (or decreasing) for $n=0,1,2,\ldots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0,R)$.*

LEMMA 3. ([6, pp. 227–229]) *We have*

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1},\ \ |x|<\pi, \tag{2.1}$$

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1},\ \ |x|<\pi/2, \tag{2.2}$$

$$\frac{1}{\sin^2 x}=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2},\ \ |x|<\pi, \tag{2.3}$$

where B_n is the Bernoulli number.

LEMMA 4. *For $p\in(0,1]$, let the function F_p be defined on $(0,\pi/2)$ by*

$$F_p(x)=\frac{\ln\frac{\sin x}{x}}{\ln(\cos px)}. \tag{2.4}$$

Then F_p is strictly increasing on $(0,\pi/2)$ for $p\in(0,1/\sqrt{5}]$ and decreasing on $(0,\pi/2)$ for $p\in[1/2,1]$. Consequently, we have

$$\frac{\ln 2-\ln\pi}{\ln(\cos(\pi p/2))}\ln(\cos px)<\ln\frac{\sin x}{x}<\frac{1}{3p^2}\ln(\cos px) \tag{2.5}$$

for $p\in(0,1/\sqrt{5}]$. The inequalities (2.5) are reversed for $p\in[1/2,1]$.

Proof. For $x \in (0, \pi/2)$, we define $f(x) = \ln \frac{\sin x}{x}$ and $g(x) = \ln(\cos px)$, where $p \in (0, 1]$. Note that $f(0^+) = g(0^+) = 0$, then $F_p(x)$ can be written as

$$F_p(x) = \frac{f(x) - f(0^+)}{g(x) - g(0^+)}.$$

Differentiation and using (2.1) and (2.2) yield

$$\frac{f'(x)}{g'(x)} = \frac{p\left(\frac{1}{x} - \cot x\right)}{\tan px} = \frac{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}}{\sum_{n=1}^{\infty} \frac{2^{2n}-1}{(2n)!} p^{2n-2} 2^{2n} |B_{2n}| x^{2n-1}} := \frac{\sum_{n=1}^{\infty} a_n x^{2n-1}}{\sum_{n=1}^{\infty} b_n x^{2n-1}},$$

where

$$a_n = \frac{2^{2n}}{(2n)!} |B_{2n}|, \quad b_n = \frac{2^{2n}-1}{(2n)!} p^{2n-2} 2^{2n} |B_{2n}|.$$

Clearly, if the monotonicity of a_n/b_n is proved, then by Lemma 2 it is deduced the monotonicity of f'/g', and then the monotonicity of the function F_p easily follows from Lemma 1. For this purpose, since a_n, $b_n > 0$ for $n \in \mathbb{N}$, we only need to show that b_n/a_n is decreasing if $0 < p \leqslant 1/\sqrt{5}$ and increasing if $1/2 \leqslant p \leqslant 1$. Indeed, elementary computation yields

$$\begin{aligned} \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} &= \left(2^{2n+2} - 1\right) p^{2n} - \left(2^{2n} - 1\right) p^{2n-2} \\ &= \left(4^{n+1} - 1\right) p^{2n-2} \left(p^2 - \frac{1}{4} + \frac{3}{4\left(4^{n+1} - 1\right)}\right). \end{aligned}$$

It is easy to obtain that for $n \in \mathbb{N}$

$$\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} \begin{cases} \leqslant 0 \text{ if } p^2 < \frac{1}{5}, \\ > 0 \text{ if } p^2 \geqslant \frac{1}{4}, \end{cases}$$

which proves the monotonicity of a_n/b_n.

By the monotonicity of the function F_p and notice that

$$F_p\left(0^+\right) = \frac{1}{3p^2} \quad \text{and} \quad F_p\left(\frac{\pi}{2}^-\right) = \frac{\ln 2 - \ln \pi}{\ln\left(\cos\left(\pi p/2\right)\right)},$$

the inequalities (2.5) follow immediately. □

REMARK 1. Lemma 4 contains many useful and interesting inequalities for trigonometric functions. For example, put $p = 1/\sqrt{3}$, $\left(2 \arccos\left(2/\pi\right)\right)/\pi \in [1/2, 1]$ in (2.5) yield the second and first inequality of (1.2), respectively; put $p = 1/2 \in [1/2, 1]$ leads to (1.6). Similarly, by virtue of Lemma 4 we will easily prove our most main results in the sequel.

LEMMA 5. *For* $x \in [0, \pi/2]$, *the function* $p \mapsto U_p(x)$ *defined by* (1.8) *is decreasing on* $(0, 1)$.

Proof. It suffices to show that $\partial U_p/\partial p > 0$ for $p \in (0,1)$. Logarithmic differentiation yields

$$\frac{3p^3}{U_p(x)}\frac{\partial U_p}{\partial p} = -2\ln(\cos px) - \frac{px\sin px}{\cos px} := V(p),$$
$$V'(p) = \frac{x}{2\cos^2 px}(\sin 2px - 2px) < 0.$$

It follows that $V(p) < V(0) = 0$, and therefore $\partial U_p/\partial p > 0$, which proves the desired result. □

LEMMA 6. *For $p \in [0,1]$, let the function f_p be defined on $(0,\pi/2)$ by*

$$f_p(x) = \ln\frac{\sin x}{x} - \frac{1}{3p^2}\ln(\cos px) \text{ if } p \in (0,1] \text{ and } f_0(x) = \ln\frac{\sin x}{x} + \frac{x^2}{6}. \tag{2.6}$$

(i) If $f_p(x) < 0$ for all $x \in (0,\pi/2)$, then $p \in [0, 1/\sqrt{5}]$.

(ii) If $f_p(x) > 0$ for all $x \in (0,\pi/2)$, then $p \in [p_1, 1]$, where $p_1 = 0.45346...$ is the unique root of equation

$$f_p\left(\frac{\pi}{2}\right) = \ln\frac{2}{\pi} - \frac{1}{3p^2}\ln\left(\cos\frac{p\pi}{2}\right) = 0 \tag{2.7}$$

on $(0,1)$.

Proof. At first, we assert that there is a unique $p_1 \in (0,1)$ to satisfy equation (2.7) such that $f_p(\pi/2) < 0$ for $p \in (0,p_1)$ and $f_p(\pi/2) > 0$ for $p \in (p_1, 1]$.

In fact, Lemma 5 indicates that $U_p(x)$ is decreasing in p on $(0,1)$, and so $p \mapsto f_p(\pi/2)$ is increasing on $(0,1)$. Since

$$f_{1/3}\left(\frac{\pi}{2}\right) = \ln\frac{2}{\pi} - 3\ln\frac{\sqrt{3}}{2} < 0,$$
$$f_{1/2}\left(\frac{\pi}{2}\right) = \ln\frac{2}{\pi} - \frac{4}{3}\ln\frac{\sqrt{2}}{2} > 0,$$

so the equation (2.7) has a unique solution p_1 on $(0,1)$ and $p_1 \in (1/3, 1/2)$ such that $f_p(\pi/2) < 0$ for $p \in (0,p_1)$ and $f_p(\pi/2) > 0$ for $p \in (p_1, 1]$. Numerical calculation yields $p_1 = 0.45346\ldots$.

Secondly, simple computations give us

$$\lim_{x\to 0^+}\frac{f_p(x)}{x^4} = \frac{1}{36}p^2 - \frac{1}{180},$$
$$f_p\left(\frac{\pi}{2}^-\right) = \ln\frac{2}{\pi} - \frac{1}{3p^2}\ln\left(\cos\frac{\pi p}{2}\right) \text{ if } p \in (0,1] \text{ and } f_{0^+}\left(\frac{\pi}{2}^-\right) = \ln\frac{2}{\pi} + \frac{1}{24}\pi^2.$$

Now, if inequality $f_p(x) < 0$ for all $x \in (0,\pi/2)$, then solving the simultaneous inequalities

$$\lim_{x\to 0^+} x^{-4} f_p(x) \leqslant 0 \text{ and } f_p\left(\frac{\pi}{2}^-\right) \leqslant 0$$

for p yields

$$p \in [0, 1/\sqrt{5}] \cap [0, p_1] = [0, 1/\sqrt{5}].$$

In the same way, if inequality $f_p(x) > 0$ for all $x \in (0, \pi/2)$, then

$$p \in [1/\sqrt{5}, 1] \cap [p_1, 1] = [p_1, 1],$$

which completes the proof. □

3. Main Results

Now we state and prove the sharp upper bound $U_p(x)$ defined by (1.8) for $(\sin x)/x$.

THEOREM 1. *The inequality*

$$\frac{\sin x}{x} < \left(\cos\frac{x}{\sqrt{5}}\right)^{5/3} \tag{3.1}$$

holds for $x \in (0, \pi/2)$, where $1/\sqrt{5}$ is the best constant. Moreover, we have

$$\left(\cos\frac{x}{\sqrt{5}}\right)^{\alpha} < \frac{\sin x}{x} < \left(\cos\frac{x}{\sqrt{5}}\right)^{5/3}, \tag{3.2}$$

where the exponents $\alpha = (\ln(2/\pi))/\ln(\cos(\sqrt{5}\pi/10)) = 1.6714...$ and $5/3 = 1.6667...$ are the best possible constants.

Proof. The second inequality of (2.5) implies that (3.1) holds for $x \in (0, \pi/2)$. The monotonicity of the function $p \longmapsto U_p(x)$ on $(0, 1)$ and the part one of Lemma 6 indicate that $1/\sqrt{5}$ is the best constant.

Put $p = 1/\sqrt{5}$ in (2.5) yields (3.2).

Thus the proof is completed. □

From the Theorem 1 and Lemma 5, we see that

$$\frac{\sin x}{x} < U_{1/\sqrt{5}}(x) < \cdots < U_0(x) = e^{-x^2/6}$$

hold for $x \in (0, \pi/2)$. Thus, in order to prove the last inequality in (1.9) holds for $p \in [0, 1/\sqrt{5}]$, we have to prove the following

THEOREM 2. *The inequality*

$$e^{-x^2/6} < \frac{2 + \cos x}{3} \tag{3.3}$$

holds for $x \in (0, \infty)$. Moreover, for $x \in (0, a)$ ($a > 0$) we have

$$\frac{2 + \cos x}{(2 + \cos a)\, e^{a^2/6}} < e^{-x^2/6} < \frac{2 + \cos x}{3}. \tag{3.4}$$

Proof. Considering the function g defined by

$$g(x) = \ln \frac{2+\cos x}{3} + \frac{x^2}{6},$$

and differentiation yields

$$g'(x) = \frac{x}{3} - \frac{\sin x}{\cos x + 2},$$
$$g''(x) = \frac{1}{3}\frac{(\cos x - 1)^2}{(\cos x + 2)^2} \geqslant 0,$$

which implies that for $x \in (0,\infty)$, $g'(x) > g'(0^+) = 0$, then, $g'(x) > 0$, that is, g is increasing on $(0,\infty)$. Hence, we have $g(x) > g(0^+) = 0$ for $x \in (0,\infty)$, that is, (3.3) is true.

For $x \in (0,a)$ we have

$$0 = g\left(0^+\right) < g(x) < g(a) = \ln\left(\frac{2+\cos a}{3} e^{a^2/6}\right),$$

which proves (3.4) and the proof is complete. □

Next we establish the sharp lower bound $U_p(x)$ defined by (1.8) for $(\sin x)/x$.

THEOREM 3. *The inequality*

$$\frac{\sin x}{x} > (\cos p_1 x)^{1/\left(3p_1^2\right)} \tag{3.5}$$

holds for all $x \in (0,\pi/2)$, *where* $p_1 = 0.45346...$ *is the best constant which is the unique root of equation (2.7) in* $p \in (0,1)$. *Moreover, we have*

$$(\cos p_1 x)^{1/\left(3p_1^2\right)} < \frac{\sin x}{x} < \beta (\cos p_1 x)^{1/\left(3p_1^2\right)}, \tag{3.6}$$

where the coefficients 1 *and* $\beta \approx 1.0002$ *are the best possible constants.*

Proof. We first prove (3.6). Clearly, it suffices to show that $0 < f_{p_1}(x) < \ln \beta$ for all $x \in (0,\pi/2)$, where f_p is defined by (2.6). To this end, we introduce an auxiliary function h defined on $(0,\pi/2)$ by

$$h(x) = \frac{f'_{p_1}(x)}{x^3} = \frac{\left(\cot x - \frac{1}{x}\right) + \frac{1}{3p_1}\tan p_1 x}{x^3}. \tag{3.7}$$

Differentiation and simplifying yield

$$x^4 h'(x) = \frac{4}{3}\frac{x}{\sin^2 2p_1 x} - \frac{1}{3}\frac{x}{\sin^2 p_1 x} + \frac{1}{x} - \frac{x}{\sin^2 x} - 3\left(\cot x - \frac{1}{x}\right) - \frac{\tan p_1 x}{p_1},$$

which, utilizing (2.1), (2.2) and (2.3), can be expanded in power series as

$$
\begin{aligned}
x^4 h'(x) &= \frac{4}{3}\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|(2p_1)^{2n-2}x^{2n-1} \\
&\quad -\frac{1}{3}\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|p_1^{2n-2}x^{2n-1} \\
&\quad -\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}-3\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1} \\
&\quad -\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|p_1^{2n-2}x^{2n-1} \\
&:= \sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{3(2n)!}u_n x^{2n-1},
\end{aligned}
$$

where

$$u_n = \left(2^{2n}-1\right)(2n-10)\,p_1^{2n-2}-3(2n-1).$$

Clearly, $u_n<0$ for $n=1,2,3,4,5$. We now show that $u_n<0$ for $n\geqslant 6$. For this purpose, we need to prove that for $n\geqslant 6$

$$p_1<\left(\frac{3(2n-1)}{(2^{2n}-1)(2n-10)}\right)^{\frac{1}{2n-2}}:=h_1(n).$$

Since $(2n-1)>(2n-10)$, we have

$$h_1(n)>\left(\frac{3}{2^{2n}-1}\right)^{\frac{1}{2n-2}}:=k(n).$$

Considering the function $k:(1,\infty)\to(0,\infty)$ defined by

$$k(x)=\left(\frac{3}{2^{2x}-1}\right)^{1/(2x-2)}, \tag{3.8}$$

and differentiation leads to

$$
\begin{aligned}
\frac{2(x-1)^2}{k(x)}k'(x) &= \ln\left(2^{2x}-1\right)-\ln 3-2\frac{(x-1)2^{2x}}{2^{2x}-1}\ln 2:=k_1(x), \\
k_1'(x) &= \frac{2^{2x+2}\ln^2 2}{(2^{2x}-1)^2}(x-1).
\end{aligned}
$$

It is revealed that k_1 is increasing on $(1,\infty)$, and so $k_1(x)>k_1(1^+)=0$, then $k'(x)>0$, that is, k is increasing on $(1,\infty)$. Therefore for $n\geqslant 6$

$$0.48583\approx 1365^{-1/10}=k(6)\leqslant k(n)<k(\infty)=\frac{1}{2}.$$

It follows that for $n \geqslant 6$

$$h_1(n) > k(n) > 0.48583 > p_1,$$

which indicates that $u_n < 0$ for $n \geqslant 6$. Thus we have $h'(x) < 0$, that is, the auxiliary function h is decreasing on $(0,\pi/2)$.

On the other hand, it is clear that

$$h\left(0^+\right) = \lim_{x\to 0+} \frac{\left(\cot x - \frac{1}{x}\right) + \frac{1}{3p_1}\tan p_1 x}{x^3} = \frac{1}{9}\left(p_1^2 - \frac{1}{5}\right) > 0.$$

And we claim that $h\left(\frac{\pi}{2}^-\right) < 0$. If $h\left(\frac{\pi}{2}^-\right) \geqslant 0$, then there must be $h(x) > 0$ for all $x \in (0,\pi/2)$, which, by (3.7), implies that $f'_{p_1}(x) > 0$, then f_{p_1} is increasing on $(0,\pi/2)$. It yields

$$f_{p_1}(x) > f_{p_1}\left(0^+\right) = 0 \text{ and } f_{p_1}(x) < f_{p_1}\left(\frac{\pi}{2}\right) = \ln\frac{2}{\pi} - \frac{1}{3p_1^2}\ln\left(\cos\frac{p_1\pi}{2}\right) = 0,$$

which is a contradiction. Consequently, $h(0^+) > 0$ and $h\left(\frac{\pi}{2}^-\right) < 0$.

Make use of the monotonicity of the auxiliary function h it is showed that there is a unique $x_0 \in (0,\pi/2)$ to satisfy $h(x_0) = 0$ such that $h(x) > 0$ for $x \in (0,x_0)$ and $h(x) < 0$ for $x \in (x_0,\pi/2)$. Then, by (3.7), it is seen that f_{p_1} is increasing on $(0,x_0)$ and decreasing on $(x_0,\pi/2)$. We conclude that

$$\begin{aligned} 0 &= f_{p_1}\left(0^+\right) < f_{p_1}(x) < f_{p_1}(x_0) \text{ for } x \in (0,x_0), \\ 0 &= f_{p_1}\left(\frac{\pi}{2}^-\right) < f_{p_1}(x) < f_{p_1}(x_0) \text{ for } x \in (x_0,\pi/2), \end{aligned}$$

that is, $0 < f_{p_1}(x) < f_{p_1}(x_0)$ for $x \in (0,\pi/2)$.

Solving the equation $h(x) = 0$ which is equivalent with

$$f'_{p_1}(x) = \left(\cot x - \frac{1}{x}\right) + \frac{1}{3p_1}\tan p_1 x = 0$$

by using mathematical computer software, we find that $x_0 \in (1.3118, 1.3119)$, and $\beta = \exp(f_{p_1}(x_0)) \approx 1.0002$, which proves (3.6).

Consequently, (3.5) is true, of course. Now, Lemma 5 and the part two of Lemma 6 reveal that p_1 can not be replaced with any smaller $p \in [0,1]$, that is, p_1 is the best constant, which completes the proof. □

Letting

$$p = 1, \frac{\sqrt{6}}{3}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, p_1 \quad \text{and} \quad \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}, \frac{1}{3}, \frac{1}{2\sqrt{3}}, \frac{1}{4}, \ldots, \to 0.$$

By Theorem 3, Theorem 1 and Theorem 2 and Lemma 5, it is easy to obtain the following

COROLLARY 1. *For $x \in (0, \pi/2)$, we have*

$$(\cos x)^{1/3} < \cdots < \left(\cos \frac{\sqrt{6}x}{3}\right)^{1/2} < \left(\cos \frac{x}{\sqrt{2}}\right)^{2/3} < \cos \frac{x}{\sqrt{3}} < \left(\cos \frac{x}{2}\right)^{4/3}$$
$$< (\cos p_1 x)^{1/(3p_1^2)} < \frac{\sin x}{x} < \left(\cos \frac{x}{\sqrt{5}}\right)^{5/3} < \left(\cos \frac{x}{\sqrt{6}}\right)^{2} < \left(\cos \frac{x}{3}\right)^{3}$$
$$< \left(\cos \frac{x}{2\sqrt{3}}\right)^{4} < \left(\cos \frac{x}{4}\right)^{16/3} < \cdots < e^{-x^2/6} < \frac{2+\cos x}{3}.$$

where $p_1 = 0.45346\ldots$ and $1/\sqrt{5}$ are the best possible constants.

Thus it can be seen that our results greatly refine the two-side inequality (1.1).

The following statement gives relative errors estimating $(\sin x)/x$ by $U_p(x)$ defined by (1.8).

THEOREM 4. *Let $p \in [0,1]$ and let f_p be defined on $(0, \pi/2)$ by (2.6). Then f_p is decreasing if $p \in [0, 1/\sqrt{5}]$ and increasing if $p \in [1/2, 1]$.*

Moreover, for $x \in (0,c)$, $c \in (0, \pi/2)$, we have

$$\gamma_{0^+}(c)\, e^{-x^2/6} < \frac{\sin x}{x} < e^{-x^2/6} \quad \text{if } p = 0, \tag{3.9}$$

$$\gamma_p(c)\,(\cos px)^{1/(3p^2)} < \frac{\sin x}{x} < (\cos px)^{1/(3p^2)} \quad \text{if } p \in (0, 1/\sqrt{5}], \tag{3.10}$$

$$(\cos px)^{1/(3p^2)} < \frac{\sin x}{x} < \gamma_p(c)\,(\cos px)^{1/(3p^2)} \quad \text{if } p \in [1/2, 1], \tag{3.11}$$

where the coefficients

$$\gamma_p(c) = c^{-1}(\sin c)(\cos pc)^{-1/(3p^2)} \text{ if } p \in (0,1] \text{ and } \gamma_{0^+}(c) = c^{-1}(\sin c)\, e^{c^2/6}$$

and 1 are the best constants.

Proof. Differentiation and using (2.1) and (2.2) yield

$$f_p'(x) = \begin{cases} \left(\cot x - \frac{1}{x}\right) + \frac{1}{3p}\tan px \text{ if } p \in (0,1], \\ \left(\cot x - \frac{1}{x}\right) + \frac{x}{3} \qquad \text{if } p = 0 \end{cases}$$
$$= -\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1} + \frac{1}{3}\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}p^{2n-2}2^{2n}|B_{2n}|x^{2n-1}$$
$$= \sum_{n=1}^{\infty} \frac{\left(2^{2n}-1\right)2^{2n}}{3\,(2n)!}|B_{2n}|\left(p^{2n-2} - \frac{3}{2^{2n}-1}\right)x^{2n-1} := \sum_{n=2}^{\infty} s_n t_n x^{2n-1},$$

where

$$s_n = \frac{\left(2^{2n}-1\right)2^{2n}|B_{2n}|}{3\,(2n)!} \frac{p^{2n-2} - \frac{3}{2^{2n}-1}}{p - \left(\frac{3}{2^{2n}-1}\right)^{1/(2n-2)}} > 0,$$
$$t_n = p - k(n)$$

for $n \geqslant 2$ and $p \in [0,1]$, here the function k is defined by (3.8). As showed in the proof of Theorem 3, k is increasing on $(1,\infty)$, and so for $n \geqslant 2$

$$1/\sqrt{5} = k(2) \leqslant k(n) < k(\infty) = \lim_{n\to\infty}\left(\frac{3}{2^{2n}-1}\right)^{1/(2n-2)} = \frac{1}{2},$$

and then, $t_n = p - k(n) \leqslant 0$ for $p \in [0,1/\sqrt{5}]$ and $t_n = p - k(n) \geqslant 0$ for $p \in [1/2,1]$. Thus, if $p \in [0,1/\sqrt{5}]$ then $f_p'(x) < 0$, which shows that f_p is decreasing on $(0,\pi/2)$, and it is derived that for $x \in (0,c)$, $c \in (0,\pi/2)$

$$\ln(\gamma_p(c)) = f_p(c) < f_p(x) < \lim_{x\to 0^+} f_p(x) = 0,$$

which yields (3.9) and (3.10).

Likewise, if $p \in [1/2,1]$ then $f_p'(x) > 0$, which implies that f_p is increasing on $(0,\pi/2)$, and (3.11) follows.

This completes the proof. □

Letting $c \to (\pi/2)^-$ and putting $p = 1/\sqrt{5},\ 1/\sqrt{6},\ 1/3,\ 0^+$ in Theorem 4, we get

COROLLARY 2. *The following inequalities*

$$\gamma_{1/\sqrt{5}}\left(\frac{\pi}{2}\right)\left(\cos\frac{x}{\sqrt{5}}\right)^{5/3} < \frac{\sin x}{x} < \left(\cos\frac{x}{\sqrt{5}}\right)^{5/3}, \tag{3.12}$$

$$\gamma_{1/\sqrt{6}}\left(\frac{\pi}{2}\right)\left(\cos\frac{x}{\sqrt{6}}\right)^{2} < \frac{\sin x}{x} < \left(\cos\frac{x}{\sqrt{6}}\right)^{2}, \tag{3.13}$$

$$\gamma_{1/3}\left(\frac{\pi}{2}\right)\left(\cos\frac{x}{3}\right)^{3} < \frac{\sin x}{x} < \left(\cos\frac{x}{3}\right)^{3}, \tag{3.14}$$

$$\gamma_{0^+}\left(\frac{\pi}{2}\right)e^{-x^2/6} < \frac{\sin x}{x} < e^{-x^2/6} \tag{3.15}$$

hold true for $x \in (0,\pi/2)$, *where* $\gamma_{1/\sqrt{5}}(\pi/2) = 0.99872...$, $\gamma_{1/\sqrt{6}}(\pi/2) = 0.99141...$, $\gamma_{1/3}(\pi/2) = 16\sqrt{3}/(9\pi)$, $\gamma_{0^+}(\pi/2) = 2e^{\pi^2/24}/\pi$ *are the best possible constants.*

Letting $c \to (\pi/2)^-$ and putting $p = 1/2$ in Theorem 4, we obtain

COROLLARY 3. *For* $x \in (0,\pi/2)$, *the double inequality*

$$\left(\cos\frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \gamma_{1/2}\left(\frac{\pi}{2}\right)\left(\cos\frac{x}{2}\right)^{4/3} \tag{3.16}$$

holds, where the coefficients 1 *and* $\gamma_{1/2}(\pi/2) = 2^{5/3}/\pi = 1.0106...$ *are the best constants.*

REMARK 2. The first inequality of (3.16) also holds for $x \in (0,\pi)$. Indeed, it can be written as

$$\left(\cos\frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} = \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\frac{x}{2}}.$$

If $x \in (0,\pi)$, that is, $x/2 \in (0,\pi/2)$, then we divide both sides by $\cos(x/2)$ to get

$$\left(\cos\frac{x}{2}\right)^{1/3} < \frac{\sin\frac{x}{2}}{\frac{x}{2}},$$

which is Adamović-Mitrinović's inequality (the left one in (1.1)) .

4. Applications

As simple applications of main results, we present some precise estimations for certain special functions and constants in this section.

For the estimations for the sine integral defined by

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}dt,$$

there has some results (see [17], [22], [23]). Now we give a general result.

PROPOSITION 1. *For* $x \in (0,\pi/2]$, *we have*

$$\sqrt{3}\sin\frac{x}{\sqrt{3}} < \mathrm{Si}\,(x) < \frac{x}{2} + \frac{\sqrt{6}}{4}\sin\frac{2x}{\sqrt{6}}. \tag{4.1}$$

Proof. By Corollary 1 we see that the inequalities

$$\cos\frac{t}{\sqrt{3}} < \frac{\sin t}{t} < \cos^2\frac{t}{\sqrt{6}} \tag{4.2}$$

hold for $t \in [0,\pi/2]$. Integrating both sides over $[0,\pi/2]$ and simple calculation yield (4.1). □

REMARK 3. By (4.1) and using (4.2) we can obtain the following

$$1.3603 \approx \frac{\sqrt{3}}{4}\pi < \int_0^{\pi/2}\frac{\sin x}{x}dx < \frac{7}{16}\pi \approx 1.3744. \tag{4.3}$$

Now we consider the error function defined as

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

By integrating both sides of (3.4) over $\left[0,\sqrt{6}x\right]$ $(x>0)$ and with $x/\sqrt{6} \to x$, we have

PROPOSITION 2. *For $x>0$ the following inequalities*

$$\frac{2}{\sqrt{\pi}}\frac{2\sqrt{6}x+\sin(\sqrt{6}x)}{2+\cos(\sqrt{6}x)}e^{-x^2}<\operatorname{erf}(x)<\frac{2}{\sqrt{\pi}}\frac{2\sqrt{6}x+\sin(\sqrt{6}x)}{3}. \tag{4.4}$$

It is known that

$$\int_0^{\pi/2}\ln(\sin x)\,dx=-\frac{\pi}{2}\ln 2.$$

We now evaluate the integral $\int_0^x \ln(\sin t)\,dt$ $(x\in(0,\pi/2))$.

PROPOSITION 3. *For $x\in(0,\pi/2)$, we have*

$$x\ln(\sin x)-x+\frac{1}{9}x^3<\int_0^x \ln(\sin t)\,dt<x\ln x-x-\frac{1}{18}x^3. \tag{4.5}$$

Proof. Utilizing (3.9) gives

$$x^{-1}(\sin x)e^{x^2/6}e^{-t^2/6}<\frac{\sin t}{t}<e^{-t^2/6}.$$

Multiplying both sides by t and taking the logarithm and next integrating $[0,x]$ yield

$$\int_0^x \ln\left(x^{-1}(\sin x)e^{x^2/6}te^{-t^2/6}\right)dt<\int_0^x \ln(\sin t)\,dt<\int_0^x \ln\left(te^{-t^2/6}\right)dt.$$

Simple integral computation leads to desired result. □

REMARK 4. From (4.5) it is easy to get

$$\tfrac{\pi^3}{72}-\tfrac{\pi}{2}<\int_0^{\pi/2}\ln(\sin t)\,dt<\tfrac{\pi}{2}\ln\tfrac{\pi}{2}-\tfrac{\pi}{2}-\tfrac{\pi^3}{144}, \tag{4.6}$$

$$\tfrac{\pi^3}{576}-\tfrac{\pi}{8}\ln 2-\tfrac{\pi}{4}<\int_0^{\pi/4}\ln(\sin t)\,dt<\tfrac{\pi}{4}\ln\tfrac{\pi}{4}-\tfrac{\pi}{4}-\tfrac{\pi^3}{1152}. \tag{4.7}$$

The Catalan constant [4]

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.9159655941772190\ldots$$

is a famous mysterious constant appearing in many places in mathematics and physics. Its integral representations [3] include the following

$$\begin{aligned}G&=\int_0^1\frac{\arctan x}{x}dx=\frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx\\&=-2\int_0^{\pi/4}\ln(2\sin x)\,dx=\frac{\pi^2}{16}-\frac{\pi}{4}\ln 2+\int_0^{\pi/4}\frac{x^2}{\sin^2 x}dx.\end{aligned}$$

By our results we can derive various estimations for G. Now, by using the third integral representation for G and (4.7), we easily obtain the following

PROPOSITION 4. *We have*

$$0.91528 \approx \frac{\pi}{2} - \frac{\pi}{2}\ln\frac{\pi}{2} + \frac{\pi^3}{576} < G < \frac{\pi}{2} - \frac{\pi}{4}\ln 2 - \frac{\pi^3}{288} \approx 0.91874. \tag{4.8}$$

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Zhen-Hang Yang
Power Supply Service Center
Zhejiang Electric Power Corporation Research Institute
Hangzhou City, Zhejiang Province, 310014 China
e-mail: yzhkm@163.com