RATE OF APPROXIMATION OF BOUNDED VARIATION FUNCTIONS
BY THE BÉZIER VARIANT OF CHLODOWSKY OPERATORS

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1. Introduction

For a function \( f \) defined on the interval \( [0, b_n] \), the Chlodowsky operators \( C_n(f, x) \) are defined by

\[ C_n(f, x) = \sum_{k=0}^{n} f \left( \frac{kb_n}{n} \right) p_{nk} \left( \frac{x}{b_n} \right), \]

where \( p_{nk}(x/b_n) = \binom{n}{k} (x/b_n)^k (1-x/b_n)^{n-k} \) and \( (b_n) \) is a sequence of increasing positive numbers, with the properties \( \lim_{n \to \infty} b_n = \infty \) and \( \lim_{n \to \infty} b_n/n = 0 \). When \( b_n \equiv 1 \), the operators \( C_n(f, x) \) become the well-known Bernstein operators

\[ B_n(f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{nk}(x). \]

In [1], the authors introduced the Bézier variant of Chlodowsky operators \( C_{n,\alpha} \) as follows:

\[ C_{n,\alpha}(f, x) = \sum_{k=0}^{n} f \left( \frac{kb_n}{n} \right) Q_{nk}^{(\alpha)} \left( \frac{x}{b_n} \right), \]


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where \( \alpha > 0 \) and

\[
Q_{nk}^{(\alpha)}(x/b_n) = J_{n,k}^{\alpha}(x/b_n) - J_{n,k+1}^{\alpha}(x/b_n), \quad J_{n,k}(x/b_n) = \sum_{j=k}^{n} p_{nj}(x/b_n).
\]

Obviously for \( \alpha = 1 \), the operators \( C_{n,\alpha} \) reduce to the operators \( C_n \).

Let

\[
K_{n,\alpha}(x/b_n, t/b_n) = \begin{cases} 
\sum_{k=b_n}^{n} Q_{nk}^{(\alpha)}(x/b_n), & 0 < t \leq b_n; \\
0, & t = 0.
\end{cases}
\]

By Lebesgue-Stieltjes integral representation, we have

\[
C_{n,\alpha}(f, x) = \int_{0}^{b_n} f(t) dt K_{n,\alpha}(x/b_n, t/b_n).
\]

In [1], H. Karsli and E. Ibikli studied the convergence rate of \( C_{n,\alpha} \) to bounded variation functions for the case \( \alpha \geq 1 \). Unfortunately, in the proof of the results, the authors made some mistakes as follows:

1. [1, Lemma 2] For all \( x \in (0, \infty) \) and \( 0 \leq t < x \), we have

\[
\lambda_{n,\alpha}(x/b_n, t/b_n) = \int_{0}^{t} K_{n,\alpha}(x/b_n, u/b_n) du \leq \frac{\alpha}{(x-t)^2} \frac{x(b_n-x)}{n}.
\]

In fact, the result should be

\[
K_{n,\alpha}(x/b_n, t/b_n) \leq \frac{\alpha}{(x-t)^2} \frac{x(b_n-x)}{n}.
\]

2. In [1, p. 439], the authors mistook \( I_{1,\alpha}(n,x) = \int_{0}^{x-x/\sqrt{n}} g_n(t) dt (\lambda_{n,\alpha}(x/b_n, t/b_n)) \), but in fact \( I_{1,\alpha}(n,x) = \int_{0}^{x-x/\sqrt{n}} g_n(t) dt (K_{n,\alpha}(x/b_n, t/b_n)) \). The representations of \( I_{2,\alpha}(n,x) \) and \( I_{3,\alpha}(n,x) \) were also wrong.

These two mistakes resulted in a lot of errors in the following proof of [1, Theorem].

In this paper, we re-discuss the pointwise approximation of \( C_{n,\alpha} \) to bounded variation functions for the case \( \alpha > 0 \) which includes \( \alpha \geq 1 \). We also mention some of the important papers on this subject by Gupta [7] and Pych-Taberska [8].

The main theorems of this paper are as follows.

**Theorem 1.** Let \( \alpha \geq 1 \), \( f \) be a function of bounded variation on every finite subinterval of \([0, \infty)\) and \( \lim_{x \to \infty} f(x) \) exists, i.e. \( f \in BV[0, \infty) \). Then for every \( x \in (0, b_n) \), we have

\[
\left| C_{n,\alpha}(f, x) - \frac{1}{2\alpha} f(x+) - \left( 1 - \frac{1}{2\alpha} \right) f(x-) \right| \leq \frac{3\alpha b_n^2}{nx(b_n-x)} \sum_{k=1}^{n} \sqrt{x+(b_n-x)/\sqrt{k}} (g_x)
\]

\[
+ \frac{\alpha b_n}{\sqrt{nx(b_n-x)}} (|f(x+)-f(x-)| + e_n(x/b_n)|f(x)-f(x-)|).
\]
Theorem 2. Let $0 < \alpha \leq 1$, $f$ be a function of bounded variation on every finite subinterval of $[0, \infty)$ and $\lim_{t \to \infty} f(x)$ exists, i.e. $f \in BV[0, \infty)$. Then for every $x \in (0, b_n)$ and $n > \frac{256 b_n^2}{255 (b_n-x)}$, we have

$$\left| C_{n, \alpha}(f, x) - \frac{1}{2\alpha} f(x^+) - \left(1 - \frac{1}{2\alpha}\right) f(x^-) \right| \leq \frac{A_{\alpha} b_n^2}{nx(b_n-x)} \sum_{k=1}^{n} \sqrt{x+\frac{(2-x)/\sqrt{k}}{x-x/\sqrt{k}}} (g_x) \tag{1}$$

$$+ \frac{b_n}{\sqrt{nx(b_n-x)}} \left| f(x^+) - f(x^-) \right| + \varepsilon_n(x/b_n) \left| f(x) - f(x^-) \right|,$$

where $A_{\alpha}$ is a positive constant depending only on $\alpha$,

$$g_x(t) = \begin{cases} f(t) - f(x^+), & x < t \leq b_n; \\ 0, & t = x; \\ f(t) - f(x^-), & 0 \leq t < x. \end{cases} \tag{2}$$

$$\varepsilon_n(x/b_n) = \begin{cases} 1, & \text{if } x = \frac{k'}{b_n}, \text{ for some } k' \in \mathbb{N}; \\ 0, & \text{if } x \neq \frac{k'}{b_n}, \text{ for all } k \in \mathbb{N}. \end{cases} \tag{3}$$

When $b_n \equiv 1$, the operators $C_{n, \alpha}(f, x)$ are just the Bernstein-Bézier operators $B_{n, \alpha}(f, x) = \sum_{k=0}^{n} f(k/n) Q_{nk}^{(\alpha)}(x)$, which were studied by Zeng [2,3]. Therefore, our theorems extend the results of Zeng. Moreover, in the case $0 < \alpha \leq 1$, Zeng [2] gave a rate of convergence of $B_{n, \alpha}$ for bounded variation functions as follows:

Let $0 < \alpha \leq 1$, $f$ be a function of bounded variation on $[0, 1]$ ($f \in BV[0, 1]$). Then for every $x \in (0, 1)$ and $n > \frac{256}{255} (1-x)^{-1}$ we have

$$\left| B_{n, \alpha}(f, x) - \frac{1}{2\alpha} f(x^+) - \left(1 - \frac{1}{2\alpha}\right) f(x^-) \right| \leq \frac{A_{\alpha}}{n(1-x)^{2-\alpha}} \sum_{k=1}^{n} \sqrt{x+\frac{(1-x)/\sqrt{k}}{x-x/\sqrt{k}}} (g_x) \tag{4}$$

$$+ \frac{1}{\sqrt{nx(1-x)}} \left| f(x^+) - f(x^-) \right| + \varepsilon_n(x) \left| f(x) - f(x^-) \right|.$$ 

Obviously, for $b_n \equiv 1$, our Theorem 2 extends and improves the result of (3).

2. Lemmas

The proof of our results are based on the following lemmas.

Lemma 1. For every $x \in (0, b_n)$ and $0 \leq k \leq n$, we have

$$p_{nk}(x/b_n) \leq \frac{b_n}{\sqrt{2enx(b_n-x)}}. \tag{5}$$

Proof. By [4, Theorem 1], we have $p_{nk}(t) < \frac{1}{\sqrt{2en(1-t)}}$ for $0 < t < 1$.

Replacing $t$ for $x/b_n$, we can get (4) easily. \qed

The following Lemma is the well-known Berry-Esseen bound for the central limit theorem of probability theory. Its proof can be found in Shiryayev [5, p. 432].
LEMMA 2. Let \( \{\xi_k\}_{k=1}^{+\infty} \) be a sequence of independent and identically distributed random variables with finite variance such that the expectation \( E(\xi_1) = a_1 \in \mathbb{R} \), the variance \( \text{Var}(\xi_1) = E(\xi_1 - a_1)^2 = b_1^2 > 0 \) and \( E|\xi_1 - E(\xi_1)|^3 < +\infty \). Then there exists a constant \( C, 1/\sqrt{2\pi} \leq C < 0.8 \), such that for all \( n \) and \( t \),

\[
\left| P\left( \frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leq t \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du \right| < C \frac{E|\xi_1 - E(\xi_1)|^3}{b_1^3\sqrt{n}} \tag{5}
\]

LEMMA 3. For \( x \in (0,b_n) \), we have

\[
\left| \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) - \frac{1}{2} \right| < \frac{0.8b_n}{\sqrt{nx(b_n-x)}}. \tag{6}
\]

Proof. Let \( \xi_1 \) be the random variable with two-point distribution \( P(\xi_1 = i) = (\frac{\alpha}{b_n})(1 - \frac{\alpha}{b_n})^{1-i} \) \( (i = 0, 1, x \in (0,b_n) \) is a parameter). Hence \( a_1 = E(\xi_1) = x/b_n \), \( b_1^2 = E(\xi_1 - a_1)^2 = \frac{x}{b_n}(1 - \frac{x}{b_n}) \), and \( E|\xi_1 - E(\xi_1)|^3 = \frac{x}{b_n}(1 - \frac{x}{b_n})^2(\frac{x}{b_n})^2 + (1 - \frac{x}{b_n})^2 \).

Let \( \{\xi_k\}_{k=1}^{+\infty} \) be a sequence of independent random variables identically distributed with \( \xi_1 \), \( \eta_n = \sum_{j=1}^{n} \xi_j \). Then the probability distribution of the random variable \( \eta_n \) is

\[
P(\eta_n = k) = \binom{n}{k}(\frac{x}{b_n})^k(1 - \frac{x}{b_n})^{n-k} = p_{nk}(x/b_n).
\]

So

\[
\sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) = P(nx/b_n < \eta_n \leq n) = 1 - P(\eta_n < nx/b_n).
\]

By (5), we get

\[
\left| \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) - \frac{1}{2} \right| = \left| P\left( \frac{\eta_n - nx/b_n}{\sqrt{n}\sqrt{x/b_n(1 - \frac{x}{b_n})}} \leq 0 \right) - \frac{1}{2} \right|
\]

\[
< C \frac{E|\xi_1 - E(\xi_1)|^3}{\sqrt{n}b_1^3} < \frac{0.8\alpha}{\sqrt{nx(b_n-x)}} < \frac{0.8b_n}{\sqrt{nx(b_n-x)}}.
\]

This completes the proof of (6). \( \square \)

LEMMA 4. For \( \alpha \geq 1 \) and \( x \in (0,b_n), k' = nx/b_n \), we have

\[
(i) \quad \left| \left( \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) \right)^{\alpha} - \frac{1}{2^\alpha} \right| \leq \frac{0.8\alpha b_n}{\sqrt{nx(b_n-x)}}, \tag{7}
\]
Hence

\[
\xi_n \quad \text{where}
\]

we get (7) from (6) easily.

(ii) Using the same method of (i), we obtain

\[
Q_{nk'}^{(\alpha)}(x/b_n) < \alpha p_{nk'}(x/b_n).
\]

(8) now follows from (4) immediately. \(\square\)

**Lemma 5.** For \(0 < \alpha \leq 1\) and \(x \in (0, b_n)\), as \(n > \frac{256b_n^2}{25x(b_n-x)}\) and \(k' = nx/b_n\), we have

(i) \[
\left| \left( \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) \right)^\alpha - \frac{1}{2^\alpha} \right| < \frac{b_n}{\sqrt{nx(b_n-x)}} \tag{9}
\]

(ii) \[
Q_{nk}^{(\alpha)}(x/b_n) < \frac{b_n}{\sqrt{nx(b_n-x)}}. \tag{10}
\]

**Proof.** (i) By mean value theorem, we have

\[
\left| \left( \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) \right)^\alpha - \frac{1}{2^\alpha} \right| = \alpha \left( \xi_{nk}(x/b_n) \right)^{\alpha-1} \left| \left( \sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) \right)^{\alpha} - \frac{1}{2} \right|,
\]

where \(\xi_{nk}(x/b_n)\) lies between \(\frac{1}{2}\) and \(\sum_{nx/b_n < k \leq n} p_{nk}(x/b_n)\).

In view of (6) and all \(n > \frac{256b_n^2}{25x(b_n-x)}\), we have

\[
\sum_{nx/b_n < k \leq n} p_{nk}(x/b_n) > \frac{1}{4} \tag{12}
\]

Hence \(\xi_{nk}(x/b_n) > \frac{1}{4}\) holds for \(n > \frac{256b_n^2}{25x(b_n-x)}\).

From (11), (6) and noting \(3.2\alpha < 4\alpha\), we get (9) immediately.

(ii) Using the mean value theorem, we get

\[
Q_{nk'}^{(\alpha)}(x/b_n) = \alpha \left( \eta_{nk'}(x/b_n) \right)^{\alpha-1} \left[ J_{n,k'}(x/b_n) - J_{n,k'+1}(x/b_n) \right]
\]

\[
= \alpha \left( \frac{1}{\eta_{nk'}(x/b_n)} \right)^{1-\alpha} p_{nk'}(x/b_n), \tag{13}
\]

where \(J_{n,k'+1}(x/b_n) < \eta_{nk'}(x/b_n) < J_{n,k'}(x/b_n)\).

But in view of (12), we know

\[
\eta_{nk'}(x/b_n) > J_{n,k'+1}(x/b_n) = \sum_{j>nx/b_n} p_{nj}(x/b_n) > \frac{1}{4}.
\]

From (13), (4) and noting \(2\alpha < 4\alpha\), we deduce that

\[
Q_{nk'}^{(\alpha)}(x/b_n) < \frac{\alpha 4^{1-\alpha} b_n}{\sqrt{2enx(b_n-x)}} < \frac{b_n}{\sqrt{nx(b_n-x)}}. \quad \square
\]
**Lemma 6.** (i) For $\alpha \geq 1$ and $0 \leq t < x$, there holds

$$K_{n,\alpha} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq \frac{\alpha x(b_n - x)}{n(x-t)^2}. \tag{14}$$

(ii) For $\alpha \geq 1$ and $0 \leq x < t$, there holds

$$1 - K_{n,\alpha} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq \frac{\alpha x(b_n - x)}{n(x-t)^2}. \tag{15}$$

**Proof.** (i) By a simple calculation, we get

$$C_n(1, x) = 1,$$

$$C_n(t, x) = x,$$

$$C_n(t^2, x) = x^2 + \frac{x(b_n - x)}{n}.$$ 

Thus

$$C_n((t - x)^2, x) = \frac{x(b_n - x)}{n}. \tag{16}$$

Now from the fact that $|x^\alpha - y^\alpha| \leq \alpha |x - y|$ with $0 \leq x, y \leq 1$ and $\alpha \geq 1$, we get

$$K_{n,\alpha} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) = \sum_{kbn \leq nt} Q_{nk}^{(\alpha)} \left( \frac{x}{b_n} \right) \leq \alpha \sum_{kbn \leq nt} p_{nk} \left( \frac{x}{b_n} \right) \leq \alpha C_n((t - x)^2, x) \frac{(t - x)^2}{n}.$$

(14) now follows from (16).

(ii) Using a similar method we can get (15) easily. $\square$

**Lemma 7.** (i) For $0 < \alpha \leq 1$ and $0 \leq t < x$, there holds

$$K_{n,\alpha} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq K_{n,1} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq \frac{x(b_n - x)}{n(x-t)^2}. \tag{17}$$

(ii) For $0 < \alpha \leq 1$ and $0 \leq x < t$, there holds

$$1 - K_{n,\alpha} \left( \frac{x}{b_n}, \frac{t}{b_n} \right) \leq A_\alpha x(b_n - x) \frac{A_\alpha x(b_n - x)}{n(x-t)^2}, \tag{18}$$

where $A_\alpha$ is a positive constant depending only on $\alpha$.

**Proof.** (i) Along the same lines of the proof of [2, Lemma 4] and the inequality of (14), we can get (17) easily.
(ii) Since $0 \leq x < t$, so $|kb_n/n - x|/|t - x| \geq 1$ for all $k \geq nt/b_n$. Thus we have

$$1 - K_{n,\alpha}(x/b_n, t/b_n) = 1 - \sum_{k \leq nt/b_n} Q_{nk}^{(\alpha)}(x/b_n) \leq \sum_{k \geq nt/b_n} Q_{nk}^{(\alpha)}(x/b_n)$$

$$= \sum_{k \geq nt/b_n} (J_{nk}^{\alpha}(x/b_n) - J_{nk+1}^{\alpha}(x/b_n)) = \left( \sum_{k \geq nt/b_n} p_{nk}(x/b_n) \right)^{\alpha}$$

$$\leq \left( \sum_{k \geq nt/b_n} \frac{kb_n/t - x}{|t - x|^2/\alpha} p_{nk}(x/b_n) \right)^{\alpha}$$

$$\leq \frac{b_n^2}{(t - x)^2} \left( \sum_{k = 0}^{\alpha} \frac{|k}{n} - x}{b_n} \frac{2/\alpha}{\alpha} p_{nk}(x/b_n) \right)^{\alpha/p}.$$ 

Then, by Hölder’s inequality with $p, q > 1$ and $1/p + 1/q = 1$, we have

$$\left( \sum_{k = 0}^{\alpha} \frac{|k}{n} - x}{b_n} \frac{2/\alpha}{\alpha} p_{nk}(x/b_n) \right)^{\alpha/p} = \left( \sum_{k = 0}^{\alpha} \frac{|k}{n} - x}{b_n} \frac{2/\alpha}{\alpha} (p_{nk}(x/b_n))^{1/p} (p_{nk}(x/b_n))^{1/q} \right)^{\alpha/p}$$

Choosing $p = \alpha[1/\alpha + 1]$, then $2p/\alpha = 2[1/\alpha + 1]$ is an even positive integer. From [6, Theorem 1.5.1], we have

$$\left( \sum_{k = 0}^{\alpha} \frac{|k}{n} - x}{b_n} \frac{2/\alpha}{\alpha} p_{nk}(x/b_n) \right)^{\alpha/p} \leq A_{\alpha} \frac{x}{b_n} \left( 1 - \frac{x}{b_n} \right)^{n-1},$$

where $A_{\alpha}$ is a positive constant depending only on $\alpha$. This completes the proof of (18). □

**Lemma 8.** (i) For $\alpha \geq 1$, $f \in BV[0, \infty)$ and $x \in (0, b_n)$, we have

$$|C_{n,\alpha}(g_s, x)| \leq \frac{3\alpha b_n^2}{nx(b_n - x)} \sum_{k = 1}^{n} \frac{x + (b_n - x)/\sqrt{k}}{x - x/\sqrt{k}} (g_s).$$

(ii) For $0 < \alpha \leq 1$, $f \in BV[0, \infty)$ and $x \in (0, b_n)$, when $n > \frac{256b_n^2}{25x(b_n - x)}$, we have

$$|C_{n,\alpha}(g_s, x)| \leq \frac{A_{\alpha} b_n^2}{nx(b_n - x)} \sum_{k = 1}^{n} \frac{x + (b_n - x)/\sqrt{k}}{x - x/\sqrt{k}} (g_s).$$

**Proof.** (i) We recall the Lebesgue-Stieltjes integral representations

$$C_{n,\alpha}(g_s, x) = \int_{0}^{b_n} g_s(t) d_{n,\alpha}(x/b_n, t/b_n).$$

(21)
Decompose the integral of (21) into three parts as follows

\[
C_{n, \alpha}(g_x, x) = \int_0^{b_n} g_x(t) d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right) = \Sigma_1 + \Sigma_2 + \Sigma_3,
\]  

(22)

where

\[
\Sigma_1 = \int_0^{x - \frac{x}{\sqrt{n}}} g_x(t) d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right), \quad \Sigma_2 = \int_0^{x + \frac{bn - x}{\sqrt{n}}} g_x(t) d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right),
\]

\[
\Sigma_3 = \int_{x - \frac{bx - x}{\sqrt{n}}}^{b_n} g_x(t) d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right).
\]

Observing that \(g_x(x) = 0\), we first have

\[
|\Sigma_2| = \int_0^{x + \frac{bn - x}{\sqrt{n}}} |g_x(t) - g_x(x)| d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leq \sqrt{\frac{(g_x)}{x - x/\sqrt{n}}} \leq \frac{1}{n - 1} \sum_{k=2}^n \sqrt{\frac{(g_x)}{x - x/\sqrt{k}}}. \tag{23}
\]

To estimate \(\Sigma_1\), let \(y = x - x/\sqrt{n}\). Using Lebesgue-Stieltjes integration by parts and (14), we have

\[
|\Sigma_1| = \left|\int_0^y g_x(t) d_t K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right)\right| = \left|g_x(y+) K_{n, \alpha} \left(\frac{x}{b_n}, \frac{y}{b_n}\right) - \int_0^y K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right) d_t g_x(t)\right| \leq \frac{\sqrt{(g_x)}}{y+} K_{n, \alpha} \left(\frac{x}{b_n}, \frac{y}{b_n}\right) + \int_0^y K_{n, \alpha} \left(\frac{x}{b_n}, \frac{t}{b_n}\right) d_t (-\sqrt{(g_x)}) \leq \frac{\alpha x (b_n - x)}{n(x - y)^2} + \frac{\alpha x (b_n - x)}{n} \int_0^y \frac{1}{(x-t)^2} d_t (-\sqrt{(g_x)}).
\]

Since

\[
\int_0^y \frac{1}{(x-t)^2} d_t (-\sqrt{(g_x)}) = -\sqrt{\frac{y^+}{(x-t)^2}} \bigg|_0^y + \int_0^y \frac{2\sqrt{y^+ (g_x)}}{(x-t)^3} d_t,
\]

we have

\[
|\Sigma_1| \leq \frac{\alpha x (b_n - x)}{nx^2} \sqrt{(g_x)} + \frac{\alpha x (b_n - x)}{n} \int_0^y \frac{2\sqrt{y^+ (g_x)}}{(x-t)^3} d_t.
\]

Putting \(t = x - x/\sqrt{u}\) for the last integral, we get

\[
|\Sigma_1| \leq \frac{\alpha x (b_n - x)}{nx^2} \sqrt{(g_x)} + \frac{\alpha x (b_n - x)}{nx^2} \sqrt{(g_x)} d_u \leq \frac{\alpha x (b_n - x)}{nx^2} \left[ \sqrt{(g_x)} + \sum_{k=1}^n \sqrt{(g_x)} \right]. \tag{24}
\]
Using the similar method and (15) to estimate $|\Sigma_3|$, we obtain

$$|\Sigma_3| \leq \frac{\alpha x(b_n - x)}{n(b_n - x)^2} \left[ b_n \sqrt{g_x} + \sum_{k=1}^{n} \sqrt{x - x/k} (g_x) \right]. \tag{25}$$

Combining the estimates of (22), (23), (24) and (25), also noting the properties of $\hat{\text{sign}}(t - x)$ and $1/(n - 1) \leq \alpha b_n^2/[nx(b_n - x)]$ for $x \in (0, b_n)$, we get

$$|C_{n, \alpha}(g_x, x)| \leq \frac{\alpha[(b_n - x)^2 + x^2]}{nx(b_n - x)} \left[ b_n \sqrt{g_x} + \sum_{k=1}^{n} \sqrt{x - x/k} (g_x) \right]$$

$$+ \frac{1}{n - 1} \sum_{k=2}^{n} \sqrt{x - x/k} (g_x)$$

$$\leq \frac{2\alpha b_n^2}{nx(b_n - x)} \sum_{k=1}^{n} \sqrt{x - x/k} (g_x) + \frac{1}{n - 1} \sum_{k=2}^{n} \sqrt{x - x/k} (g_x)$$

$$\leq \frac{3\alpha b_n^2}{nx(b_n - x)} \sum_{k=1}^{n} \sqrt{x - x/k} (g_x).$$

This completes the proof of (19).

(ii) Using the same method and (17), (18), we can also get (20) easily. □

3. Proof of Theorem 1 and Theorem 2

Let $f$ satisfy the conditions of Theorem 1 and Theorem 2. We can decompose $f(t)$ into four parts as

$$f(t) = \frac{1}{2\alpha} f(x+) + (1 - \frac{1}{2\alpha}) f(x-) + g_\alpha(t) + \frac{f(x+) - f(x-)}{2\alpha} \hat{\text{sign}}(t - x)$$

$$+ \delta_x(t) \left[ f(x) - \frac{1}{2\alpha} f(x+) - \left(1 - \frac{1}{2\alpha}\right) f(x-) \right],$$

where

$$\hat{\text{sign}}(t - x) = \begin{cases} 2\alpha - 1, & t > x; \\ 0, & t = x; \\ -1, & t < x. \end{cases}$$

$$\delta_x(t) = \begin{cases} 1, & t = x; \\ 0, & t \neq x. \end{cases}$$
\(g_x(t)\) is defined in (1). Therefore,

\[
\begin{align*}
|C_{n, \alpha}(f, x) - \frac{1}{2\alpha} f(x+) - \left(1 - \frac{1}{2\alpha}\right) f(x-) - \left(1 - \frac{1}{2\alpha}\right) f(x-) | & \\
\leq |C_{n, \alpha}(g_x, x)| + \left| \frac{f(x+) - f(x-)}{2\alpha} C_{n, \alpha}(\text{sign}(t-x), x) \right| + \left[ f(x) - \frac{1}{2\alpha} f(x+) - \left(1 - \frac{1}{2\alpha}\right) f(x-) \right] C_{n, \alpha}(\delta_x, x) \right|.
\end{align*}
\]

(26)

By direct calculation, we get

\[
C_{n, \alpha}(\delta_x, x) = \epsilon_n(x/b_n) Q_{nk}^{\alpha}(x/b_n)
\]

and

\[
C_{n, \alpha}(\text{sign}(t-x), x) = \sum_{k > nx/b_n} (2\alpha - 1) Q_{nk}^{\alpha}(x/b_n) + \sum_{k < nx/b_n} (-1) Q_{nk}^{\alpha}(x/b_n)
\]

\[
= 2\alpha \sum_{k > nx/b_n} Q_{nk}^{\alpha}(x/b_n) - 1 + \epsilon_n(x/b_n) Q_{nk'}^{\alpha}(x/b_n)
\]

\[
= 2\alpha \left( \sum_{k > nx/b_n} p_{nk}(x/b_n) \right)^{\alpha} - 1 + \epsilon_n(x/b_n) Q_{nk'}^{\alpha}(x/b_n),
\]

where \(\epsilon_n(x/b_n)\) is defined in (2).

Therefore, we have

\[
\frac{f(x+) - f(x-)}{2\alpha} C_{n, \alpha}(\text{sign}(t-x), x) + \left[ f(x) - \frac{1}{2\alpha} f(x+) - \left(1 - \frac{1}{2\alpha}\right) f(x-) \right] C_{n, \alpha}(\delta_x, x)
\]

\[
= \left[ f(x+) - f(x-) \right] \left( \sum_{\frac{nx}{b_n} < k} p_{nk}(x/b_n) \right)^{\alpha} - \frac{1}{2\alpha} \left[ f(x) - f(x-) \right] \epsilon_n(x/b_n) Q_{nk'}^{\alpha}(x/b_n) \right).
\]

(27)

By combining the estimates given by (26), (19), (27), (7) and (8), we obtain Theorem 1; and by combining the estimates given by (26), (20), (27), (9) and (10), we obtain Theorem 2.

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**REFERENCES**


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