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RATE OF APPROXIMATION OF BOUNDED VARIATION FUNCTIONS BY THE BÉZIER VARIANT OF CHLODOWSKY OPERATORS

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Abstract. In this paper the pointwise approximation of the Bézier variant of Chlodowsky operators for bounded variation functions is studied. By means of the analysis techniques and some results of probability theory, we obtain an estimate formula on this type approximation. Our results correct the mistake of Karsli and Ibikli [H. Karsli and E. Ibikli, Convergence rate of a new Bézier variant of Chlodowsky operators to bounded variation functions, J. Comput. Appl. Math 212 (2008) 431–443], and also extend the work of Zeng [X. M. Zeng, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions II, J. Approx. Theory 104 (2000) 330–344].

1. Introduction

For a function f defined on the interval $[0,b_n]$, the Chlodowsky operators $C_n(f,x)$ are defined by

$$C_n(f,x)=\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)p_{nk}\left(\frac{x}{b_n}\right),$$

where $p_{nk}(x/b_n)=\binom{n}{k}(x/b_n)^k(1-x/b_n)^{n-k}$ and (b_n) is a sequence of increasing positive numbers, with the properties $\lim_{n\to\infty} b_n=\infty$ and $\lim_{n\to\infty} b_n/n=0$. When $b_n\equiv 1$, the operators $C_n(f,x)$ become the well-know Bernstein operators

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{nk}(x).$$

In [1], the authors introduced the Bézier variant of Chlodowsky operators $C_{n,\alpha}$ as follows:

$$C_{n,\alpha}(f,x)=\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)Q_{nk}^{(\alpha)}\left(\frac{x}{b_n}\right),$$

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where $\alpha > 0$ and

$$Q_{nk}^{(\alpha)}(x/b_n) = J_{n,k}^{\alpha}(x/b_n) - J_{n,k+1}^{\alpha}(x/b_n), \qquad J_{n,k}(x/b_n) = \sum_{j=k}^{n} p_{nj}(x/b_n).$$

Obviously for $\alpha = 1$, the operators $C_{n,\alpha}$ reduce to the operators C_n.

Let

$$K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) = \begin{cases} \sum_{kb_n \leqslant nt} Q_{nk}^{(\alpha)}\left(\frac{x}{b_n}\right), & 0 < t \leqslant b_n; \\ 0, & t = 0. \end{cases}$$

By Lebesgue-Stieltjes intergral representation, we have

$$C_{n,\alpha}(f,x) = \int_0^{b_n} f(t) d_t K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right).$$

In [1], H. Karsli and E. Ibikli studied the convergence rate of $C_{n,\alpha}$ to bounded variation functions for the case $\alpha \geqslant 1$. Unfortunately, in the proof of the results, the authors made some mistakes as follows:

(1) [1, Lemma 2] For all $x \in (0,\infty)$ and $0 \leqslant t < x$, we have

$$\lambda_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) = \int_0^t K_{n,\alpha}\left(\frac{x}{b_n}, \frac{u}{b_n}\right) du \leqslant \frac{\alpha}{(x-t)^2} \frac{x(b_n - x)}{n}.$$

In fact, the result should be

$$K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant \frac{\alpha}{(x-t)^2} \frac{x(b_n - x)}{n}.$$

(2) In [1, p. 439], the authors mistook $I_{1,\alpha}(n,x) = \int_0^{x-x/\sqrt{n}} g_x(t) d_t (\lambda_{n,\alpha}(\frac{x}{b_n}, \frac{t}{b_n}))$, but in fact $I_{1,\alpha}(n,x) = \int_0^{x-x/\sqrt{n}} g_x(t) d_t (K_{n,\alpha}(\frac{x}{b_n}, \frac{t}{b_n}))$. The representations of $I_{2,\alpha}(n,x)$ and $I_{3,\alpha}(n,x)$ were also wrong.

These two mistakes resulted in a lot of errors in the following proof of [1, Theorem].

In this paper, we re-discuss the pointwise approximation of $C_{n,\alpha}$ to bounded variation functions for the case $\alpha > 0$ which includes $\alpha \geqslant 1$. We also mention some of the important papers on this subject by Gupta [7] and Pych-Taberska [8].

The main theorems of this paper are as follows.

THEOREM 1. *Let $\alpha \geqslant 1$, f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists, i.e. $f \in BV[0,\infty)$. Then for every $x \in (0,b_n)$, we have*

$$\left|C_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right| \leqslant \frac{3\alpha b_n^2}{nx(b_n - x)} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}} (g_x)$$

$$+ \frac{\alpha b_n}{\sqrt{nx(b_n - x)}} \left(|f(x+) - f(x-)| + \varepsilon_n(x/b_n)|f(x) - f(x-)|\right).$$

THEOREM 2. *Let* $0 < \alpha \leqslant 1$, f *be a function of bounded variation on every finite subinterval of* $[0,\infty)$ *and* $\lim_{x\to\infty} f(x)$ *exists, i.e.* $f \in BV[0,\infty)$. *Then for every* $x \in (0,b_n)$ *and* $n > \frac{256b_n^2}{25x(b_n-x)}$, *we have*

$$\left|C_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \leqslant \frac{A_\alpha b_n^2}{nx(b_n-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x)$$

$$+\frac{b_n}{\sqrt{nx(b_n-x)}}\big(|f(x+)-f(x-)|+\varepsilon_n(x/b_n)|f(x)-f(x-)|\big),$$

where A_α *is a positive constant depending only on* α,

$$g_x(t) = \begin{cases} f(t)-f(x+), & x<t\leqslant b_n; \\ 0, & t=x; \\ f(t)-f(x-), & 0\leqslant t<x. \end{cases} \tag{1}$$

$$\varepsilon_n(x/b_n) = \begin{cases} 1, & \text{if } x=\frac{k'b_n}{n}, \quad \text{for some } k'\in N; \\ 0, & \text{if } x\neq\frac{kb_n}{n}, \quad \text{for all } k\in N. \end{cases} \tag{2}$$

When $b_n \equiv 1$, the operators $C_{n,\alpha}(f,x)$ are just the Bernstein-Bézier operators $B_{n,\alpha}(f,x) = \sum_{k=0}^{n} f(\frac{k}{n})Q_{nk}^{(\alpha)}(x)$, which were studied by Zeng [2,3]. Therefore, our theorems extend the results of Zeng. Moreover, in the case $0<\alpha\leqslant 1$, Zeng [2] gave a rate of convergence of $B_{n,\alpha}$ for bounded variation functions as follows:

Let $0<\alpha\leqslant 1$, f be a function of bounded variation on $[0,1]$ $(f\in BV[0,1])$. Then for every $x\in(0,1)$ and $n>\frac{256}{25}(x(1-x))^{-1}$ we have

$$\left|B_{n,\alpha}(f,x) - \frac{1}{2^\alpha}f(x+) - \left(1-\frac{1}{2^\alpha}\right)f(x-)\right| \leqslant \frac{A_\alpha}{n(x(1-x))^{2-\alpha}}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$

$$+\frac{1}{\sqrt{nx(1-x)}}\big(|f(x+)-f(x-)|+\varepsilon_n(x)|f(x)-f(x-)|\big). \tag{3}$$

Obviously, for $b_n\equiv 1$, our Theorem 2 extends and improves the result of (3).

2. Lemmas

The proof of our results are based on the following lemmas.

LEMMA 1. *For every* $x\in(0,b_n)$ *and* $0\leqslant k\leqslant n$, *we have*

$$p_{nk}(x/b_n) \leqslant \frac{b_n}{\sqrt{2enx(b_n-x)}}. \tag{4}$$

Proof. By [4, Theorem 1], we have $p_{nk}(t) < \frac{1}{\sqrt{2ent(1-t)}}$ for $0<t<1$. Replacing t for x/b_n, we can get (4) easily. □

The following Lemma is the well-known Berry-Esseen bound for the central limit theorem of probability theory. Its proof can be found in Shiryayev [5, p. 432].

LEMMA 2. *Let $\{\xi_k\}_{k=1}^{+\infty}$ be a sequence of independent and identically distributed random variables with finite variance such that the expectation $E(\xi_1)=a_1\in R$, the variance $Var(\xi_1)=E(\xi_1-a_1)^2=b_1^2>0$ and $E|\xi_1-E(\xi_1)|^3<+\infty$. Then there exists a constant C, $1/\sqrt{2}\pi\leqslant C<0.8$, such that for all n and t,*

$$\left|P\left(\frac{1}{b_1\sqrt{n}}\sum_{k=1}^{n}(\xi_k-a_1)\leqslant t\right)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-u^2/2}du\right|<C\frac{E|\xi_1-E(\xi_1)|^3}{b_1^3\sqrt{n}} \tag{5}$$

LEMMA 3. *For $x\in(0,b_n)$, we have*

$$\left|\sum_{nx/b_n<k\leqslant n}p_{nk}(x/b_n)-\frac{1}{2}\right|<\frac{0.8b_n}{\sqrt{nx(b_n-x)}}. \tag{6}$$

Proof. Let ξ_1 be the random variable with two-point distribution $P(\xi_1=i)=(\frac{x}{b_n})^i(1-\frac{x}{b_n})^{1-i}$ ($i=0,1$, $x\in(0,b_n)$ is a parameter). Hence $a_1=E(\xi_1)=x/b_n$, $b_1^2=E(\xi_1-a_1)^2=\frac{x}{b_n}(1-\frac{x}{b_n})$, and $E|\xi_1-E(\xi_1)|^3=\frac{x}{b_n}(1-\frac{x}{b_n})[(\frac{x}{b_n})^2+(1-\frac{x}{b_n})^2]$. Let $\{\xi_k\}_{k=1}^{+\infty}$ be a sequence of independent random variables identically distributed with ξ_1, $\eta_n=\sum_{j=1}^{n}\xi_j$. Then the probability distribution of the random variable η_n is

$$P(\eta_n=k)=\binom{n}{k}(x/b_n)^k(1-x/b_n)^{n-k}=p_{nk}(x/b_n).$$

So

$$\sum_{nx/b_n<k\leqslant n}p_{nk}(x/b_n)=P(nx/b_n<\eta_n\leqslant n)=1-P(\eta_n\leqslant nx/b_n)$$
$$=1-P\left(\frac{\eta_n-nx/b_n}{\sqrt{n}\sqrt{\frac{x}{b_n}(1-\frac{x}{b_n})}}\leqslant 0\right).$$

By (5), we get

$$\left|\sum_{nx/b_n<k\leqslant n}p_{nk}(x/b_n)-\frac{1}{2}\right|=\left|P\left(\frac{\eta_n-nx/b_n}{\sqrt{n}\sqrt{\frac{x}{b_n}(1-\frac{x}{b_n})}}\leqslant 0\right)-\frac{1}{2}\right|$$
$$<\frac{C}{\sqrt{n}}\frac{E|\xi_1-E(\xi_1)|^3}{b_1^3}<\frac{0.8[(\frac{x}{b_n})^2+(1-\frac{x}{b_n})^2]b_n}{\sqrt{nx(b_n-x)}}<\frac{0.8b_n}{\sqrt{nx(b_n-x)}}.$$

This completes the proof of (6). □

LEMMA 4. *For $\alpha\geqslant 1$ and $x\in(0,b_n)$, $k'=nx/b_n$, we have*

$$\text{(i)}\qquad \left|\left(\sum_{nx/b_n<k\leqslant n}p_{nk}(x/b_n)\right)^{\alpha}-\frac{1}{2^{\alpha}}\right|\leqslant\frac{0.8\alpha b_n}{\sqrt{nx(b_n-x)}}\,, \tag{7}$$

$$\text{(ii)} \qquad Q_{nk'}^{(\alpha)}(x/b_n) < \frac{\alpha b_n}{\sqrt{2enx(b_n - x)}} . \tag{8}$$

Proof. (i) From the fact that $|x^\alpha - y^\alpha| \leqslant \alpha|x-y|$ with $0 \leqslant x,\ y \leqslant 1$ and $\alpha \geqslant 1$, we get (7) from (6) easily.

(ii) Using the same method of (i), we obtain $Q_{nk'}^{(\alpha)}(x/b_n) \leqslant \alpha p_{nk'}(x/b_n)$.

(8) now follows from (4) immediately. □

LEMMA 5. *For* $0 < \alpha \leqslant 1$ *and* $x \in (0, b_n)$, *as* $n > \frac{256b_n^2}{25x(b_n - x)}$ *and* $k' = nx/b_n$, *we have*

$$\text{(i)} \qquad \left| \left(\sum_{nx/b_n < k} p_{nk}(x/b_n) \right)^\alpha - \frac{1}{2^\alpha} \right| < \frac{b_n}{\sqrt{nx(b_n - x)}} , \tag{9}$$

$$\text{(ii)} \qquad Q_{nk'}^{(\alpha)}(x/b_n) < \frac{b_n}{\sqrt{nx(b_n - x)}} . \tag{10}$$

Proof. (i) By mean value theorem, we have

$$\left| \left(\sum_{nx/b_n < k \leqslant n} p_{nk}(x/b_n) \right)^\alpha - \frac{1}{2^\alpha} \right| = \alpha \left(\xi_{nk}(x/b_n)\right)^{\alpha-1} \left| \left(\sum_{nx/b_n < k \leqslant n} p_{nk}(x/b_n) \right) - \frac{1}{2} \right|, \tag{11}$$

where $\xi_{nk}(x/b_n)$ lies between $\frac{1}{2}$ and $\sum_{nx/b_n < k \leqslant n} p_{nk}(x/b_n)$.

In view of (6) and all $n > \frac{256b_n^2}{25x(b_n - x)}$, we have

$$\sum_{nx/b_n < k \leqslant n} p_{nk}(x/b_n) > \frac{1}{4} . \tag{12}$$

Hence $\xi_{nk}(x/b_n) > \frac{1}{4}$ holds for $n > \frac{256b_n^2}{25x(b_n - x)}$.

From (11), (6) and noting $3.2\alpha < 4^\alpha$, we get (9) immediately.

(ii) Using the mean value theorem, we get

$$\begin{aligned} Q_{nk'}^{(\alpha)}(x/b_n) &= \alpha\big(\eta_{nk'}(x/b_n)\big)^{\alpha-1}[J_{n,k'}(x/b_n) - J_{n,k'+1}(x/b_n)] \\ &= \alpha\Big(\frac{1}{\eta_{nk'}(x/b_n)}\Big)^{1-\alpha} p_{n,k'}(x/b_n) , \end{aligned} \tag{13}$$

where $J_{n,k'+1}(x/b_n) < \eta_{nk'}(x/b_n) < J_{n,k'}(x/b_n)$.

But in view of (12), we know

$$\eta_{nk'}(x/b_n) > J_{n,k'+1}(x/b_n) = \sum_{j > nx/b_n} p_{nj}(x/b_n) > \frac{1}{4}.$$

From (13),(4) and noting $2\alpha < 4^\alpha$, we deduce that

$$Q_{nk'}^{(\alpha)}(x/b_n) < \frac{\alpha 4^{1-\alpha} b_n}{\sqrt{2enx(b_n - x)}} < \frac{b_n}{\sqrt{nx(b_n - x)}}. \quad \square$$

LEMMA 6. *(i) For $\alpha \geqslant 1$ and $0 \leqslant t < x$, there holds*

$$K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant \frac{\alpha x(b_n - x)}{n(x-t)^2}. \tag{14}$$

(ii) For $\alpha \geqslant 1$ and $0 \leqslant x < t$, there holds

$$1 - K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant \frac{\alpha x(b_n - x)}{n(x-t)^2}. \tag{15}$$

Proof. (i) By a simple calculation, we get

$$C_n(1,x) = 1,$$

$$C_n(t,x) = x,$$

$$C_n(t^2,x) = x^2 + \frac{x(b_n - x)}{n}.$$

Thus

$$C_n((t-x)^2,x) = \frac{x(b_n - x)}{n}. \tag{16}$$

Now from the fact that $|x^\alpha - y^\alpha| \leqslant \alpha|x-y|$ with $0 \leqslant x,\ y \leqslant 1$ and $\alpha \geqslant 1$, we get

$$\begin{aligned} K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) &= \sum_{kb_n \leqslant nt} Q_{nk}^{(\alpha)}\left(\frac{x}{b_n}\right) \leqslant \alpha \sum_{kb_n \leqslant nt} p_{nk}\left(\frac{x}{b_n}\right) \\ &\leqslant \alpha \sum_{kb_n \leqslant nt} \frac{(kb_n/n - x)^2}{(t-x)^2} p_{nk}\left(\frac{x}{b_n}\right) \leqslant \alpha \frac{C_n((t-x)^2,x)}{(t-x)^2}. \end{aligned}$$

(14) now follows from (16).

(ii) Using a similar method we can get (15) easily. □

LEMMA 7. *(i) For $0 < \alpha \leqslant 1$ and $0 \leqslant t < x$, there holds*

$$K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant K_{n,1}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant \frac{x(b_n - x)}{n(x-t)^2}. \tag{17}$$

(ii) For $0 < \alpha \leqslant 1$ and $0 \leqslant x < t$, there holds

$$1 - K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) \leqslant \frac{A_\alpha x(b_n - x)}{n(x-t)^2}, \tag{18}$$

where A_α is a positive constant depending only on α.

Proof. (i) Along the same lines of the proof of [2, Lemma 4] and the inequality of (14), we can get (17) easily.

(ii) Since $0 \leqslant x < t$, so $|\frac{kb_n}{n} - x|/|t-x| \geqslant 1$ for all $k \geqslant nt/b_n$. Thus we have

$$\begin{aligned}
1 - K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right) &= 1 - \sum_{k \leqslant nt/b_n} Q_{nk}^{(\alpha)}(x/b_n) \leqslant \sum_{k \geqslant nt/b_n} Q_{nk}^{(\alpha)}(x/b_n) \\
&= \sum_{k \geqslant nt/b_n} \left(J_{n,k}^{\alpha}(x/b_n) - J_{n,k+1}^{\alpha}(x/b_n)\right) = \left(\sum_{k \geqslant\ nt/b_n} p_{nk}(x/b_n)\right)^{\alpha} \\
&\leqslant \left(\sum_{k \geqslant nt/b_n} \frac{|\frac{kb_n}{n} - x|^{2/\alpha}}{|t-x|^{2/\alpha}} p_{nk}(x/b_n)\right)^{\alpha} \\
&\leqslant \frac{b_n^2}{(t-x)^2} \left(\sum_{k=0}^{n} \left|\frac{k}{n} - \frac{x}{b_n}\right|^{2/\alpha} p_{nk}(x/b_n)\right)^{\alpha}.
\end{aligned}$$

Then, by Hölder's inequality with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned}
\left(\sum_{k=0}^{n} \left|\frac{k}{n} - \frac{x}{b_n}\right|^{2/\alpha} p_{nk}(x/b_n)\right)^{\alpha} &= \left(\sum_{k=0}^{n} \left|\frac{k}{n} - \frac{x}{b_n}\right|^{2/\alpha} (p_{nk}(x/b_n))^{1/p} (p_{nk}(x/b_n))^{1/q}\right)^{\alpha} \\
&\leqslant \left(\sum_{k=0}^{n} \left|\frac{k}{n} - \frac{x}{b_n}\right|^{2p/\alpha} p_{nk}(x/b_n)\right)^{\alpha/p}.
\end{aligned}$$

Choosing $p = \alpha[1/\alpha + 1]$, then $2p/\alpha = 2[1/\alpha + 1]$ is an even positive integer. From [6, Theorem 1.5.1], we have

$$\left(\sum_{k=0}^{n} \left|\frac{k}{n} - \frac{x}{b_n}\right|^{2/\alpha} p_{nk}(x/b_n)\right)^{\alpha} \leqslant A_{\alpha} \frac{x}{b_n}\left(1 - \frac{x}{b_n}\right) n^{-1},$$

where A_{α} is a positive constant depending only on α. This completes the proof of (18). □

LEMMA 8. *(i) For $\alpha \geqslant 1$, $f \in BV[0,\infty)$ and $x \in (0, b_n)$, we have*

$$\left|C_{n,\alpha}(g_x, x)\right| \leqslant \frac{3\alpha b_n^2}{nx(b_n - x)} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x + (b_n - x)/\sqrt{k}} (g_x). \tag{19}$$

(ii) For $0 < \alpha \leqslant 1$, $f \in BV[0,\infty)$ and $x \in (0, b_n)$, when $n > \frac{256 b_n^2}{25x(b_n - x)}$, we have

$$\left|C_{n,\alpha}(g_x, x)\right| \leqslant \frac{A_{\alpha} b_n^2}{nx(b_n - x)} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x + (b_n - x)/\sqrt{k}} (g_x). \tag{20}$$

Proof. (i) We recall the Lebesgue-Stieltjes integral representations

$$C_{n,\alpha}(g_x, x) = \int_0^{b_n} g_x(t) d_t K_{n,\alpha}\left(\frac{x}{b_n}, \frac{t}{b_n}\right). \tag{21}$$

Decompose the integral of (21) into three parts as follows

$$C_{n,\alpha}(g_x,x)=\int_0^{b_n} g_x(t)d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)=\Sigma_1+\Sigma_2+\Sigma_3\,, \tag{22}$$

where

$$\Sigma_1=\int_0^{x-\frac{x}{\sqrt{n}}} g_x(t)d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right), \qquad \Sigma_2=\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} g_x(t)d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right),$$

$$\Sigma_3=\int_{x+\frac{b_n-x}{\sqrt{n}}}^{b_n} g_x(t)d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right).$$

Observing that $g_x(x)=0$, we first have

$$\begin{aligned}\left|\Sigma_2\right| &= \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}} |g_x(t)-g_x(x)|d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\\ &\leqslant \bigvee_{x-x/\sqrt{n}}^{x+(b_n-x)/\sqrt{n}}(g_x)\leqslant \frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x).\end{aligned} \tag{23}$$

To estimate Σ_1, let $y=x-x/\sqrt{n}$. Using Lebesgue-Stieltjes integration by parts and (14), we have

$$\begin{aligned}\left|\Sigma_1\right| &= \left|\int_0^{y} g_x(t)d_tK_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\right|\\ &= \left|g_x(y+)K_{n,\alpha}\left(\frac{x}{b_n},\frac{y}{b_n}\right)-\int_0^{y}K_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)d_tg_x(t)\right|\\ &\leqslant \bigvee_{y+}^{x}(g_x)K_{n,\alpha}\left(\frac{x}{b_n},\frac{y}{b_n}\right)+\int_0^{y}K_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)d_t(-\bigvee_{t}^{x}(g_x))\\ &\leqslant \bigvee_{y+}^{x}(g_x)\frac{\alpha x(b_n-x)}{n(x-y)^2}+\frac{\alpha x(b_n-x)}{n}\int_0^{y}\frac{1}{(x-t)^2}d_t(-\bigvee_{t}^{x}(g_x)).\end{aligned}$$

Since

$$\int_0^{y}\frac{1}{(x-t)^2}d_t(-\bigvee_{t}^{x}(g_x))=-\frac{\bigvee_t^x(g_x)}{(x-t)^2}\Big|_0^{y+}+\int_0^{y}\frac{2\bigvee_t^x(g_x)}{(x-t)^3}dt,$$

we have

$$\left|\Sigma_1\right|\leqslant\frac{\alpha x(b_n-x)}{nx^2}\bigvee_{0}^{x}(g_x)+\frac{\alpha x(b_n-x)}{n}\int_0^{y}\frac{2\bigvee_t^x(g_x)}{(x-t)^3}dt.$$

Putting $t=x-x/\sqrt{u}$ for the last integral, we get

$$\begin{aligned}\left|\Sigma_1\right| &\leqslant \frac{\alpha x(b_n-x)}{nx^2}\bigvee_{0}^{x}(g_x)+\frac{\alpha x(b_n-x)}{nx^2}\int_1^{n}\bigvee_{x-x/\sqrt{u}}^{x}(g_x)du\\ &\leqslant \frac{\alpha x(b_n-x)}{nx^2}\left[\bigvee_{0}^{x}(g_x)+\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x)\right].\end{aligned} \tag{24}$$

Using the similar method and (15) to estimate $|\Sigma_3|$, we obtain

$$|\Sigma_3| \leqslant \frac{\alpha x(b_n - x)}{n(b_n - x)^2}\left[\bigvee_x^{b_n}(g_x) + \sum_{k=1}^{n}\bigvee_{x}^{x+(b_n-x)/\sqrt{k}}(g_x)\right]. \tag{25}$$

Combining the estimates of (22), (23), (24) and (25), also noting the properties of $\bigvee_a^b(f)$ and $1/(n-1) \leqslant \alpha b_n^2/[nx(b_n-x)]$ for $x \in (0, b_n)$, we get

$$\begin{aligned}
|C_{n,\alpha}(g_x, x)| &\leqslant \frac{\alpha[(b_n-x)^2 + x^2]}{nx(b_n-x)}\left[\bigvee_0^{b_n}(g_x) + \sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x)\right] \\
&\quad + \frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x) \\
&\leqslant \frac{2\alpha b_n^2}{nx(b_n-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x) + \frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x) \\
&\leqslant \frac{3\alpha b_n^2}{nx(b_n-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(b_n-x)/\sqrt{k}}(g_x).
\end{aligned}$$

This completes the proof of (19).

(ii) Using the same method and (17), (18), we can also get (20) easily. □

3. Proof of Theorem 1 and Theorem 2

Let f satisfy the conditions of Theorem 1 and Theorem 2. We can decompose $f(t)$ into four parts as

$$\begin{aligned}
f(t) = &\frac{1}{2^\alpha}f(x+) + (1 - \frac{1}{2^\alpha})f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha}\widehat{sign}(t-x) \\
&+ \delta_x(t)\left[f(x) - \frac{1}{2^\alpha}f(x+) - \left(1 - \frac{1}{2^\alpha}\right)f(x-)\right],
\end{aligned}$$

where

$$\widehat{sign}(t-x) = \begin{cases} 2^\alpha - 1, & t > x; \\ 0, & t = x; \\ -1, & t < x. \end{cases}$$

$$\delta_x(t) = \begin{cases} 1, & t = x; \\ 0, & t \neq x. \end{cases}$$

$g_x(t)$ is defined in (1). Therefore,

$$\begin{aligned}&\left|C_{n,\alpha}(f,x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right|\\ &\leqslant |C_{n,\alpha}(g_x,x)|+\left|\frac{f(x+)-f(x-)}{2^\alpha}C_{n,\alpha}\left(\widehat{sign}(t-x),x\right)\right.\\ &\quad\left.+\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right]C_{n,\alpha}(\delta_x,x)\right|. \qquad (26)\end{aligned}$$

By direct calculation, we get

$$C_{n,\alpha}(\delta_x,x)=\varepsilon_n(x/b_n)Q_{nk'}^{(\alpha)}(x/b_n)$$

and

$$\begin{aligned}C_{n,\alpha}\left(\widehat{sign}(t-x),x\right)&=\sum_{k>nx/b_n}(2^\alpha-1)Q_{nk}^{(\alpha)}(x/b_n)+\sum_{k<nx/b_n}(-1)Q_{nk}^{(\alpha)}(x/b_n)\\ &=2^\alpha\sum_{k>nx/b_n}Q_{nk}^{(\alpha)}(x/b_n)-1+\varepsilon_n(x/b_n)Q_{nk'}^{(\alpha)}(x/b_n)\\ &=2^\alpha\left(\sum_{k>nx/b_n}p_{nk}(x/b_n)\right)^\alpha-1+\varepsilon_n(x/b_n)Q_{nk'}^{(\alpha)}(x/b_n),\end{aligned}$$

where $\varepsilon_n(x/b_n)$ is defined in (2).

Therefore, we have

$$\begin{aligned}&\left|\frac{f(x+)-f(x-)}{2^\alpha}C_{n,\alpha}\left(\widehat{sign}(t-x),x\right)+\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right]C_{n,\alpha}(\delta_x,x)\right|\\ &=\left|[f(x+)-f(x-)]\left[\left(\sum_{\frac{nx}{b_n}<k}p_{nk}(x/b_n)\right)^\alpha-\frac{1}{2^\alpha}\right]+[f(x)-f(x-)]\varepsilon_n(x/b_n)Q_{nk'}^{(\alpha)}(x/b_n)\right|. \qquad (27)\end{aligned}$$

By combining the estimates given by (26), (19), (27), (7) and (8), we obtain Theorem 1; and by combining the estimates given by (26), (20), (27), (9) and (10), we obtain Theorem 2.

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