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REFINEMENTS OF BOUNDS FOR THE FIRST AND SECOND SEIFFERT MEANS

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Abstract. In this paper, we find the greatest values α, λ and the least values β, μ such that the double inequalities $\alpha[5A(a,b)/6+H(a,b)/6]+(1-\alpha)A^{5/6}(a,b)H^{1/6}(a,b) < P(a,b) < \beta[5A(a,b)/6+H(a,b)/6]+(1-\beta)A^{5/6}(a,b)H^{1/6}(a,b)$ and $\lambda[A(a,b)/3+2Q(a,b)/3]+(1-\lambda)A^{1/3}(a,b)Q^{2/3}(a,b) < T(a,b) < \mu[A(a,b)/3+2Q(a,b)/3]+(1-\mu)A^{1/3}(a,b)Q^{2/3}(a,b)$ hold for all $a,b>0$ with $a\neq b$. Here $A(a,b)$, $H(a,b)$, $Q(a,b)$, $P(a,b)$ and $T(a,b)$ denote the arithmetic, harmonic, quadratic, first Seiffert and second Seiffert means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a\neq b$, the first and second Seiffert means $P(a,b)$ [13] and $T(a,b)$ [14] are defined by

$$P(a,b)=\frac{a-b}{4\arctan(\sqrt{a/b})-\pi} \tag{1.1}$$

and

$$T(a,b)=\frac{a-b}{2\arctan[(a-b)/(a+b)]}, \tag{1.2}$$

respectively.

Recently, both means P and T have been the subject of intensive research. In particular, many remarkable inequalities for P and T can be found in the literature [2, 4, 5, 7–12, 14–17]. The first Seiffert mean $P(a,b)$ can be rewritten as (see [9], Eq. (2.4))

$$P(a,b)=\frac{a-b}{2\arcsin[(a-b)/(a+b)]}. \tag{1.3}$$

Let $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, $H(a,b)=2ab/(a+b)$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $L(a,b)=(b-a)/(\log b-\log a)$, and $L_r(a,b)=(a^{r+1}+$

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$b^{r+1})/(a^r+b^r)$ be the arithmetic, geometric, harmonic, quadratic, identric, logarithmic and r-th Lehmer means of two positive real numbers a and b with $a \neq b$. Then

$$\min\{a,b\} < H(a,b) = L_{-1}(a,b) < G(a,b) = L_{-1/2}(a,b) < L(a,b) < I(a,b) \\ < A(a,b) = L_0(a,b) < Q(a,b) < \max\{a,b\}.$$

Seiffert [13–15] established that

$$L(a,b) < P(a,b) < I(a,b), \tag{1.4}$$

$$A(a,b) < T(a,b) < Q(a,b),$$

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)}, \tag{1.5}$$

$$P(a,b) > \frac{A(a,b)G(a,b)}{L(a,b)}, \tag{1.6}$$

$$P(a,b) > \frac{2}{\pi}A(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [6], Jagers proved that the inequality

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b) \tag{1.7}$$

holds for $a,b>0$ with $a \neq b$, where $M_r(a,b) = [(a^r+b^r)/2]^{1/r}$ $(r \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ denotes the r-th power mean of a and b.

According to Carlson [1] and Pfaff [3], Sándor [11] found that the first Seiffert mean $P(a,b)$ is the common limit of the sequences given by

$$x_0 = G(a,b), \quad y_0 = A(a,b), \quad x_{n+1} = \frac{x_n+y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1}y_n}, (n \geqslant 0),$$

and by using the sequential method, the following more general results were given:

$$x_n < P(a,b) < y_n, \tag{1.8}$$

$$\sqrt[3]{x_n y_n^2} < P(a,b) < \frac{x_n+2y_n}{3} \tag{1.9}$$

for all $n \geqslant 0$ and $a,b>0$ with $a \neq b$. In particular, for $n=1$ from (1.8) and $n=0$ from (1.9) one has

$$\frac{A(a,b)+G(a,b)}{2} < P(a,b) < \sqrt{\frac{A(a,b)+G(a,b)}{2}A(a,b)},$$

$$A(a,b)^{2/3}G(a,b)^{1/3} < P(a,b) < \frac{2A(a,b)+G(a,b)}{3}. \tag{1.10}$$

The lower bound in (1.10) are better than that in (1.5) and (1.6), and the upper bound in (1.10) are better than that in (1.4) and (1.7) (see [11]). In fact, infinitely many refinements for $P(a,b)$ have been proved by use of (1.8) and (1.9).

Wang and Chu [16] proved that the inequality

$$P(a,b) > A^{5/6}(a,b)H^{1/6}(a,b) \tag{1.11}$$

holds for all $a,b>0$ with $a \neq b$.

Indeed, inequality (1.11) is exactly the left side of inequality (1.10) because of $A(a,b)H(a,b) = G^2(a,b)$. Therefore, it due to Sándor [11].

In [5, 7], the authors given the bounds for P and T in terms of power mean as follows

$$M_{\log 2/\log \pi}(a,b) < P(a,b),$$

$$M_{\log 2/\log(\pi/2)}(a,b) < T(a,b) < M_{5/3}(a,b)$$

for all $a,b>0$ with $a \neq b$.

Recently, Chu et al. [2, 17] proved that the inequalities

$$L_{-1/6}(a,b) < P(a,b) < L_0(a,b),$$

$$L_0(a,b) < T(a,b) < L_{1/3}(a,b)$$

and

$$\frac{2}{\pi}A(a,b) + \left(1 - \frac{2}{\pi}\right)H(a,b) < P(a,b) < \frac{5}{6}A(a,b) + \frac{1}{6}H(a,b) \tag{1.12}$$

hold for $a,b>0$ with $a \neq b$.

In [12], Sándor found that $T(a,b)$ is the common limit of the sequences $\{u_n\}$ and $\{v_n\}$ given by

$$u_0 = A(a,b), \quad v_0 = Q(a,b), \quad u_{n+1} = \frac{u_n + v_n}{2}, \quad v_{n+1} = \sqrt{u_{n+1}v_n}, \quad (n \geqslant 0),$$

and established a more general inequality:

$$\sqrt[3]{u_n v_n^2} < T(a,b) < \frac{u_n + 2v_n}{3} \tag{1.13}$$

for all $n \geqslant 0$ and $a,b>0$ with $a \neq b$. Particular, for $n=0$ and $n=1$ from (1.13) we get

$$Q^{2/3}(a,b)A^{1/3}(a,b) < T(a,b) < \frac{2}{3}Q(a,b) + \frac{1}{3}A(a,b), \tag{1.14}$$

$$\sqrt[3]{Q(a,b)\left(\frac{Q(a,b)+A(a,b)}{2}\right)^2} < T(a,b)$$

$$< \frac{1}{3}\left(\frac{Q(a,b)+A(a,b)}{2} + 2\sqrt{\frac{Q(a,b)+A(a,b)}{2}Q(a,b)}\right).$$

In fact, infinitely many refinements for $T(a,b)$ have been proved by use of (1.13).

Motivated by inequalities (1.10), (1.12) and (1.14), it is natural to ask what are the greatest values α, λ and the least values β, μ such that the double inequalities

$$\alpha[5A(a,b)/6+H(a,b)/6]+(1-\alpha)A^{5/6}(a,b)H^{1/6}(a,b)<P(a,b)$$
$$<\beta[5A(a,b)/6+H(a,b)/6]+(1-\beta)A^{5/6}(a,b)H^{1/6}(a,b)$$

and

$$\lambda\,[A(a,b)/3+2Q(a,b)/3]+(1-\lambda)A^{1/3}(a,b)Q^{2/3}(a,b)<T(a,b)$$
$$<\mu\,[A(a,b)/3+2Q(a,b)/3]+(1-\mu)A^{1/3}(a,b)Q^{2/3}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

The purpose of this paper is to answer these questions. All numerical computations are carried out using MATHEMATICA software.

2. Lemmas

In order to establish our main results we need two lemmas, which we present in this section.

LEMMA 2.1. *Let* $f(x)=-p^2x^{12}-2p^2x^{11}-3p^2x^{10}-(4p^2+6p)x^9-(5p^2+12p)x^8+(6p^2-30p)x^7+(7p^2-48p)x^6+(8p^2-66p)x^5+(9p^2-108p+24)x^4+(10p^2-108p+48)x^3-(25p^2+36p-36)x^2+24(1-p)x+12(1-p)$. *Then the following statements are true:*

(1) If $p=8/25$, *then* $f(x)>0$ *for* $x\in(0,1)$;

(2) If $p=12/(5\pi)$, *then there exists* $\gamma\in(0,1)$ *such that* $f(x)>0$ *for* $x\in(0,\gamma)$ *and* $f(x)<0$ *for* $x\in(\gamma,1)$.

Proof. Part (1) follows easily from

$$f(x)=\frac{4}{625}(1-x)(16x^{11}+48x^{10}+96x^9+460x^8+1140x^7+2544x^6$$
$$+4832x^5+8004x^4+9510x^3+7250x^2+3825x+1275)>0$$

for all $x\in(0,1)$ if $p=8/25$.

For part (2), if $p=12/(5\pi)$, then simple computations lead to

$$6p^2-30p=\frac{72(12-25\pi)}{25\pi^2}<0, \tag{2.1}$$

$$7p^2-48p=\frac{144(7-20\pi)}{25\pi^2}<0, \tag{2.2}$$

$$8p^2-66p=\frac{144(8-25\pi)}{25\pi^2}<0, \tag{2.3}$$

$$9p^2-108p+24=\frac{24(54-270\pi+25\pi^2)}{25\pi^2}<0, \tag{2.4}$$

$$10p^2 - 108p + 48 = \frac{48(6 - 27\pi + 5\pi^2)}{5\pi^2} < 0, \tag{2.5}$$

$$-25p^2 - 36p + 36 = \frac{36(-20 - 12\pi + 5\pi^2)}{5\pi^2} < 0, \tag{2.6}$$

$$f(0) = 12(1-p) > 0, \tag{2.7}$$

$$f(1) = 144 - 450p = 144 - \frac{1080}{\pi} < 0, \tag{2.8}$$

$$\begin{aligned} f'(x) =& -12p^2x^{11} - 22p^2x^{10} - 30p^2x^9 - 9(4p^2+6p)x^8 - 8(5p^2+12p)x^7 \\ &+ 7(6p^2-30p)x^6 + 6(7p^2-48p)x^5 + 5(8p^2-66p)x^4 \\ &+ 4(9p^2-108p+24)x^3 + 3(10p^2-108p+48)x^2 \\ &- 2(25p^2+36p-36)x + 24(1-p), \end{aligned}$$

$$f'(0) = 24(1-p) > 0, \tag{2.9}$$

$$f'(1) = 336 - 1830p = \frac{24(14\pi - 183)}{\pi} < 0 \tag{2.10}$$

and

$$\begin{aligned} f''(x) =& -132p^2x^{10} - 220p^2x^9 - 270p^2x^8 - 72(4p^2+6p)x^7 - 56(5p^2+12p)x^6 \\ &+ 42(6p^2-30p)x^5 + 30(7p^2-48p)x^4 + 20(8p^2-66p)x^3 \\ &+ 12(9p^2-108p+24)x^2 + 6(10p^2-108p+48)x \\ &- 2(25p^2+36p-36). \end{aligned} \tag{2.11}$$

It follows from (2.1)–(2.6) and (2.11) that

$$f''(x) < 0 \tag{2.12}$$

for $x \in (0,1)$. Hence $f'(x)$ is strictly decreasing on $(0,1)$.

Inequalities (2.9) and (2.10) together with the monotonicity of $f'(x)$ lead to the conclusion that there exists $x_0 \in (0,1)$, such that $f'(x) > 0$ for $x \in (0,x_0)$ and $f'(x) < 0$ for $x \in (x_0,1)$. Thus $f(x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on $(x_0,1)$.

Therefore, part (2) follows from (2.7) and (2.8) together with the piecewise monotonicity of $f(x)$. $\square$

LEMMA 2.2. *Let* $g(x) = 3(1-q)x^6 + 6(1-q)x^5 - (4q^2+6q-9)x^4 + (4q^2 - 18q+12)x^3 + (3q^2-12q+6)x^2 - 2q^2x - q^2$. *Then the following statements are true:*

(1) If $q = 4/5$, *then* $g(x) > 0$ *for* $x \in (1, \sqrt[6]{2})$;

(2) If $q = [12/\pi - 3\sqrt[3]{2}]/[2\sqrt{2} + 1 - 3\sqrt[3]{2}] = 0.821\cdots$, *then there exists* $\xi \in (1, \sqrt[6]{2})$ *such that* $g(x) < 0$ *for* $x \in (1,\xi)$ *and* $g(x) > 0$ *for* $x \in (\xi, \sqrt[6]{2})$.

Proof. Part (1) follows easily from

$$g(x) = \frac{1}{25}(x-1)(15x^5 + 45x^4 + 86x^3 + 90x^2 + 48x + 16) > 0$$

for all $x \in (1, \sqrt[6]{2})$ if $q = 4/5$.

For part (2), if $q = [12/\pi - 3\sqrt[3]{2}]/[2\sqrt{2} + 1 - 3\sqrt[3]{2}] = 0.821\cdots$, then simple computations lead to

$$-4q^2 - 6q + 9 = 1.377\cdots > 0, \tag{2.13}$$

$$4q^2 - 18q + 12 = -0.0823\cdots < 0, \tag{2.14}$$

$$3q^2 - 12q + 6 = -1.8302\cdots < 0, \tag{2.15}$$

$$g(1) = 36 - 45q < 0, \tag{2.16}$$

$$g(\sqrt[6]{2}) = 0.563\cdots > 0 \tag{2.17}$$

and

$$\begin{aligned} g'(x) =& 18(1-q)x^5 + 30(1-q)x^4 - 4(4q^2 + 6q - 9)x^3 + 3(4q^2 - 18q + 12)x^2 \\ &+ 2(3q^2 - 12q + 6)x - 2q^2. \end{aligned} \tag{2.18}$$

It follows from (2.13)–(2.15) and (2.18) that

$$\begin{aligned} g'(x) >& 18(1-q)x^2 + 30(1-q)x^2 - 4(4q^2 + 6q - 9)x^2 + 3(4q^2 - 18q + 12)x^2 \\ &+ 2(3q^2 - 12q + 6)x^2 - 2q^2x^2 \\ =& 6(22 - 25q)x^2 > 0 \end{aligned} \tag{2.19}$$

for $x \in (1, \sqrt[6]{2})$. Hence $g(x)$ is strictly increasing on $(1, \sqrt[6]{2})$.

Therefore, part (2) follows from (2.16) and (2.17) together with the monotonicity of $g(x)$. $\square$

3. Main results

THEOREM 3.1. *The double inequality*

$$\begin{aligned} &\alpha\left[\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)\right] + (1-\alpha)A^{5/6}(a,b)H^{1/6}(a,b) < P(a,b) \\ &< \beta\left[\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)\right] + (1-\beta)A^{5/6}(a,b)H^{1/6}(a,b) \end{aligned}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant 8/25$ *and* $\beta \geqslant 12/(5\pi)$.

Proof. Firstly, we prove that the inequalities

$$P(a,b) > \frac{8}{25}\left[\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)\right] + \frac{17}{25}A^{5/6}(a,b)H^{1/6}(a,b), \tag{3.1}$$

$$P(a,b) < \frac{12}{5\pi}\left[\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)\right] + \left(1 - \frac{12}{5\pi}\right)A^{5/6}(a,b)H^{1/6}(a,b) \tag{3.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Since $P(a,b)$, $A(a,b)$ and $H(a,b)$ are symmetric and homogenous of degree 1. Without loss generality, we assume that $a>b$. Let $r=(a-b)/(a+b)$, $r'=\sqrt{1-r^2}$ and $p\in\{8/25, 12/(5\pi)\}$. Then $r\in(0,1)$,

$$\frac{P(a,b)}{A(a,b)}=\frac{r}{\arcsin(r)}, \quad \frac{H(a,b)}{A(a,b)}=1-r^2,$$

$$\frac{P(a,b)-A^{5/6}(a,b)H^{1/6}(a,b)}{5A(a,b)/6+H(a,b)/6-A^{5/6}(a,b)H^{1/6}(a,b)}=\frac{r/\arcsin r-(1-r^2)^{1/6}}{5/6+(1-r^2)/6-(1-r^2)^{1/6}}, \tag{3.3}$$

$$\lim_{r\to 0^+}\frac{r/\arcsin r-(1-r^2)^{1/6}}{5/6+(1-r^2)/6-(1-r^2)^{1/6}}=\frac{8}{25}, \tag{3.4}$$

$$\lim_{r\to 1^-}\frac{r/\arcsin r-(1-r^2)^{1/6}}{5/6+(1-r^2)/6-(1-r^2)^{1/6}}=\frac{12}{5\pi}, \tag{3.5}$$

$$\begin{aligned}
&P(a,b)-p\left[\frac{5}{6}A(a,b)+\frac{1}{6}H(a,b)\right]-(1-p)A^{5/6}(a,b)H^{1/6}(a,b)\\
=&A(a,b)\left[\frac{r}{\arcsin r}-p\left(1-\frac{1}{6}r^2\right)-(1-p)r'^{1/3}\right]\\
=&\frac{A(a,b)[5p+pr'^2+6(1-p)r'^{1/3}]}{6\arcsin r}\left[\frac{6r}{5p+pr'^2+6(1-p)r'^{1/3}}-\arcsin r\right].
\end{aligned} \tag{3.6}$$

Let

$$F(r)=\frac{6r}{5p+pr'^2+6(1-p)r'^{1/3}}-\arcsin r. \tag{3.7}$$

Then simple computations yield

$$F(0)=0, \tag{3.8}$$

$$F(1)=\frac{6}{5p}-\frac{\pi}{2}, \tag{3.9}$$

$$F'(r)=\frac{(1-r'^{1/3})^2 f(r'^{1/3})}{r'^{5/3}\left[5p+pr'^2+6(1-p)r'^{1/3}\right]^2}, \tag{3.10}$$

where the function $f(\cdot)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1 $p=8/25$. Then from (3.10) and Lemma 2.1(1) we clearly see that $F'(r)>0$ for $r\in(0,1)$. Thus $F(r)$ is strictly increasing on $(0,1)$.

Therefore, inequality (3.1) follows from (3.6)–(3.8) together with the monotonicity of $F(r)$.

Case 2 $p=12/(5\pi)$. Then from (3.10) and Lemma 2.1(2) we know that there exists $\lambda_0\in(0,1)(=\sqrt{1-\gamma^6})$ such that $F'(r)<0$ for $r\in(0,\lambda_0)$ and $F'(r)>0$ for $r\in(\lambda_0,1)$. Hence $F(r)$ is strictly decreasing on $(0,\lambda_0)$ and strictly increasing on $(\lambda_0,1)$.

Note that equation (3.9) becomes

$$F(1) = 0. \tag{3.11}$$

Therefore, inequality (3.2) follows from (3.6)–(3.8) and (3.11) together with the piecewise monotonicity of $F(r)$, and Theorem 3.1 follows from (3.1) and (3.2) in conjunction with the following statements.

- If $\alpha > 8/25$, then equations (3.3) and (3.4) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $P(a,b) < \alpha[5A(a,b)/6 + H(a,b)/6] + (1-\alpha)A^{5/6}(a,b)H^{1/6}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_1)$.
- If $\beta < 12/(5\pi)$, then equations (3.3) and (3.5) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $P(a,b) > \beta[5A(a,b)/6 + H(a,b)/6] + (1-\beta)A^{5/6}(a,b)H^{1/6}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

THEOREM 3.2. *The double inequality*

$$\begin{aligned} \lambda\left[\frac{1}{3}A(a,b) + \frac{2}{3}Q(a,b)\right] + (1-\lambda)A^{1/3}(a,b)Q^{2/3}(a,b) < T(a,b) \\ < \mu\left[\frac{1}{3}A(a,b) + \frac{2}{3}Q(a,b)\right] + (1-\mu)A^{1/3}(a,b)Q^{2/3}(a,b) \end{aligned}$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\lambda \leqslant 4/5$ *and* $\mu \geqslant \mu_0 = [12/\pi - 3\sqrt[3]{2}]/[2\sqrt{2} + 1 - 3\sqrt[3]{2}] = 0.82104\cdots$.

Proof. Firstly, we prove that the inequalities

$$T(a,b) > \frac{4}{5}\left[\frac{1}{3}A(a,b) + \frac{2}{3}Q(a,b)\right] + \frac{1}{5}A^{1/3}(a,b)Q^{2/3}(a,b), \tag{3.12}$$

$$T(a,b) < \mu_0\left[\frac{1}{3}A(a,b) + \frac{2}{3}Q(a,b)\right] + (1-\mu_0)A^{1/3}(a,b)Q^{2/3}(a,b) \tag{3.13}$$

hold for all $a,b > 0$ with $a \neq b$.

Since $T(a,b)$, $A(a,b)$ and $Q(a,b)$ are symmetric and homogenous of degree 1. Without loss generality, we assume that $a > b$. Let $r = (a-b)/(a+b)$, $r^* = \sqrt{1+r^2}$ and $q \in \{4/5, \mu_0\}$. Then $r^* \in (1, \sqrt{2})$,

$$\frac{T(a,b)}{A(a,b)} = \frac{r}{\arctan(r)}, \quad \frac{Q(a,b)}{A(a,b)} = \sqrt{1+r^2},$$

$$\frac{T(a,b) - A^{1/3}(a,b)Q^{2/3}(a,b)}{A(a,b)/3 + 2Q(a,b)/3 - A^{1/3}(a,b)Q^{2/3}(a,b)} = \frac{r/\arctan r - (1+r^2)^{1/3}}{1/3 + 2\sqrt{1+r^2}/3 - (1+r^2)^{1/3}}, \tag{3.14}$$

$$\lim_{r\to 0^+} \frac{r/\arctan r - (1+r^2)^{1/3}}{1/3 + 2\sqrt{1+r^2}/3 - (1+r^2)^{1/3}} = \frac{4}{5}, \tag{3.15}$$

$$\lim_{r\to 1^-} \frac{r/\arctan r-(1+r^2)^{1/3}}{1/3+2\sqrt{1+r^2}/3-(1+r^2)^{1/3}}=\mu_0, \tag{3.16}$$

$$\begin{aligned}
&T(a,b)-q\left[\frac{1}{3}A(a,b)+\frac{2}{3}Q(a,b)\right]-(1-q)A^{1/3}(a,b)Q^{2/3}(a,b)\\
=&A(a,b)\left[\frac{r}{\arctan r}-\frac{q}{3}(1+2r^*)-(1-q)r^{*2/3}\right]\\
=&\frac{A(a,b)[2qr^*+q+3(1-q)r^{*2/3}]}{3\arctan r}\left[\frac{3r}{2qr^*+q+3(1-q)r^{*2/3}}-\arctan r\right].
\end{aligned} \tag{3.17}$$

Let

$$G(r)=\frac{3r}{2qr^*+q+3(1-q)r^{*2/3}}-\arctan r. \tag{3.18}$$

Then simple computations yield

$$G(0)=0, \tag{3.19}$$

$$G(1)=\frac{3}{2\sqrt{2}q+q+3(1-q)\sqrt[3]{2}}-\frac{\pi}{4}, \tag{3.20}$$

$$G'(r)=\frac{(1-r^{*1/3})^2 g(r^{*1/3})}{r^{*2}\left[2qr^*+q+3(1-q)r^{*2/3}\right]^2}, \tag{3.21}$$

where the function $g(\cdot)$ is defined as in Lemma 2.2.

We divide the proof into two cases.

Case A $q=4/5$. Then from (3.21) and Lemma 2.2(1) we clearly see that $G'(r)>0$ for $r\in(0,1)$. Thus $G(r)$ is strictly increasing on $(0,1)$.

Therefore, inequality (3.12) follows from (3.17)–(3.19) together with the monotonicity of $G(r)$.

Case B $q=\mu_0$. Then from (3.21) and Lemma 2.2(2) we know that there exists $\lambda_0^*\in(0,1)(=\sqrt{\xi^6-1})$ such that $G'(r)<0$ for $r\in(0,\lambda_0^*)$ and $G'(r)>0$ for $r\in(\lambda_0^*,1)$. Hence $G(r)$ is strictly decreasing on $(0,\lambda_0^*)$ and strictly increasing on $(\lambda_0^*,1)$.

Note that equation (3.19) reduces to

$$G(1)=0. \tag{3.22}$$

Therefore, inequality (3.13) follows from (3.17)–(3.19) and (3.22) together with the piecewise monotonicity of $G(r)$, and Theorem 3.2 follows from (3.12) and (3.13) in conjunction with the following statements.

- If $\lambda>4/5$, then equations (3.14) and (3.15) lead to the conclusion that there exists $0<\delta_3<1$ such that $T(a,b)<\lambda\,[A(a,b)/3+2Q(a,b)/3]+(1-\lambda)A^{1/3}(a,b)Q^{2/3}(a,b)$ for all $a,b>0$ with $(a-b)/(a+b)\in(0,\delta_3)$.

- If $\mu<\mu_0$, then equations (3.14) and (3.16) imply that there exists $0<\delta_4<1$ such that $T(a,b)>\mu\,[A(a,b)/3+2Q(a,b)/3]+(1-\mu)A^{1/3}(a,b)Q^{2/3}(a,b)$ for all $a,b>0$ with $(a-b)/(a+b)\in(1-\delta_4,1)$. $\square$

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