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THE STEINER SYMMETRIZATION OF LOG–CONCAVE FUNCTIONS AND ITS APPLICATIONS

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Abstract. In this paper, we give a new definition of functional Steiner symmetrizations on log-concave functions. Using the functional Steiner symmetrization, we give a new proof of the classical Prékopa-Leindler inequality on log-concave functions.

1. Introduction

Functional Steiner symmetrization, as a kind of important rearrangement of functions, has been studied in [1, 3, 4, 5, 6, 7, 8]. For a nonnegative measurable function f, the familiar definition of its Steiner symmetrization (see [3, 4, 5, 8]) is defined as follows:

DEFINITION 1.1. For a nonnegative measurable function f on $\mathbb{R}^n$ vanishing at infinity, its Steiner symmetrization is defined as

$$\bar{S}_u f(x) = \int_0^\infty \mathscr{X}_{\bar{S}_u E(t)}(x)dt, \tag{1.1}$$

where $\bar{S}_u E(t)$ is the Steiner symmetrization of the level set $E(t) := \{x \in \mathbb{R}^n : f(x) > t\}$ about the hyperplane $u^\perp$ and $\mathscr{X}_{\bar{S}_u E(t)}$ denotes the characteristic function of $\bar{S}_u E(t)$.

In this paper, for log-concave functions, we give a new definition of the functional Steiner symmetrization. Our definition provides a new approach to the original defintion, but we do not use geometric Steiner symmetrization and our approach is more suitable for certain functional problems.

DEFINITION 1.2. For an integrable log-concave function $f : \mathbb{R}^n \to \mathbb{R}$ and a hyperplane $H = u^\perp$ $(u \in S^{n-1})$ in $\mathbb{R}^n$, for any $x = x' + tu \in \mathbb{R}^n$, where $x' \in H$ and $t \in \mathbb{R}$, we define the *Steiner symmetrization* $S_u f$ (or $S_H f$) of f about H by

$$(S_u f)(x) = \inf_{\lambda \in [0,1]} \sup_{t_1 \in \mathbb{R}} [f(x' + t_1 u)^\lambda f(x' + (t_1 + 2t)u)^{1-\lambda}]. \tag{1.2}$$

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By Remark 1, $S_u f$ is also log-concave.

A central inequality connected with the Minkowski sum of two bodies $A,B\subset\mathbb{R}^n$ and a parameter $0\leqslant\lambda\leqslant 1$ is the Brunn-Minkowski inequality:

$$Vol_n(\lambda A+(1-\lambda)B)\geqslant Vol_n(A)^{\lambda}Vol_n(B)^{1-\lambda}.$$

The Prékopa-Leindler inequality (e.g., [10]) is the functional analogue of the Brunn-Minkowski inequality: For given log-concave functions $f,g\in L^1:\mathbb{R}^n\to\mathbb{R}^+$ and $0\leqslant\lambda\leqslant 1$,

$$\int_{\mathbb{R}^n}\sup\{f(x)^{\lambda}g(y)^{1-\lambda}:\lambda x+(1-\lambda)y=z\}dz\geqslant\left(\int_{\mathbb{R}^n}f(z)dz\right)^{\lambda}\left(\int_{\mathbb{R}^n}g(z)dz\right)^{1-\lambda}.$$

Prékopa-Leindler inequality is closely related to the reverse Young's inequality and a number of other important and classical inequalities in analysis. Prékopa-Leindler inequality is a reverse form of Hölder's inequality and a particular case of the reverse Brascamp-Lieb inequalities (see [2, 9]). In this paper, using the functional Steiner symmetrization, we give a new proof of the Prékopa-Leindler inequality on the log-concave functions.

2. The equivalence between new and original definitions

In this section, for log-concave functions, we prove the new definition is equivalent to the original definition. First, we give the Steiner symmetrization on convex sets.

DEFINITION 2.1. Let K be a non-empty convex set in $\mathbb{R}^n$ and let H be a hyperplane in $\mathbb{R}^n$ with unit normal vector u. The *Steiner symmetrization* $S_H K$ of K about H is defined as:

$$S_H K=\{x'+\frac{1}{2}(t_1-t_2)u:\ x'\in P_H(K),\ t_i\in I_K(x')\text{ for }i=1,2\},\tag{2.1}$$

where $P_H(K)=\{x'\in H:\ x'+tu\in K\text{ for some }t\in\mathbb{R}\}$ is the projection of K onto the hyperplane H and $I_K(x')=\{t\in\mathbb{R}:\ x'+tu\in K\}$.

It is well-known that the Steiner symmetrization of the subgraph of f is equivalent to the subgraph of the Steiner symmetrization of f, i.e.,

$$S_u(\mathscr{S}_f)=\mathscr{S}_{\bar{S}_u f},$$

where $\mathscr{S}_f=\{(x,t)\in\mathbb{R}^{n+1}:0<t\leqslant f(x)\}$ denotes the subgraph of f and $\bar{S}_u f$ is given by Definition 1.1. Therefore, in order to prove the equivalence between Definition 1.1 and Definition 1.2, it is sufficient to prove that

$$S_u(\mathscr{S}_f)=\mathscr{S}_{S_u f},\tag{2.2}$$

where $S_u f$ is given in Definition 1.2.

PROPOSITION 2.1. *For an integrable log-concave function* $f:\mathbb{R}^n\to\mathbb{R}$ *and a hyperplane* $H=u^{\perp}$ *(*$u\in S^{n-1}$*) in* $\mathbb{R}^n$*, for any* $x'\in H$*, let* $\phi_1(t)=S_uf(x'+tu)$ *and* $\phi(t)=f(x'+tu)$*, then* ϕ_1 *is an even log-concave function about* $t\in\mathbb{R}$ *and for any* $s\geqslant 0$

$$Vol_1([\phi_1\geqslant s])=Vol_1([\phi\geqslant s]).$$

Proof. By equality (1.2), it is clear that ϕ_1 is an even function. For any $s\geqslant 0$, if $[\phi\geqslant s]=[t_0,t_0+2t]$, next we prove that ϕ_1 is log-concave and $[\phi_1\geqslant s]=[-t,t]$. First we prove

$$\phi_1(t)=\phi(t_0)=\phi(t_0+2t)=s, \tag{2.3}$$

where $\phi(t_0)=\phi(t_0+2t)=s$ is clear. By (1.2), we have

$$\begin{aligned}\phi_1(t)&=\inf_{\lambda\in[0,1]}\sup_{t_1\in\mathbb{R}}[f(x'+t_1u)^{\lambda}f(x'+(t_1+2t)u)^{1-\lambda}]\\&\geqslant\inf_{\lambda\in[0,1]}[f(x'+t_0u)^{\lambda}f(x'+(t_0+2t)u)^{1-\lambda}]\\&=s.\end{aligned}\tag{2.4}$$

On the other hand, we prove that there is some $\lambda\in(0,1)$ such that

$$\sup_{t_1\in\mathbb{R}}[\phi(t_1)^{\lambda}\phi(t_1+2t)^{1-\lambda}]=s.$$

Since ϕ is a log-concave function defined in $\mathbb{R}$ and by Theorem 1.5.2 in [11], both the right derivative ϕ_r' and the left derivative ϕ_l' exist. It is clear that $\phi_r'(t_0+2t)\leqslant 0$ and $\phi_r'(t_0)\geqslant 0$, if $\phi_r'(t_0)-\phi_r'(t_0+2t)\neq 0$, then let $\lambda_0=\frac{-\phi_r'(t_0+2t)}{\phi_r'(t_0)-\phi_r'(t_0+2t)}$; if $\phi_r'(t_0)-\phi_r'(t_0+2t)=0$, since $\phi_r'(t_0+2t)\leqslant 0$ and $\phi_r'(t_0)\geqslant 0$, we have $\phi_r'(t_0+2t)=0$ and $\phi_r'(t_0)=0$, for this case, let λ_0 be any real number on $(0,1)$. Let $\Phi(t')=\phi(t')^{\lambda_0}\phi(t'+2t)^{1-\lambda_0}$, then $\Phi(t')$ is also log-concave, its right derivative at $t'=t_0$ satisfies

$$\Phi_r'(t_0)=\lambda_0\phi_r'(t_0)+(1-\lambda_0)\phi_r'(t_0+2t)=0.$$

Thus

$$\sup_{t_1\in\mathbb{R}}[\phi(t_1)^{\lambda_0}\phi(t_1+2t)^{1-\lambda_0}]=[\phi(t_0)^{\lambda_0}\phi(t_0+2t)^{1-\lambda_0}]=s.$$

Thus, $\phi_1(t)=s$.

Next, we prove that ϕ_1 is log-concave. Since ϕ_1 is even, it suffices to prove that for any $0\leqslant t_1<t_2$ and $0<\alpha<1$,

$$\phi_1(\alpha t_1+(1-\alpha)t_2)\geqslant\phi_1^{\alpha}(t_1)\phi_1^{1-\alpha}(t_2). \tag{2.5}$$

By (2.3), there are t_0, t_1' and t_2' satisfying

$$\phi_1(t_1)=\phi(t_1')=\phi(t_1'-2t_1), \tag{2.6}$$

$$\phi_1(t_2) = \phi(t_2') = \phi(t_2' - 2t_2) \tag{2.7}$$

and

$$\phi_1(\alpha t_1 + (1-\alpha)t_2) = \phi(t_0) = \phi(t_0 - 2(\alpha t_1 + (1-\alpha)t_2)). \tag{2.8}$$

Since ϕ is log-concave, we have

$$\phi(\alpha t_1' + (1-\alpha)t_2') \geqslant \phi(t_1')^{\alpha}\phi(t_2')^{1-\alpha} = \phi_1(t_1)^{\alpha}\phi_1(t_2)^{1-\alpha} \tag{2.9}$$

and

$$\begin{aligned}\phi(\alpha t_1' + (1-\alpha)t_2' - 2(\alpha t_1 + (1-\alpha)t_2)) &\geqslant \phi(t_1' - 2t_1)^{\alpha}\phi(t_2' - 2t_2)^{1-\alpha} \\ &= \phi_1(t_1)^{\alpha}\phi_1(t_2)^{1-\alpha}.\end{aligned} \tag{2.10}$$

Since the distance between $\alpha t_1' + (1-\alpha)t_2'$ and $\alpha t_1' + (1-\alpha)t_2' - 2(\alpha t_1 + (1-\alpha)t_2)$ is $2(\alpha t_1 + (1-\alpha)t_2)$ and ϕ is log-concave, by the second equality in (2.8), we have

$$\phi(t_0) \geqslant \phi(\alpha t_1' + (1-\alpha)t_2') \tag{2.11}$$

or

$$\phi(t_0 - 2(\alpha t_1 + (1-\alpha)t_2)) \geqslant \phi(\alpha t_1' + (1-\alpha)t_2' - 2(\alpha t_1 + (1-\alpha)t_2)). \tag{2.12}$$

By (2.8)-(2.12), we have

$$\phi_1(\alpha t_1 + (1-\alpha)t_2) \geqslant \phi_1^{\alpha}(t_1)\phi_1^{1-\alpha}(t_2),$$

which implies that ϕ_1 is log-concave.

Since ϕ_1 is even and log-concave and (2.3), we have $[\phi_1 \geqslant s] = [-t, t]$, which implies that for any $s \geqslant 0$, $Vol_1([\phi_1 \geqslant s]) = Vol_1([\phi \geqslant s])$. □

REMARK 1. By the similar proof of Proposition 2.1, we can prove that $S_u f$ given in (1.2) is also log-concave. By Proposition 2.1, for any $x' \in u^{\perp}$, if $\phi(t) = f(x' + tu)$, then $S_u(\mathscr{S}_{\phi}) = \mathscr{S}_{S_u\phi}$. Since $x' \in u^{\perp}$ is arbitrary, we have $S_u(\mathscr{S}_f) = \mathscr{S}_{S_u f}$.

3. Proof of Prékopa-Leindler inequality

LEMMA 3.1. *Let $0 < \lambda < 1$ and let $f,\ g \in L^1 : \mathbb{R}^+ \to \mathbb{R}^+$ be log-concave functions, then*

$$\begin{aligned}&\int_0^{+\infty} \sup\{f(x)^{\lambda}g(y)^{1-\lambda} : x \geqslant 0, y \geqslant 0, \lambda x + (1-\lambda)y = z\}dz \\ &\geqslant \left(\int_0^{+\infty} f(z)dz\right)^{\lambda}\left(\int_0^{+\infty} g(z)dz\right)^{1-\lambda}.\end{aligned} \tag{3.1}$$

Proof. We can assume without loss of generality that f and g are bounded with

$$\sup_{x\in\mathbb{R}^+} f(x) = \sup_{x\in\mathbb{R}^+} g(x) = 1.$$

If $t \geqslant 0$, $x \geqslant 0$ and $y \geqslant 0$, $f(x) \geqslant t$, and $g(y) \geqslant t$, let

$$\Phi(z) = \sup\{f(x)^{\lambda} g(y)^{1-\lambda} : \ x \geqslant 0,\ y \geqslant 0,\ \lambda x + (1-\lambda)y = z\},$$

then

$$\Phi(\alpha x + (1-\alpha)y) \geqslant f(x)^{\lambda} g(y)^{1-\lambda} \geqslant t.$$

With the notation for upper level sets,

$$[\Phi \geqslant t] \supset \alpha[f \geqslant t] + (1-\alpha)[g \geqslant t],$$

where $[f \geqslant t] = \{x \in \mathbb{R}^+ : f(x) \geqslant t\}$. The sets on the right-hand side are nonempty, so by Fubini's theorem and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
\int_0^{+\infty} \Phi(z)dz &\geqslant \int_0^1 Vol_1([\Phi \geqslant t])dt \geqslant \int_0^1 Vol_1(\alpha[f \geqslant t] + (1-\alpha)[g \geqslant t])dt \\
&= \alpha \int_0^1 Vol_1([f \geqslant t])dt + (1-\alpha)\int_0^1 Vol_1([g \geqslant t])dt \\
&= \alpha \int_0^{+\infty} f(z)dz + (1-\alpha)\int_0^{+\infty} g(z)dz \\
&\geqslant \left(\int_0^{+\infty} f(z)dz\right)^{\alpha} \left(\int_0^{+\infty} g(z)dz\right)^{1-\alpha}. \qquad (3.2)
\end{aligned}$$

□

LEMMA 3.2. *If h_1, h_2 are one-dimensional increasing convex functions defined on $[0,+\infty)$, then, for $0 < \lambda < 1$,*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup\{e^{-[\lambda h_1(|x|)+(1-\lambda)h_2(|y|)]} : \lambda x + (1-\lambda)y = z\}dz \\
&\geqslant \left(\int_{\mathbb{R}^n} e^{-h_1(|z|)}dz\right)^{\lambda} \left(\int_{\mathbb{R}^n} e^{-h_2(|z|)}dz\right)^{1-\lambda}.
\end{aligned}$$

Proof. By the polar coordinate transformation and the monotonicity of functions h_1 and h_2, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup\{e^{-[\lambda h_1(|x|)+(1-\lambda)h_2(|y|)]} : \lambda x + (1-\lambda)y = z\}dz \\
&= \int_{S^{n-1}} \int_0^{+\infty} \sup\{e^{-[\lambda h_1(|x|)+(1-\lambda)h_2(|y|)]} : \lambda x + (1-\lambda)y = r\theta\} r^{n-1} dr d\theta \\
&= \int_{S^{n-1}} \int_0^{+\infty} \sup\{e^{-[\lambda h_1(r_1)+(1-\lambda)h_2(r_2)]} : \lambda r_1\theta_1 + (1-\lambda)r_2\theta_2 = r\theta\} r^{n-1} dr d\theta \\
&= \omega_n \int_0^{+\infty} \sup\{e^{-[\lambda h_1(r_1)+(1-\lambda)h_2(r_2)]} : \lambda r_1 + (1-\lambda)r_2 = r\} r^{n-1} dr. \qquad (3.3)
\end{aligned}$$

For $r_1 \geqslant 0$ and $r_2 \geqslant 0$ such that $\lambda r_1 + (1-\lambda) r_2 = r$, we have

$$r_1^{\lambda} r_2^{1-\lambda} \leqslant \lambda r_1 + (1-\lambda) r_2 = r.$$

Hence, we have

$$\begin{aligned}
&\sup\left\{ \left(e^{-h_1(r_1)} r_1^{n-1}\right)^{\lambda} \left(e^{-h_2(r_2)} r_2^{n-1}\right)^{1-\lambda} : \lambda r_1 + (1-\lambda) r_2 = r,\ r_1 \geqslant 0,\ r_2 \geqslant 0\right\} \\
\leqslant\ & r^{n-1} \sup\left\{ e^{-[\lambda h_1(r_1) + (1-\lambda) h_2(r_2)]} : \lambda r_1 + (1-\lambda) r_2 = r,\ r_1 \geqslant 0,\ r_2 \geqslant 0\right\}.
\end{aligned}$$

Therefore, by (3.3) and Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup\{e^{-[\lambda h_1(|x|) + (1-\lambda) h_2(|y|)]} : \lambda x + (1-\lambda) y = z\} dz \\
\geqslant\ & \omega_n \int_0^{+\infty} \sup\left\{ \left(e^{-h_1(r_1)} r_1^{n-1}\right)^{\lambda} \left(e^{-h_2(r_2)} r_2^{n-1}\right)^{1-\lambda} : \lambda r_1 + (1-\lambda) r_2 = r,\right. \\
&\qquad\qquad \left. r_1 \geqslant 0,\ r_2 \geqslant 0\right\} dr. \\
\geqslant\ & \left(\omega_n \int_0^{+\infty} e^{-h_1(r)} r^{n-1} dr\right)^{\lambda} \left(\omega \int_0^{+\infty} e^{-h_2(r)} r^{n-1} dr\right)^{1-\lambda} \\
=\ & \left(\int_{\mathbb{R}^n} e^{-h_1(|z|)} dz\right)^{\lambda} \left(\int_{\mathbb{R}^n} e^{-h_2(|z|)} dz\right)^{1-\lambda}.
\end{aligned}$$

this completes the proof. □

LEMMA 3.3. *Let $u \in S^{n-1}$ and $\lambda \in (0,1)$, if $\phi_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $e^{-\phi_i} \in L_1$, $i = 1,2$ then for every z*

$$\begin{aligned}
&\inf\{\lambda S_u \phi_1(x) + (1-\lambda) S_u \phi_2(y);\ \lambda x + (1-\lambda) y = z\} \\
\geqslant\ & S_u \left(\inf\{\lambda \phi_1(x) + (1-\lambda) \phi_2(y);\ \lambda x + (1-\lambda) y = z\}\right).
\end{aligned} \tag{3.4}$$

Proof. Let $x = x' + t_1 u$, $y = y' + t_2 u$ and $z = z' + tu$, where $x', y', z' \in u^{\perp}$, then we have

$$\begin{aligned}
& S_u \left(\inf\{\lambda \phi_1(x) + (1-\lambda) \phi_2(y);\ \lambda x + (1-\lambda) y = z\}\right) \\
=\ & \sup_{\alpha \in [0,1]} \inf_{t' \in \mathbb{R}} \left[\alpha \inf\left\{\lambda \phi_1(x) + (1-\lambda) \phi_2(y);\ \lambda x + (1-\lambda) y = z' + t'u\right\}\right. \\
& \left. + (1-\alpha) \inf\left\{\lambda \phi_1(x) + (1-\lambda) \phi_2(y);\ \lambda x + (1-\lambda) y = z' + (t' + 2t) u\right\}\right] \\
=\ & \sup_{\alpha \in [0,1]} \inf_{t' \in \mathbb{R}} \left[\alpha \inf\left\{\lambda \phi_1(x' + t_1' u) + (1-\lambda) \phi_2(y' + t_2' u);\ \lambda x' + (1-\lambda) y' = z',\right.\right. \\
& \left. \lambda t_1' + (1-\lambda) t_2' = t'\right\} + (1-\alpha) \inf\left\{\lambda \phi_1(x' + t_1'' u) + (1-\lambda) \phi_2(y' + t_2'' u);\right. \\
& \left.\left. \lambda x' + (1-\lambda) y' = z', \lambda t_1'' + (1-\lambda) t_2'' = t' + 2t\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\alpha\in[0,1]} \inf_{t'\in\mathbb{R}} \Bigg[\alpha \inf\left\{\lambda\phi_1(x'+t_1'u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\frac{t'-\lambda t_1'}{1-\lambda}u\right);\, x'\in u^\perp, t_1'\in\mathbb{R}\right\}\\
&\quad +(1-\alpha)\inf\left\{\lambda\phi_1(x'+t_1''u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\frac{t'+2t-\lambda t_1''}{1-\lambda}u\right);\, x'\in u^\perp, t_1''\in\mathbb{R}\right\}\Bigg]\\
&\leqslant \sup_{\alpha\in[0,1]} \inf_{(t',t_1',t_1'')\in\mathbb{R}^3} \inf_{x'\in u^\perp} \Bigg[\alpha\left\{\lambda\phi_1(x'+t_1'u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\frac{t'-\lambda t_1'}{1-\lambda}u\right)\right\}\\
&\quad +(1-\alpha)\left\{\lambda\phi_1(x'+t_1''u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\frac{t'+2t-\lambda t_1''}{1-\lambda}u\right)\right\}\Bigg]. \qquad (3.5)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\inf\{\lambda S_u\phi_1(x)+(1-\lambda)S_u\phi_2(y);\ \lambda x+(1-\lambda)y=z\}\\
&= \inf\left\{\lambda S_u\phi_1(x'+t_1u)+(1-\lambda)S_u\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\frac{t-\lambda t_1}{1-\lambda}u\right);\, x'\in u^\perp, t_1\in\mathbb{R}\right\}\\
&= \inf_{\{x'\in u^\perp, t_1\in\mathbb{R}\}} \Bigg\{\lambda \sup_{\alpha\in[0,1]}\inf_{t_1'\in\mathbb{R}}[\alpha\phi_1(x'+t_1'u)+(1-\alpha)\phi_1(x'+(t_1'+2t_1)u)]\\
&\quad +(1-\lambda)\sup_{\alpha\in[0,1]}\inf_{t_2'\in\mathbb{R}}\left[\alpha\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+t_2'u\right)+(1-\alpha)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+\left(t_2'+2\frac{t-\lambda t_1}{1-\lambda}\right)u\right)\right]\Bigg\}\\
&\geqslant \inf_{\{x'\in u^\perp,\, t_1\in\mathbb{R}\}}\sup_{\alpha\in[0,1]} \Bigg\{\lambda\inf_{t_1'\in\mathbb{R}}[\alpha\phi_1(x'+t_1'u)+(1-\alpha)\phi_1(x'+(t_1'+2t_1)u)]\\
&\quad +(1-\lambda)\inf_{t_2'\in\mathbb{R}}\left(\alpha\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+t_2'u\right)+(1-\alpha)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+(t_2'+2\frac{t-\lambda t_1}{1-\lambda})u\right)\right)\Bigg\}\\
&\geqslant \sup_{\alpha\in[0,1]}\inf_{\{x'\in u^\perp,\, t_1\in\mathbb{R}\}} \Bigg\{\lambda\inf_{t_1'\in\mathbb{R}}[\alpha\phi_1(x'+t_1'u)+(1-\alpha)\phi_1(x'+(t_1'+2t_1)u)]\\
&\quad +(1-\lambda)\inf_{t_2'\in\mathbb{R}}\left(\alpha\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+t_2'u\right)+(1-\alpha)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+(t_2'+2\frac{t-\lambda t_1}{1-\lambda})u\right)\right)\Bigg\}\\
&= \sup_{\alpha\in[0,1]}\inf_{\{x'\in u^\perp,\, (t_1,t_1',t_2')\in\mathbb{R}^3\}} \Bigg\{\lambda[\alpha\phi_1(x'+t_1'u)+(1-\alpha)\phi_1(x'+(t_1'+2t_1)u)]\\
&\quad +(1-\lambda)\left(\alpha\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+t_2'u\right)+(1-\alpha)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+(t_2'+2\frac{t-\lambda t_1}{1-\lambda})u\right)\right)\Bigg\}\\
&= \sup_{\alpha\in[0,1]}\inf_{\{x'\in u^\perp,\, (t_1,t_1',t_2')\in\mathbb{R}^3\}} \Bigg\{\alpha\left(\lambda\phi_1(x'+t_1'u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+t_2'u\right)\right)\\
&\quad +(1-\alpha)\left(\lambda\phi_1(x'+(t_1'+2t_1)u)+(1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda}+(t_2'+2\frac{t-\lambda t_1}{1-\lambda})u\right)\right)\Bigg\}. \qquad (3.6)
\end{aligned}
$$

In the above inequality, let $\begin{bmatrix} t_1' \\ t_2' \\ t_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\lambda}{1-\lambda} & 0 & \frac{1}{1-\lambda} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} T_1' \\ T_2' \\ T_1 \end{bmatrix}$, by (3.5) and (3.6), we have

$$\begin{aligned}
&\inf\{\lambda S_u\phi_1(x) + (1-\lambda)S_u\phi_2(y);\ \lambda x + (1-\lambda)y = z\} \\
\geqslant &\sup_{\alpha\in[0,1]} \inf_{(T_1,T_1',T_2')\in\mathbb{R}^3} \inf_{x'\in u^\perp} \left[\alpha\left\{\lambda\phi_1(x'+T_1'u) + (1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda} + \frac{T_1-\lambda T_1'}{1-\lambda}u\right)\right\}\right. \\
&\left.+(1-\alpha)\left\{\lambda\phi_1(x'+T_2'u) + (1-\lambda)\phi_2\left(\frac{z'-\lambda x'}{1-\lambda} + \frac{T_1+2t-\lambda T_2'}{1-\lambda}u\right)\right\}\right] \\
\geqslant &S_u(\inf\{\lambda\phi_1(x) + (1-\lambda)\phi_2(y);\ \lambda x + (1-\lambda)y = z\}). \qquad (3.7)
\end{aligned}$$

This completes the proof. □

THEOREM 3.4. *Let $0 < \lambda < 1$ and let $f,\ g \in L^1 : \mathbb{R}^n \to \mathbb{R}^+$ be log-concave functions, then*

$$\int_{\mathbb{R}^n} \sup\{f(x)^\lambda g(y)^{1-\lambda} : \lambda x + (1-\lambda)y = z\}dz \geqslant \left(\int_{\mathbb{R}^n} f(z)dz\right)^\lambda \left(\int_{\mathbb{R}^n} g(z)dz\right)^{1-\lambda}.$$

Proof. By the uniform convergence and integral invariance of functional Steiner symmetrizations (see, e.g., [3, 4, 8] for references), there is a sequence of directions $\{u_i\}$ so that

$$\lim_{i\to\infty} \|S_{u_i}\cdots S_{u_1}f(x) - e^{-h_1(|x|)}\|_1 = 0$$

and

$$\lim_{i\to\infty} \|S_{u_i}\cdots S_{u_1}g(x) - e^{-h_2(|x|)}\|_1 = 0,$$

where h_1 and h_2 are one-dimensional increasing convex functions and $\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} e^{-h_1(|x|)}dx$, $\int_{\mathbb{R}^n} g(x)dx = \int_{\mathbb{R}^n} e^{-h_2(|x|)}dx$. Taking limit $i\to\infty$, by the continuity of integral in $L^1(\mathbb{R}^n)$ and Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup\{f(x)^\lambda g(y)^{1-\lambda} : \lambda x + (1-\lambda)y = z\}dz \\
\geqslant &\int_{\mathbb{R}^n} \sup\{e^{-[\lambda h_1(|x|)+(1-\lambda)h_2(|y|)]} : \lambda x + (1-\lambda)y = z\}dz \\
\geqslant &\left(\int_{\mathbb{R}^n} e^{-h_1(|z|)}dz\right)^\lambda \left(\int_{\mathbb{R}^n} e^{-h_2(|z|)}dz\right)^{1-\lambda} \\
= &\left(\int_{\mathbb{R}^n} f(z)dz\right)^\lambda \left(\int_{\mathbb{R}^n} g(z)dz\right)^{1-\lambda}.
\end{aligned}$$

This completes the proof. □

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