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REMARK ON OZEKI INEQUALITY FOR CONVEX POLYGONS

IGOR Ž. MILOVANOVIĆ, EMINA I. MILOVANOVIĆ AND TATJANA Z. MIRKOVIĆ

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Abstract. This paper gives proof of a discrete inequality that represents Ozeki's inequality for convex polygons and its converse. The proof is based on determining eigenvalues of one nearly tridiagonal symmetric matrix.

1. Introduction

If $x_1, x_2, \ldots, x_{n+1}$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers, and $x_1 = x_{n+1}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \pi$, then the following inequality holds

$$\cos\frac{\pi}{4}\sum_{k=1}^{n} x_k^2 \geqslant \sum_{k=1}^{n} x_k x_{k+1}\cos\alpha_k. \tag{1}$$

This inequality was proved by N. Ozeki in [10]. It represents generalization of Wolstenholm [12], Lenhard [5], and Erdös-Mordell [4, 9] inequalities for convex polygons. This is probably the reason why it is met in the literature under different names: Wolstenholm inequality, Wolstenholm-Lenhard inequality or Erdös-Mordell inequality.

Inequality (1), as well as its generalization, has applications in solving various problems in geometry (see for example [8]). Therefore it was proved over and over again and has a lot of generalizations (see [3, 11, 13]). In this paper we are going to prove a discrete inequality which contains inequality (1) as its part. The proof, which is short, is based on computing eigenvalues of one symmetric nearly-tridiagonal matrix. The proof itself may be interesting in solving problems in other mathematical disciplines, such as differential and difference equations [1], matrix theory [2, 6], and inequalities [7].

2. The main result

LEMMA 1. *Suppose $x_1, x_2, \ldots, x_{n+1}$, $x_1 = x_{n+1}$ are real numbers. Then the following inequality is valid*

$$\cos\frac{(2\lceil\frac{n}{2}\rceil-1)\pi}{n}\sum_{k=1}^{n} x_k^2 \leqslant \sum_{k=1}^{n-1} x_k x_{k+1} - x_1 x_n \leqslant \cos\frac{\pi}{n}\sum_{k=1}^{n} x_k^2. \tag{2}$$

Equality on the right (left) side of inequality (2) holds if and only if

$$x_k = C\cdot\sin\frac{k\pi}{n}, \quad \left(x_k = C\cdot\sin\frac{k(2\lceil\frac{n}{2}\rceil-1)\pi}{n}\right), \qquad C>0.$$

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Proof. For the given real numbers $x_1, x_2, \ldots, x_{n+1}$, $x_1 = x_{n+1}$ we form the expression

$$F(\vec{x}) = \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n-1} x_k x_{k+1} - x_1 x_n = (H\vec{x}, \vec{x}), \tag{3}$$

where $\vec{x} = \begin{bmatrix} x_1 \; x_2 \; \ldots \; x_n \end{bmatrix}^T$, while H_n is a symmetric nearly-tridiagonal matrix of the form

$$H_n = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \ldots & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & & 0 & 0 \\ 0 & \frac{1}{2} & 1 & & 0 & 0 \\ \vdots & & & & & \\ -\frac{1}{2} & 0 & 0 & \ldots & \frac{1}{2} & 1 \end{bmatrix}.$$

Let $P_n(\lambda) = \det(\lambda I - H_n)$ be a characteristic polynom of matrix H_n, and $Q_n(\lambda) = 2^n \cdot P_n(\lambda)$, where each row of determinant $P_n(\lambda)$ is multiplied by 2. Further, let D_n be a tridiagonal determinant contained in $Q_n(\lambda)$. It satisfies the following three term recurrent relation

$$D_n = (2\lambda - 2)D_{n-1} + D_{n-2}, \quad D_0 = 1, \quad D_1 = 2\lambda - 2,$$

wherefrom we have that $D_k = \frac{\sin(k+1)\theta}{\sin\theta}$, $\theta \neq k\pi$, where $2\lambda - 2 = \cos\theta$. By developing determinant $Q_n(\lambda) = 2^n \cdot P_n(\lambda)$ over the elements of the first row and first column, we obtain

$$Q_n(\lambda) = (2\lambda - 2)D_{n-1} - 2D_{n-2} + 2 = 2\cos\theta + 2.$$

From the equality $Q_n(\lambda) = 0$, which is equivalent to $P_n(\lambda) = 0$, eigenvalues of matrix H_n are

$$\lambda_k = 1 + \cos\frac{(2k-1)\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

Since $\lambda_k \geqslant 0$, for each k, $\min\lambda_k = 1 + \cos\frac{(2\lceil\frac{n}{2}\rceil - 1)\pi}{n}$ and $\max\lambda_k = 1 + \cos\frac{\pi}{n}$, according to (3) we have

$$\left(1 + \cos\frac{(2\lceil\frac{n}{2}\rceil - 1)\pi}{n}\right)\sum_{k=1}^{n} x_k^2 \leqslant F(\vec{x}) \leqslant \left(1 + \cos\frac{\pi}{n}\right)\sum_{k=1}^{n} x_k^2.$$

From the above inequalities directly follows inequality (2). $\square$

THEOREM 1. *If $x_1, x_2, \ldots, x_{n+1}$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers, and $x_1 = x_{n+1}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \pi$, then the following inequality holds*

$$\cos\frac{(2\lceil\frac{n}{2}\rceil - 1)\pi}{n}\sum_{k=1}^{n} x_k^2 \leqslant \sum_{k=1}^{n} x_k x_{k+1}\cos\alpha_k \leqslant \cos\frac{\pi}{n}\sum_{k=1}^{n} x_k^2. \tag{4}$$

Proof. If we substitute $x_k := x_k \sin\beta_k$ and $x_k := x_k\cos\beta_k$, $k = 1, 2, \ldots, n$, $\beta_1 = \beta_{n+1}$ in (2), and then sum up the obtained inequalities, we obtain

$$\cos\frac{(\lceil\frac{n}{2}\rceil - 1)\pi}{n}\sum_{k=1}^{n} x_k^2 \leqslant \sum_{k=1}^{n} x_k x_{k+1}\cos(\beta_k - \beta_{k+1}) \leqslant \cos\frac{\pi}{2}\sum_{k=1}^{n} x_k^2.$$

Now, with the substitutions $\beta_k - \beta_{k+1} = \alpha_k$, for $k = 1, 2, \ldots, n-1$, and $\beta_n - \beta_1 = \alpha_n - \pi$, in the above inequality, we obtain the required result, i.e. the assertion of Theorem 1. $\square$

3. Conclusion

In this paper we have proved a discrete inequality for real sequences, which represents the extension of Ozeki's inequality for convex polygons.

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Igor Ž. Milovanović
Faculty of Electronic Engineering
A. Medvedeva 14, P. O. Box 73, 18000 Niš, Serbia
e-mail: igor@elfak.ni.ac.rs

Emina I. Milovanović
Faculty of Electronic Engineering
A. Medvedeva 14, P. O. Box 73, 18000 Niš, Serbia
e-mail: ema@elfak.ni.ac.rs

Tatjana Z. Mirković
State University of Novi Pazar, Novi Pazar, Serbia
e-mail: tmirkovic@np.ac.rs