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ESTIMATES FOR NEUMAN–SÁNDOR MEAN BY POWER MEANS AND THEIR RELATIVE ERRORS

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Abstract. For $a,b>0$ with $a\neq b$, let $NS(a,b)$ denote the Neuman-Sándor mean defined by

$$NS(a,b)=\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}$$

and $A_p(a,b)$, $\mathscr{L}_p(a,b)$ denote the r-order power and Lehmer means. Based on our earlier worker [27], we prove that

$$\alpha_p A_p<NS<A_p \text{ and } A_p<NS\leqslant\beta_p A_p$$

holds if and only if $p\geqslant 4/3$ and $p\leqslant p_0$, respectively, where

$$\alpha_p=\left(2^{1/p-1}\right)/\ln(1+\sqrt{2}) \text{ if } p\in[1/4/3,\infty),$$

$$\beta_p=\begin{cases} NS(1,x_0)/A_p(1,x_0) & \text{if } p\in(1,p_0], \\ 2^{1/p-1}/\ln\left(1+\sqrt{2}\right) & \text{if } p\in(0,1], \\ \infty & \text{if } p\in(-\infty,0] \end{cases}$$

are the best constants, here x_0 is the unique root of the equation

$$NS(1,x)=\frac{A(1,x)A_2(1,x)}{\mathscr{L}_{p_0-1}(1,x)}$$

on $(0,1)$, and $p\mapsto\alpha_p A_p$ is decreasing on $(0,\infty)$. Also, we have

$$\alpha_{4/3}A_{4/3}<A_{p_0}<NS<A_{4/3}<\alpha_{4/3}^{-1}A_{p_0}.$$

Our results clearly are generations of known ones.

1. Introduction

Throughout the paper, we assume that $a,b\in(0,\infty):=\mathbb{R}_+$ with $a\neq b$. The classical power mean of order p and Lehmer mean of a and b are defined by

$$A_p=A_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p} \text{ if } p\neq 0 \text{ and } A_0=A_0(a,b)=\sqrt{ab}, \tag{1.1}$$

$$\mathscr{L}_p=\mathscr{L}_p(a,b)=\frac{a^{p+1}+b^{p+1}}{a^p+b^p}, \tag{1.2}$$

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respectively. It is well-known that the functions $p \mapsto A_p(a,b)$ is continuous and strictly increasing on $\mathbb{R}$ (see [1], [9]) and log-convex on $(-\infty,0)$ and log-concave on $(0,\infty)$ (see [23, Conclusion 1]). While the function $p \mapsto \mathscr{L}_p(a,b)$ is also continuous and increasing on $\mathbb{R}$ (see [9], [22, Conclusion 1]) and log-convex on $(-\infty,-1/2)$ and log-concave on $(-1/2,\infty)$ (see [22, Conclusion 2]).

As special cases of power mean, the arithmetic mean, geometric mean and quadratic mean are $A = A(a,b) = A_1(a,b)$, $G = G(a,b) = A_0(a,b)$ and $Q = Q(a,b) = A_2(a,b)$, respectively. Clearly, the Lehmer mean can be expressed by power means as

$$\mathscr{L}_p = A_{p+1}^{p+1} A_p^{-p} \tag{1.3}$$

In 2003, Neuman and Sándor defined in [12] a new mean

$$NS = NS(a,b) = \frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}} = \frac{a-b}{2\ln\frac{a-b+\sqrt{2(a^2+b^2)}}{a+b}}, \tag{1.4}$$

and first established

$$G < L < P < A < NS < T < Q \tag{1.5}$$

and

$$\frac{\pi}{2}P > A > \operatorname{arcsinh}(1) NS > \frac{\pi}{2}T, \tag{1.6}$$

where L is the well-known logarithmic mean, P and T stand for the first and second Seiffert means (see [17], [16], [18]). Lately, Constin and Toader [5, Theorem 1] showed that $A_{3/2}$ can be put between NS and T, that is,

$$NS < A_{3/2} < T, \tag{1.7}$$

and they obtained the following nice chain of inequalities for certain means:

$$G < L < A_{1/2} < P < A < NS < A_{3/2} < T < A_2. \tag{1.8}$$

In 2012, Yang [27] first established the optimal evaluations for Neuman-Sándor mean by power means

$$A_{\frac{\ln 2}{\ln\ln(3+2\sqrt{2})}} < NS < A_{4/3}, \tag{1.9}$$

where $(\ln 2)/\ln\ln\left(3+2\sqrt{2}\right)$ and $4/3$ are the best possible constants, and obtained a more nice chain of inequalities for bivariate means:

$$\begin{aligned} A_0 &< L < A_{1/3} < A_{\ln_\pi 2} < P < A_{2/3} < I < A_{\ln 2} \\ &< A_{\frac{\ln 2}{\ln\ln(3+2\sqrt{2})}} < NS < A_{4/3} < A_{\log_{\pi/2} 2} < T < A_{5/3}, \end{aligned}$$

where I is the identric (exponential) mean of positive numbers a and b (also see [11], [20], [15], [8], [6], [7], [26]). Very recently, Constin and Toader in [4] and Chu and Long in [2] gave other proofs of (1.9), respectively.

Other inequalities for Neuman-Sándor mean can be found in [2], [3], [10], [13], [14], [21], [28], [29].

This paper is based on our earlier work [27] and treats mainly the estimates for Neuman-Sándor mean NS by power means A_p and their relative errors for all $p \in \mathbb{R}$. In Section 2, we give some useful lemmas. The main results are contained Section 3, which not only solve the best estimate problems for Neuman-Sándor mean NS by power means A_p, but also give relative errors of estimates for all $p \in \mathbb{R}$. Also, as by-products, we establish a chain of Ky Fan type inequalities involving the two means and another best estimate for NS. In the last section, we give several remarks on bounds for Neuman-Sándor mean in terms of power means by using the monotonicity and log-convexity of the function $p \mapsto A_p$ and $p \mapsto 2^{1/p}A_p$ on $(0,\infty)$. An optimal estimate for Neuman-Sándor mean NS by Lehmer mean $\mathscr{L}_p$ is incidentally presented.

2. Lemmas

In the sequel, the function g_p defined on $(0,1)$ by

$$g_p(x) = x^{p+2} + x^{p+1} + 2x^p - x^{2-p} - x^{3-p} - 2x^{4-p} + (p-1)x^4 - x^3 + x - p + 1 \quad (2.1)$$

play an important role, where $p \in \mathbb{R}$. We first deal with the sign of $g_p(x)$.

LEMMA 1. *For real number $p \in \mathbb{R}$, let the function g_p be defined on $(0,1)$ by (2.1). Then $g_p(x) < 0$ for $x \in (0,1)$ if and only if $p \geqslant 4/3$ and $g_p(x) > 0$ if and only if $p \leqslant 1$.*

Proof. Firstly we show that $p \mapsto g_p(x)$ is decreasing on $\mathbb{R}$ for $x \in (0,1)$. Indeed, differentiation leads to

$$\frac{\partial g_p(x)}{\partial p} = \left(x^{p+1} + x^{p+2} + x^{2-p} + x^{3-p} + 2x^{4-p} + 2x^p\right)\ln x + \left(x^4 - 1\right) < 0$$

for $x \in (0,1)$.

Secondly, we need two limits $\lim_{x\to 1^-}(1-x)^{-1} g_p(x)$ and $\lim_{x\to 0^+} g_p(x)$. Application of L'Hospital's rule gives

$$\lim_{x\to 1^-} \frac{g_p(x)}{1-x} = 16 - 12p. \quad (2.2)$$

To obtain the second limit, we write $g_p(x)$ as

$$g_p(x) = \left(x^2 + x + 2\right)x^p - x^{2-p}\left(1 + x + 2x^2\right) + (p-1)x^4 - x^3 + x - p + 1,$$

which easily yields

$$\lim_{x\to 0^+} g_p(x) = \begin{cases} -\infty & \text{if } p > 2, \\ -2 & \text{if } p = 2, \\ -p+1 & \text{if } 0 < p < 2, \\ 3 & \text{if } p = 0, \\ \infty & \text{if } p < 0. \end{cases} \quad (2.3)$$

Now we prove that $g_p(x)<0$ for $x\in(0,1)$ if and only if $p\geqslant 4/3$. If $g_p(x)\leqslant 0$ for $x\in(0,1)$, then we have

$$\lim_{x\to 1^-}\frac{g_p(x)}{1-x}\leqslant 0 \text{ and } \lim_{x\to 0+} g_p(x)\leqslant 0,$$

which, by (2.2) and (2.3), reveals that $p\geqslant 4/3$, that is, the condition $p\geqslant 4/3$ is necessary.

Next we show that $p\geqslant 4/3$ is sufficient. Since $p\mapsto g_p(x)$ is decreasing on $\mathbb{R}$ for $x\in(0,1)$, it suffices to prove that $g_p(x)<0$ for $x\in(0,1)$ when $p=4/3$. We have

$$g_{4/3}(x)=x-x^3+\frac{1}{3}x^4-x^{2/3}+2x^{4/3}-x^{5/3}+x^{7/3}-2x^{8/3}+x^{10/3}-\frac{1}{3},$$

and therefore

$$3g_{4/3}\left(x^3\right)=x^{12}+3x^{10}-3x^9-6x^8+3x^7-3x^5+6x^4+3x^3-3x^2-1.$$

Factoring yields that for $x\in(0,1)$

$$3g_{4/3}\left(x^3\right)=(x-1)^3(x+1)\left(x^8+2x^7+7x^6+9x^5+9x^4+9x^3+7x^2+2x+1\right)<0,$$

which proves the sufficiency.

Lastly, we prove that $g_p(x)>0$ for $x\in(0,1)$ if and only if $p\leqslant 1$. Similarly, the necessary condition easily follows from

$$\lim_{x\to 1^-}\frac{g_p(x)}{1-x}\geqslant 0 \text{ and } \lim_{x\to 0+} g_p(x)\geqslant 0,$$

which, by (2.2) and (2.3), yields $p\leqslant 1$.

If $p\leqslant 1$, then by the monotonicity of $p\mapsto g_p(x)$ it is derived that

$$g_p(x)\geqslant g_1(x)=2x\left(1-x^2\right)>0,$$

which proves the sufficiency and the whole proof is complete. □

LEMMA 2. *Let the function g_p be defined on $(0,1)$ by (2.1). Then there is a unique $x_1\in(0,1)$ such that $g_p(x)<0$ for $x\in(0,x_1)$ and $g_p(x)>0$ for $x\in(x_1,1)$ if $p\in(1,4/3)$.*

Proof. We prove the desired result stepwise.

Step 1: We have $g_p^{(4)}(x)>0$ for $x\in(0,1)$ when $p\in(1,4/3)$.

Differentiations yield

$$\begin{aligned} g_p'(x) &= (p+2)x^{p+1}+(p+1)x^p+2px^{p-1}+(p-2)x^{1-p} \\ &\quad +(p-3)x^{2-p}+2(p-4)x^{3-p}+4(p-1)x^3-3x^2+1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} g_p''(x) &= (p+1)(p+2)x^p+p(p+1)x^{p-1}+2p(p-1)x^{p-2} \\ &\quad -(p-1)(p-2)x^{-p}-(p-2)(p-3)x^{1-p} \\ &\quad -2(p-3)(p-4)x^{2-p}+12(p-1)x^2-6x, \end{aligned} \tag{2.5}$$

$$\begin{aligned} g_p'''(x) &= p(p+1)(p+2)x^{p-1}+p(p-1)(p+1)x^{p-2} \\ &\quad +2p(p-1)(p-2)x^{p-3}+p(p-1)(p-2)x^{-p-1} \\ &\quad +(p-1)(p-2)(p-3)x^{-p}+2(p-2)(p-3)(p-4)x^{1-p} \\ &\quad +24(p-1)x-6, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{g_p^{(4)}(x)}{p-1} &= p(p+1)(p+2)x^{p-2} \\ &\quad +p(p+1)(p-2)x^{p-3}+2p(p-2)(p-3)x^{p-4} \\ &\quad -p(p+1)(p-2)x^{-p-2}-p(p-2)(p-3)x^{-p-1} \\ &\quad -2(p-2)(p-3)(p-4)x^{-p}+24 \\ &:= I_1+I_2+I_3+I_4, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} I_1 &= p(p+1)(p+2)x^{p-2}>0, \\ I_2 &= p(p+1)(p-2)x^{p-3}+2p(p-2)(p-3)x^{p-4} \\ &= p(2-p)x^{p-4}(2(3-p)-(p+1)x) \\ &> p(2-p)x^{p-3}(2(3-p)-(p+1)) = p(2-p)x^{p-3}(5-3p)>0, \\ I_3 &= -p(p+1)(p-2)x^{-p-2}-p(p-2)(p-3)x^{-p-1} \\ &= p(2-p)x^{-p-2}((p+1)-(3-p)x) \\ &> p(2-p)x^{-p-2}((p+1)-(3-p)) = 2p(2-p)x^{-p-2}(p-1)>0, \\ I_4 &= 2(2-p)(3-p)(4-p)x^{-p}+24>0 \end{aligned}$$

Hence, $g_p^{(4)}(x)>0$ for $x\in(0,1)$ when $p\in(1,4/3)$.

Step 2: Let $p_1\approx 1.2102$ be the root of equation

$$h(p)=8p^3-30p^2+94p-84=0.$$

Then $g_p'''(x)<0$ for $x\in(0,1)$ when $p\in(1,p_1)$, and there is unique $x_3\in(0,1)$ such that $g_p'''(x)<0$ for $x\in(0,x_3)$ and $g_p'''(x)>0$ for $x\in(x_3,1)$ when $p\in(p_1,4/3)$.

Differentiation yields

$$h'(p)=24p^2-60p+94=\frac{3}{2}(4p-5)^2+\frac{113}{2}>0.$$

In view of $h(1) = -12 < 0$ and $h(4/3) = 188/27 > 0$, the equation $h(p) = 0$ has a unique root $p_1 \approx 1.2102$ such that $h(p) < 0$ for $p \in (1, p_1)$ and $h(p) > 0$ for $p \in (p_1, 4/3)$.

1) When $p \in (1, p_1)$, from $g_p^{(4)}(x) > 0$ for $x \in (0,1)$ it is deduced that

$$g_p'''(x) < g_p'''(1) = 8p^3 - 30p^2 + 94p - 84 = h(p) < 0.$$

2) When $p \in (p_1, 4/3)$, to prove the part two of this step, it suffices to verify that $g_p'''(0^+) < 0$ and $g_p'''(1) > 0$. Simple computation yields

$$\begin{aligned} \operatorname{sgn} g_p'''\left(0^+\right) &= \operatorname{sgn}\left(p(p-1)(p-2)\right) < 0, \\ g_p'''(1) &= 8p^3 - 30p^2 + 94p - 84 = h(p) > 0, \end{aligned}$$

which proves the desired result.

Step 3: There is a unique $x_2 \in (0,1)$ such that $g_p''(x) > 0$ for $x \in (0, x_2)$ and $g_p''(x) < 0$ for $x \in (x_2, 1)$ when $p \in (1, 4/3)$.

We distinguish two cases to prove this step.

In the case of $p \in (1, p_1)$, we see that $g_p'''(x) < 0$ for $x \in (0,1)$. It together with

$$\begin{aligned} \operatorname{sgn} g_p''\left(0^+\right) &= \operatorname{sgn}\left(-(p-1)(p-2)\right) > 0, \\ g_p''(1) &= 12(3p-4) < 0, \end{aligned}$$

leads to the desired assertion.

In the case of $p \in (p_1, 4/3)$, we see that there is a unique $x_3 \in (0,1)$ such that $g_p'''(x) < 0$ for $x \in (0, x_3)$ and $g_p'''(x) > 0$ for $x \in (x_3, 1)$. It follows that $g_p''(x) < g_p''(1) < 0$ for $x \in (x_3, 1)$, which in combination with $g_p''(0^+) > 0$ reveals that there is a unique $x_2 \in (0, x_3)$ such that $g_p''(x) > 0$ for $x \in (0, x_2)$ and $g_p''(x) < 0$ for $x \in (x_2, 1)$. This completes the step.

Step 4: There are two numbers $x_{11} \in (0, x_2), x_{12} \in (x_2, 1)$ such that $g_p'(x) < 0$ for $x \in (0, x_{11}) \cup (x_{12}, 1)$ and $g_p'(x) > 0$ for $x \in (x_{11}, x_{12})$ when $p \in (1, 4/3)$.

Due to Step 3 and the facts that

$$\begin{aligned} \operatorname{sgn} g_p'\left(0^+\right) &= \operatorname{sgn}\left(p-2\right) < 0, \\ g_p'(1) &= 4(3p-4) < 0, \end{aligned}$$

in order to prove this step, it is enough to verify that $g_p'(x_2) > 0$.

In fact, if $g_p'(x_2) < 0$ then $g_p'(x) < g_p'(x_2) < 0$ for $x \in (0, x_2)$ and $g_p'(x) < g_p'(x_2) < 0$ for $x \in (x_2, 1)$, and then $g_p'(x) < 0$ for $x \in (0,1)$. It follows that $g_p(x) > g_p(1) = 0$, which, by Lemma 1, leads to $p \leqslant 1$. It is clearly a contradiction. Hence there must be $g_p'(x_2) > 0$, which completes the Step 4.

Step 5: There is a unique $x_1 \in (x_{11}, x_{12})$ such that $g_p(x) < 0$ for $x \in (0, x_1)$ and $g_p(x) > 0$ for $x \in (x_1, 1)$ if $p \in (1, 4/3)$.

From Step 4 and the facts that

$$g_p\left(0^+\right) = 1 - p < 0, \quad g_p\left(1^-\right) = 0,$$

we have the following variance table of $g_p(x)$:

x	0^+	$(0,x_{11})$	x_{11}	(x_{11},x_{12})	x_{12}	$(x_{12},1)$	1
$g_p'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$g_p(x)$	$-$	$\searrow$	$-$	$\nearrow$	$+$	$\searrow$	0

where

$$g_p(x_{11}) < g_p(0^+) = 1-p < 0 \quad \text{and} \quad g_p(x_{12}) > g_p(1) = 0.$$

Thus the step follows. $\square$

For real number $p \in \mathbb{R}$, let the function f_p be defined on $(0,1)$ by

$$f_p(x) = \operatorname{arcsinh}\frac{x-1}{x+1} - \sqrt{2}\frac{x-1}{(x+1)\sqrt{x^2+1}}\frac{x^p+1}{x^{p-1}+1}. \tag{2.8}$$

Differentiating $f_p(x)$ and simplifying lead to

$$f_p'(x) = \frac{\sqrt{2}(1-x)x^p}{\left(\sqrt{x^2+1}\right)^3(x+1)^2(x+x^p)^2}g_p(x), \tag{2.9}$$

where $g_p(x)$ is defined by (2.1). By Lemma 1 and 2, we easily obtain

LEMMA 3. *Let f_p be defined on $(0,1)$ by (2.8). Then*
(i) f_p is decreasing on $(0,1)$ if and only if $p \geqslant 4/3$;
(ii) f_p is increasing on $(0,1)$ if and only if $p \leqslant 1$;
(iii) there is a unique $x_1 \in (0,1)$ such that f_p is decreasing on $(0,x_1)$ and increasing on $(x_1,1)$ if $p \in (1,4/3)$.

Using Lemma 3 we can prove the following

LEMMA 4. *Let f_p be defined on $(0,1)$ by (2.8). Then*
(i) $f_p(x) > 0$ for $x \in (0,1)$ if and only if $p \geqslant 4/3$;
(ii) $f_p(x) < 0$ for $x \in (0,1)$ if and only if $p \leqslant 1$;
(iii) there is a unique $x_0 \in (0,x_1)$ to satisfy $f_p(x_0) = 0$ such that $f_p(x) > 0$ for $x \in (0,x_0)$ and $f_p(x) < 0$ for $x \in (x_0,1)$ if $p \in (1,4/3)$.

Proof. We first show two limits as follows:

$$\lim_{x\to 1^-}\frac{f_p(x)}{(1-x)^3} = \frac{1}{8}\left(p-\frac{4}{3}\right), \tag{2.10}$$

$$f_p(0^+) = \lim_{x\to 0^+} f_p(x) = \begin{cases} \ln\left(\sqrt{2}-1\right)+\sqrt{2} & \text{if} \quad p>1, \\ \ln\left(\sqrt{2}-1\right)+\frac{\sqrt{2}}{2} & \text{if} \quad p=1, \\ \ln\left(\sqrt{2}-1\right) & \text{if} \quad p<1. \end{cases} \tag{2.11}$$

In fact, application of L'Hospital rule leads to (2.10), and direct computation yields (2.11).

(i) If $f_p(x) > 0$ for $x \in (0,1)$ then there must be $\lim_{x\to 1^-}(1-x)^{-3} f_p(x) \geqslant 0$ and $\lim_{x\to 0^+} f_p(x) \geqslant 0$, which by (2.10) and (2.11) indicates $p \geqslant 4/3$.

We now prove $f_p(x) > 0$ for $x \in (0,1)$ if $p \geqslant 4/3$. From part one of Lemma 3 it is deduced that $f_p(x) > f_p(1) = 0$.

(ii) Similarly, if $f_p(x) < 0$ for $x \in (0,1)$ then we have $\lim_{x\to 1^-}(1-x)^{-3} f_p(x) \leqslant 0$ and $\lim_{x\to 0^+} f_p(x) \leqslant 0$, which by (2.10) and (2.11) yields $p \leqslant 1$. Conversely, if $p \leqslant 1$, then by part one of Lemma 3 it is derived that $f_p(x) < f_p(1) = 0$.

(iii) By part three of Lemma 3, when $p \in (1, 4/3)$, f_p is decreasing on $(0, x_1)$ and increasing on $(x_1, 1)$, then $f_p(x) < f_p(1) = 0$ for $x \in (x_1, 1)$ but $f_p(0^+) = \ln\left(\sqrt{2}-1\right) + \sqrt{2} > 0$. This indicates the desired result.

This proves the lemma. □

Now let us consider the function F_p be defined on $(0,1)$ by

$$F_p(x) = \ln\frac{NS(1,x)}{A_p(1,x)} = \begin{cases} \ln\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}} - \frac{1}{p}\ln\frac{x^p+1}{2} & \text{if} \quad p \neq 0, \\ \ln\frac{x-1}{2\operatorname{arcsinh}\frac{x-1}{x+1}} - \frac{1}{2}\ln x & \text{if} \quad p = 0. \end{cases} \tag{2.12}$$

Differentiation yields

$$F_p'(x) = \frac{x^{p-1}+1}{x^p+1}\frac{1}{2(x-1)\operatorname{arcsinh}\frac{x-1}{x+1}} \times f_p(x), \tag{2.13}$$

where $f_p(x)$ is defined by (2.8). By Lemma 4 the following is immediate.

LEMMA 5. *Let F_p be defined on $(0,1)$ by(2.12). Then*
(i) F_p is increasing on $(0,1)$ if and only if $p \geqslant 4/3$;
(ii) F_p is decreasing on $(0,1)$ if and only if $p \leqslant 1$;
(iii) there is a unique $x_0 \in (0,1)$ to satisfy $f_p(x_0) = 0$ such that F_p is increasing on $(0, x_0)$ and decreasing on $(x_0, 1)$ if $p \in (1, 4/3)$.

3. Main results

Now we are in a position to state and prove our main results, which are contained the following three theorems. The first one gives the right estimate for Neuman-Sándor mean by power mean and its relative error.

THEOREM 1. *For $a, b > 0$ with $a \neq b$, the inequality $NS < A_p$ holds if and only if $p \geqslant 4/3$. Moreover, we have*

$$\alpha_p A_p < NS < A_p \tag{3.1}$$

for $p \geqslant 4/3$, where $\alpha_p = \left(2^{1/p-1}\right)/\ln(1+\sqrt{2})$ is the best possible constant.

Proof. In order to prove the desired result, we need to verify two limit relations:

$$\lim_{x\to 1^-}\frac{F_p(x)}{(x-1)^2}=-\frac{1}{24}(3p-4), \tag{3.2}$$

$$F_p\left(0^+\right)=\lim_{x\to 0^+}F_p(x)=\begin{cases}\ln\frac{2^{1/p-1}}{\ln\left(1+\sqrt{2}\right)} & \text{if } p>0,\\ \infty & \text{if } p\leqslant 0.\end{cases} \tag{3.3}$$

In fact, using power series expansion gives

$$F_p(x)=-\frac{1}{24}(3p-4)(x-1)^2+O\left((x-1)^3\right),$$

which yields (3.2). While (3.3) easily follows by direct calculations.

Now we prove the inequality $NS<A_p$ holds for all $a,b>0$ with $a\neq b$ if and only if $p\geqslant 4/3$. By symmetry, we assume that $b>a>0$. Then inequality $NS<A_p$ is equivalent to

$$\ln NS(1,x)-\ln A_p(1,x)=F_p(x)<0,$$

where $x=a/b\in(0,1)$.

The necessity easily follows from $\lim_{x\to 1^-}(x-1)^{-2}F_p(x)\leqslant 0$ and $\lim_{x\to 0^+}F_p(x)\leqslant 0$. Solving the simultaneous inequalities for p gives $p\geqslant 4/3$.

The sufficiency can be obtained by part one of Lemma 5. Indeed, we have $F_p(x)<F_p(1)=0$ for $x\in(0,1)$ if $p\geqslant 4/3$.

Utilizing the monotonicity of the function F_p on $(0,1)$, we have

$$\ln\alpha_p=\ln\frac{2^{1/p-1}}{\ln\left(1+\sqrt{2}\right)}=F_p\left(0^+\right)<F_p(x)<F_p\left(1^-\right)=0,$$

which implies (3.1).

Thus the proof of Theorem 1 is finished. □

As a consequence of Theorem 1, we have

COROLLARY 1. *For $a,b>0$ with $a\neq b$, the following estimate inequalities hold:*

$$\alpha_{4/3}A_{4/3}<NS<A_{4/3}, \tag{3.4}$$

$$\alpha_2 Q<NS<Q, \tag{3.5}$$

$$\alpha_\infty\max(a,b)<NS<\max(a,b), \tag{3.6}$$

where $\alpha_{4/3}=(\sqrt[4]{2}\ln(1+\sqrt{2}))^{-1}\approx 0.954$, $\alpha_2=(\sqrt{2}\ln(1+\sqrt{2}))^{-1}\approx 0.802$ and $\alpha_\infty=(2\ln(1+\sqrt{2}))^{-1}\approx 0.567$ are the best possible constants.

Proof. Putting $p=4/3,2$ in Theorem 1 yields (3.4) and (3.5). By Theorem 1, in order to show (3.6), it suffices to verify that

$$\lim_{p\to\infty}A_p=\max(a,b)\ \text{ and }\ \alpha_\infty=\frac{1}{2\ln\left(1+\sqrt{2}\right)}.$$

For this purpose, we assume that $b>a>0$. Then by a simple computation we have

$$\lim_{p\to\infty}\ln A_p=\ln b+\lim_{p\to\infty}\frac{\ln\left((a/b)^p+1\right)-\ln 2}{p}=\ln b,$$

which in conjunction with

$$\alpha_\infty=\lim_{p\to\infty}\frac{2^{1/p-1}}{\ln\left(1+\sqrt{2}\right)}=\frac{1}{2\ln\left(1+\sqrt{2}\right)}$$

gives (3.6). □

Next we establish the left estimate for Neuman-Sándor mean by power mean and give its relative error.

THEOREM 2. *For $a,b>0$ with $a\neq b$, the inequality $NS>A_p$ holds if and only if $p\leqslant p_0=(\ln 2)/(\ln\ln(3+2\sqrt{2}))\approx 1.223$. Moreover, we have*

$$A_p<NS\leqslant\beta_p A_p \tag{3.7}$$

for $p\leqslant p_0$, where

$$\beta_p=\begin{cases}\frac{NS(1,x_0)}{A_p(1,x_0)} & \text{if } p\in(1,p_0],\\ \frac{2^{1/p-1}}{\ln\left(1+\sqrt{2}\right)} & \text{if } p\in(0,1],\\ \infty & \text{if } p\in(-\infty,0]\end{cases}$$

is the best possible constant, here x_0 is the unique root of the equation $f_p(x)=0$ on $(0,1)$, $f_p(x)$ is defined by (2.8).

Proof. Clearly, the inequality $NS>A_p$ is equivalent to $\ln NS(1,x)-\ln A_p(1,x)=F_p(x)>0$, where $x=a/b\in(0,1)$. Now we show that $F_p(x)>0$ holds for all $x\in(0,1)$ if and only if $p\leqslant p_0$.

From the simultaneous inequalities $\lim_{x\to 1^-}(x-1)^{-2}F_p(x)\geqslant 0$ and $\lim_{x\to 0^+}F_p(x)\geqslant 0$ together with (3.2) and (3.3) it is deduced that $p\leqslant p_0$, which implies the necessity.

We now prove the condition $p\leqslant p_0$ is sufficient. To this end, we distinguish two cases to prove it.

In the case of $p\leqslant 1$, since F_p is decreasing on $(0,1)$ by part two of Lemma 5, it follows that

$$0=F_p(1)<F_p(x)\leqslant F_p\left(0^+\right)=\begin{cases}\frac{1}{p}\ln 2-\ln\ln\left(3+2\sqrt{2}\right) & \text{if } p>0,\\ \infty & \text{if } p\leqslant 0.\end{cases} \tag{3.8}$$

In the case of $p\in(1,p_0]$, by part three of Lemma 5, we see that there is a unique $x_0\in(0,1)$ to satisfy $f_p(x_0)=0$ such that F_p is increasing on $(0,x_0)$ and decreasing on $(x_0,1)$. It is acquired that

$$0\leqslant\frac{1}{p}\ln 2-\ln\ln\left(3+2\sqrt{2}\right)=F_p\left(0^+\right)<F_p(x)<F_p(x_0)\ \text{for } x\in(0,x_0)$$
$$0=F_p(1)<F_p(x_3)<F_p(x_0)\ \text{for } x\in(x_0,1),$$

that is,

$$0 < F_p(x) \leqslant F_p(x_0) \tag{3.9}$$

for all $x \in (0,1)$, which proves the sufficiency.

Inequalities (3.7) follows from (3.8) and (3.9), which completes the proof. □

Letting $p = p_0 = (\ln 2)/(\ln\ln(3+2\sqrt{2}))$ in Theorem 2 and solving the equation $f_{p_0}(x) = 0$ on $(0,1)$ by mathematical computation software, we find that $x_0 \in (0.1580, 0.1581)$, and then, $\beta_{p_0} = NS(1,x_0)/A_p(1,x_0) \approx 1.014$. Letting $p = 1,\ 1/2$ in Theorem 2, we have

COROLLARY 2. *For $a,b>0$ with $a \neq b$, the following estimate inequalities hold*

$$A_{p_0} < NS < \beta_{p_0} A_{p_0}, \tag{3.10}$$

$$A < NS < \beta_1 A, \tag{3.11}$$

$$\frac{A+G}{2} < NS < \beta_{1/2}\frac{A+G}{2}, \tag{3.12}$$

where $\beta_{p_0} \approx 1.014$, $\beta_1 = 1/\ln(1+\sqrt{2}) \approx 1.135$ and $\beta_{1/2} = 2/\ln(1+\sqrt{2}) \approx 2.269$ are the best constants.

REMARK 1. The estimate inequalities (3.11) is due to Neuman [12, (2.15)].

Now let us observe the estimate for Neuman-Sándor mean by power mean and its relative error when $p \in (p_0, 4/3)$, where $p_0 = (\ln 2)/(\ln\ln(3+2\sqrt{2}))$.

THEOREM 3. *Let $0 < a < b$. Then, when $p \in (p_0, 4/3)$, there is a number $c_0 \in (0,1)$ such that*

$$\alpha_p A_p < NS < A_p \quad \text{for } 0 < a < c_0 b, \tag{3.13}$$

$$A_p < NS < \gamma_p A_p \quad \text{for } c_0 b < a < b, \tag{3.14}$$

where $\alpha_p = \left(2^{1/p-1}\right)/\ln(1+\sqrt{2})$ and $\gamma_p = NS(1,x_0)/A_p(1,x_0)$ are the best constants, here x_0 is the unique root of the equation $f_p(x) = 0$ on $(0,1)$, $f_p(x)$ is defined by (2.8).

Proof. The part three of Lemma 5 tells us that there is a unique $x_0 \in (0,1)$ to satisfy $f_p(x_0) = 0$ such that F_p defined on $(0,1)$ by (2.12) is increasing on $(0,x_0)$ and decreasing on $(x_0,1)$ if $p \in (1,4/3)$. Therefore, when $p \in (p_0,4/3)$ we have $F_p(x) > F_p(1) = 0$ for $x \in (x_0,1)$. This together with the fact

$$F_p\left(0^+\right) = \frac{1}{p}\ln 2 - \ln\ln\left(3+2\sqrt{2}\right) < 0$$

yields that there is a number $c_0 \in (0,1)$ such that

$$F_p\left(0^+\right) < F_p(x) < 0 \text{ for } x \in (0,c_0),$$
$$0 < F_p(x) < F_p(x_0) \text{ for } x \in (c_0,1).$$

Letting $x = a/b$ implies (3.13) and (3.14), which completes the proof. □

Additionally, it is worth pointing out that Lemmas 3, 4 and 5 not only serve for the proof of our main results previous, but also can deduce some sharp inequalities. For example, Lemma 5 implies a chain of Ky Fan type inequalities for Neuman-Sándor mean and power mean.

THEOREM 4. *For all $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$, the following Ky Fan type inequalities*

$$\frac{A_p(a_2,b_2)}{A_p(a_1,b_1)} < \frac{NS(a_2,b_2)}{NS(a_1,b_1)} < \frac{A_q(a_2,b_2)}{A_q(a_1,b_1)} \tag{3.15}$$

hold if and only if $p \geqslant 4/3$ and $q \leqslant 1$.

Proof. By Lemma 5 it is seen that F_p is increasing on $(0,1)$ if and only if $p \geqslant 4/3$ and decreasing if and only if $p \leqslant 1$. Therefore, for $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$, we have

$$\ln\frac{NS(1,a_1/b_1)}{A_p(1,a_1/b_1)} < \ln\frac{NS(1,a_2/b_2)}{A_p(1,a_2/b_2)} \text{ if and only if } p \geqslant 4/3,$$
$$\ln\frac{NS(1,a_1/b_1)}{A_q(1,a_1/b_1)} > \ln\frac{NS(1,a_2/b_2)}{A_q(1,a_2/b_2)} \text{ if and only if } q \leqslant 1,$$

which imply the desired result. □

Noting that $f_p(x)$ defined by (2.8) can be written as

$$f_p(x) = \frac{1}{2}(1-x)\left(\frac{\mathscr{L}_{p-1}(1,x)}{A(1,x)Q(1,x)} - \frac{1}{NS(1,x)}\right), \tag{3.16}$$

and next utilizing Lemma 4, we have

THEOREM 5. *For $a, b > 0$ with $a \neq b$, the inequalities*

$$\frac{AQ}{\mathscr{L}_{p-1}} < NS < \frac{AQ}{\mathscr{L}_{q-1}} \tag{3.17}$$

hold if and only if $p \geqslant 4/3$ and $q \leqslant 1$, where A, Q stand for arithmetic mean, quadratic mean, and $\mathscr{L}_r$ is the Lehmer mean defined by (1.2). Particularly, we have

$$\frac{AQ}{\mathscr{L}_{1/3}} < NS < Q. \tag{3.18}$$

Likewise, from Lemma 3 we see that f_p is decreasing on $(0,1)$ if $p \geqslant 4/3$ and increasing if $p \leqslant 1$. It is obtained that

$$f_p(1) < f_p(x) < f_p\left(0^+\right) \text{ if } p \geqslant 4/3,$$
$$f_p\left(0^+\right) < f_p(x) < f_p(1) \text{ if } p \leqslant 1.$$

Using (2.11) and (3.16) and letting $x = a/b$, we get

THEOREM 6. *For $0<a<b$, the inequalities hold:*

$$0<\frac{\mathscr{L}_{p-1}}{AQ}-\frac{1}{NS}<\frac{\sqrt{2}-\ln\left(\sqrt{2}+1\right)}{b-a} \text{ if } p\geqslant 4/3, \tag{3.19}$$

$$-\frac{1}{2}\frac{2\ln\left(\sqrt{2}+1\right)-\sqrt{2}}{b-a}<\frac{1}{Q}-\frac{1}{NS}<0 \text{ if } p=1, \tag{3.20}$$

$$-\frac{\ln\left(\sqrt{2}+1\right)}{b-a}<\frac{\mathscr{L}_{p-1}}{AQ}-\frac{1}{NS}<0 \text{ if } p<1. \tag{3.21}$$

4. Remarks

In this section, we give several remarks on bounds for Neuman-Sándor mean by power means. To this end, we also need the monotonicity and log-convexity results for ratio of Stolarsky means, which are from [24, Theorem 3.5] and [25, Theorem 3.6], respectively, where Stolarsky means are defined in [19] by

$$S_{p,q}(a,b)=\begin{cases}\left(\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{1/(p-q)} & \text{if} \quad p\neq q,\ pq\neq 0,\\ \left(\dfrac{a^p-b^p}{p(\ln a-\ln b)}\right)^{1/p} & \text{if} \quad p\neq 0,\ q=0,\\ \left(\dfrac{a^q-b^q}{q(\ln a-\ln b)}\right)^{1/q} & \text{if} \quad p=0,\ q\neq 0,\\ \exp\left(\dfrac{a^p\ln a-b^p\ln b}{a^p-b^p}-\dfrac{1}{p}\right) & \text{if} \quad p=q\neq 0,\\ \sqrt{ab} & \text{if} \quad p=q=0.\end{cases} \tag{4.1}$$

PROPOSITION 1. *Let $a,b,c,d>0$ with $b/a>d/c\geqslant 1$. Then for fixed $r,s\in\mathbb{R}$,*

(i) $p\mapsto S_{pr,ps}(a,b)/S_{pr,ps}(c,d)$ is strictly increasing on $\mathbb{R}$ if $r+s>0$ and decreasing on $\mathbb{R}$ if $r+s<0$;

(ii) $p\mapsto S_{pr,ps}(a,b)/S_{pr,ps}(c,d)$ is strictly log-concave in p on $(0,\infty)$ and log-convex on $(-\infty,0)$ if $r+s>0$, and strictly log-convex on $(0,\infty)$ and log-concave on $(-\infty,0)$ if $r+s<0$.

Assume that $p,r,s>0$ and let $a\to 0^+$ in the above proposition. Then $S_{pr,ps}(a,b)/S_{pr,ps}(c,d)=b/\mathscr{H}_D(pr,ps;c,d)$, where

$$\mathscr{H}_D(pr,ps;c,d)=\begin{cases}\left(\dfrac{c^{pr}-d^{pr}}{c^{ps}-d^{ps}}\right)^{1/(pr-ps)} & \text{if } r\neq s, p,r,s>0,\\ e^{1/(pr)}I^{1/(pr)}\left(c^{pr},d^{pr}\right) & \text{if} \quad r=s,p,r,s>0,\end{cases} \tag{4.2}$$

here $I(x,y)$ is the identric (exponential) mean of positive numbers x and y. By Proposition 1 we have

LEMMA 6. *Let $p,r,s,c,d>0$ with $c\neq d$. Then for fixed $r,s>0$, the function $p\mapsto \mathscr{H}_D(pr,ps;c,d)$ defined by (4.2) is strictly decreasing on $(0,\infty)$ and log-convex on $(0,\infty)$.*

Particularly, the function $p\mapsto \mathscr{H}_D(2p,p;c,d)=2^{1/p}A_p$ is strictly decreasing on $(0,\infty)$ and log-convex on $(0,\infty)$.

Now we remark the best bounds for Neuman-Sándor mean NS in terms of power means.

REMARK 2. Using the monotonicity of the function $p\mapsto A_p$ and $p\mapsto 2^{1/p}A_p$, we see that

$$p\mapsto \alpha_p A_p=\frac{2^{1/p-1}}{\ln(1+\sqrt{2})}A_p=\frac{1}{2\ln(1+\sqrt{2})}\left(2^{1/p}A_p\right)$$

is decreasing in p on $(0,\infty)$. Thus the Corollary 1 can be improved as

$$\alpha_\infty \max(a,b)<\cdots<\alpha_2 Q<\alpha_{4/3}A_{4/3}<NS<A_{4/3}<Q<\cdots<\max(a,b). \tag{4.3}$$

In the same way, for $p\in(0,1]$ the Corollary 2 can be partly improved as

$$G<\cdots<\frac{A+G}{2}<A<NS<\alpha_1 A<\beta_{1/2}\frac{A+G}{2}<\cdots<\infty. \tag{4.4}$$

The last problem is that for $p\in(1,p_0]$ whether the right bound for NS in (3.7) $\beta_p A_p$ is decreasing with p. We guess that the answer is positive, which is posed as a conjecture.

CONJECTURE 1. *Let $p_0=(\ln 2)/(\ln\ln(3+2\sqrt{2}))$ for $p\in(1,p_0]$. Then the function $p\mapsto \beta_p A_p$ is decreasing with p on $(1,p_0]$, where $\beta_p=NS(1,x_0)/A_p(1,x_0)$, x_0 is the unique root of the equation $f_p(x)=0$ on $(0,1)$, $f_p(x)$ is defined by (2.8).*

REMARK 3. For the best bounds for Neuman-Sándor mean NS in terms of power means given in (3.4) and (3.7), we have

$$\alpha_{4/3}A_{4/3}<A_{p_0}<NS<A_{4/3}<\alpha_{4/3}^{-1}A_{p_0}, \tag{4.5}$$

where $\alpha_{4/3}=(\sqrt[4]{2}\ln(1+\sqrt{2}))^{-1}$, $p_0=(\ln 2)/(\ln\ln(3+2\sqrt{2}))$.

In fact, since $p\mapsto 2^{1/p}A_p$ is strictly decreasing on $(0,\infty)$, we have

$$2^{1/p_0}A_{p_0}>2^{3/4}A_{4/3}, \tag{4.6}$$

which implies that $A_{p_0}>2^{-1/p_0}2^{3/4}A_{4/3}$ and $A_{4/3}<2^{1/p_0}2^{-3/4}A_{p_0}$. To show (4.5), it is enough to verify that $2^{-1/p_0}2^{3/4}=\alpha_{4/3}$. We have

$$2^{-1/p_0}2^{3/4}=2^{3/4}\exp\left(-\tfrac{\ln 2}{p_0}\right)=2^{3/4}\exp\left(-\ln 2\tfrac{\ln\ln(3+2\sqrt{2})}{\ln 2}\right)=\tfrac{2^{3/4}}{2\ln(1+\sqrt{2})}=\alpha_{4/3}.$$

It should be noted that the relative error of estimate for NS by A_{p_0} given by (3.7) is clearly superior to another ones given by (4.5), since $\alpha_{4/3}^{-1}=(\sqrt[4]{2}\ln(1+\sqrt{2}))\approx 1.048>\beta_{p_0}\approx 1.014$, but the later can avoid some complicated computations and is also rather small.

REMARK 4. Using the log-convexity of the function $p \mapsto A_p$ and $p \mapsto 2^{1/p} A_p$ we can also give the best estimate for NS by Lehmer mean:

$$(2\ln(1+\sqrt{2}))^{-1}\mathscr{L}_{1/6} < NS < \mathscr{L}_{1/6}. \tag{4.7}$$

By $NS < A_{4/3}$ given in (3.4) and $\mathscr{L}_r = A_{r+1}^{r+1} A_r^{-r}$ given by (1.3), for showing $NS < \mathscr{L}_{1/6}$, it suffices to check $A_{4/3} < A_{7/6}^{7/6} A_{1/6}^{-1/6}$, that is, $A_{4/3}^{6/7} A_{1/6}^{1/7} < A_{7/6}$, which follows by that the function $p \mapsto A_p$ is log-concave on $(0,\infty)$ (see [23, Conclusion 1]). On the other hand, since $p \mapsto 2^{1/p} A_p$ is log-convex on $(0,\infty)$ by Lemma 6, we get

$$\left(2^{3/4} A_{4/3}\right)^{6/7} \left(2^6 A_{1/6}\right)^{1/7} > 2^{6/7} A_{7/6},$$

which can be simplified to $2^{3/4} A_{4/3} > A_{7/6}^{7/6} A_{1/6}^{-1/6} = \mathscr{L}_{1/6}$. And, by (3.4) it is acquired that

$$NS > \alpha_{4/3} A_{4/3} = 2^{-3/4} \alpha_{4/3} \left(2^{3/4} A_{4/3}\right) > 2^{-3/4} \alpha_{4/3} \mathscr{L}_{1/6} = (2\ln(1+\sqrt{2}))^{-1} \mathscr{L}_{1/6}.$$

Further, we assert that the right estimate for NS by Lehmer mean $\mathscr{L}_r$ is the best. In fact, assume that $0 < a < b$ and let $x = a/b \in (0,1)$. Using power series expansion gives

$$\frac{NS(1,x)}{\mathscr{L}_p(1,x)} = -\frac{1}{24}(6p-1)(x-1)^2 + O\left((x-1)^3\right),$$

which indicates that $p \geqslant 1/6$ is necessary for $NS < \mathscr{L}_p$ to hold.

Meanwhile, for ensuring that $NS > \theta \mathscr{L}_{1/6}$ is true for $x \in (0,1)$, it has to satisfy that $\lim_{x \to 0^+} \left(NS - \theta \mathscr{L}_{1/6}\right) = (2\ln(1+\sqrt{2}))^{-1} - \theta \geqslant 0$. That is, $\theta = (2\ln(1+\sqrt{2}))^{-1}$ is the best.

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