Journal of
Mathematical
Inequalities
Volume 7, Number 4 (2013), 739–750 doi:10.7153/jmi-07-67

n–EXPONENTIAL CONVEXITY OF HARDY–TYPE AND BOAS–TYPE FUNCTIONALS

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(*Communicated by A. Aglić Aljinović*)

Abstract. In this paper, we discuss and prove n-exponential convexity of the linear functionals obtained by taking the positive difference of Hardy-type and Boas-type inequalities. Also, we give some examples related to our main results.

1. Introduction

Let us recall the classical Hardy inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1 \tag{1.1}$$

where f is a non-negative function, such that $f \in L^p(\mathbb{R}_+)$.

We also note that (1.1) shows that the Hardy operator H, defined by setting

$$(Hf)(x) := \frac{1}{x}\int_0^x f(t)\,dt,$$

maps L^p into itself with operator norm $p/(p-1)$.

R. P. Boas [2], proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.2}$$

holds for all continuous convex functions $\Phi\colon [0,\infty) \to \mathbb{R}$, measurable non–negative functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and non–decreasing bounded functions $m\colon [0,\infty) \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1.2) is the Lebesgue–Stieltjes integral with respect to m. After its author, relation (1.2) was named

Mathematics subject classification (2010): 26D15, 26D10, 26D99.
Keywords and phrases: Hardy-type inequalities, Boas-type inequality, convex function, linear functionals, n-exponentially convexity, divided difference.

the Boas inequality. In the case of a concave function Φ, it holds with reversed sign of inequality.

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0\cdot\infty, \frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Further, we set $\mathbb{N}_k = \{1, 2, \ldots, k\}$ for $k \in \mathbb{N}$. Moreover, by a weight $u = u(x)$ we mean a non-negative measurable function on the actual interval or more general set.

In the sequel let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and let operator A_k be defined as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y), \tag{1.3}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\, d\mu_2(y) < \infty, \quad x \in \Omega_1. \tag{1.4}$$

Let $U(k)$ denote the class of measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

In [9] this result is given:

THEOREM 1.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.4). Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by*

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)}\, d\mu_1(x) < \infty. \tag{1.5}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x) \leqslant \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y) \tag{1.6}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$ for all $y \in \Omega_2$, where A_k is defined by (1.3).

Under assumptions of Theorem 1.1, we define a linear functional by taking the positive difference of the inequality stated in (1.6) as:

$$\Delta_1(\Phi) = \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x). \tag{1.7}$$

If we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by f_1/f_2, where $f_i:\Omega_2\to\mathbb{R}$, $(i=1,2)$ are measurable functions in Theorem 1.1 we obtain the following result (for details see [7]).

THEOREM 1.2. *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1 and k be a non-negative measurable function on $\Omega_1\times\Omega_2$. Suppose that the function $x\mapsto u(x)\frac{k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$, and that v is defined on Ω_2 by*

$$v(y):=f_2(y)\int_{\Omega_1}\frac{u(x)k(x,y)}{g_2(x)}d\mu_1(x)<\infty.$$

If Φ is a convex function on the interval $I\subseteq\mathbb{R}$, then the inequality

$$\int_{\Omega_1}u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right)d\mu_1(x)\leqslant\int_{\Omega_2}v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)d\mu_2(y) \tag{1.8}$$

holds for all measurable functions $f_i:\Omega_2\to\mathbb{R}$, such that $\frac{f_1(y)}{f_2(y)}\in I$, and $g_i\in U(k)$, $(i=1,2)$.

REMARK 1.1. If we take $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$ and $d\mu_2(y)=dy$ the inequality (1.8) becomes the inequality given in [6, Theorem 2.1].

Under assumptions of the Theorem 1.2, we define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality given in (1.8) as:

$$\Delta_2(\Phi)=\int_{\Omega_2}v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)d\mu_2(y)-\int_{\Omega_1}u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right)d\mu_1(x). \tag{1.9}$$

The discrete results about Hardy-type inequalities are given in [3]. Here, we consider a special case of [3, Theorem 2.1], that is for convex functions this result holds.

THEOREM 1.3. *Let $M,N\in\mathbb{N}$, and let non-negative real numbers u_m, v_n, k_{mn}, where $m\in\mathbb{N}_M$, $n\in\mathbb{N}_N$, be such that*

$$K_m=\sum_{n=1}^{N}k_{mn}>0,\quad m\in\mathbb{N}_M, \tag{1.10}$$

and

$$v_n=\sum_{m=1}^{M}u_m\frac{k_{mn}}{K_m},\quad n\in\mathbb{N}_N. \tag{1.11}$$

If Φ is a convex function on the interval $I\subseteq\mathbb{R}$, then the inequality

$$\sum_{n=1}^{N}v_n\Phi(a_n)-\sum_{m=1}^{M}u_m\Phi(A_m)\geqslant 0 \tag{1.12}$$

holds for all real numbers $a_n \in I$, *for* $n \in \mathbb{N}_N$, *where*

$$A_m = \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n.$$

We define linear functional from (1.12) as:

$$\Delta_3(\Phi) = \sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \tag{1.13}$$

Now, we give result related to general Boas-type inequality.

Let λ be finite Borel measure on $\mathbb{R}_+$. By $\operatorname{supp}\lambda$ we mean its support, that is, the set of all $t \in \mathbb{R}_+$ such that $\lambda(N_t)$ holds for all open neighborhoods N_t of t. Hence,

$$L = \int_{\operatorname{supp}\lambda} d\lambda(t) = \int_0^\infty d\lambda(t) = \lambda(\mathbb{R}_+) < \infty. \tag{1.14}$$

Furthermore, let X be a topological space equipped with a continuous scaler multiplication $(a, \mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$ and $\mathbf{x} \in X$, such that

$$1\mathbf{x} = \mathbf{x}, \quad a(b\mathbf{x}) = (ab)\mathbf{x}, \quad \mathbf{x} \in X, \quad a, b \in \mathbb{R}_+.$$

Let a Borel set $\Omega \subseteq X$ be λ-balanced, that is, let $t\Omega = \{t\mathbf{x} : \mathbf{x} \in \Omega\} \subseteq \Omega$ hold for all $t \in \operatorname{supp}\lambda$. For a Borel measurable function $f : \Omega \to \mathbb{R}$, we define its Hardy-Littlewood average, Af, as

$$Af(\mathbf{x}) = \frac{1}{L} \int_0^\infty f(t\mathbf{x}) d\lambda t, \quad \mathbf{x} \in \Omega. \tag{1.15}$$

Finally, suppose that μ and ν are σ-finite Borel measure on X. For all $t > 0$ and Borel set $S \subseteq X$, we define

$$\mu_t(S) = \mu\left(\frac{1}{t} S\right). \tag{1.16}$$

Obviously, μ_t is a σ-finite Borel measure on X for all $t \in \mathbb{R}_+$. Throughout this paper, we assume that μ_t are absolutely continuous with respect to measure ν, that is $\mu_t \ll \nu$, for each $t \in \operatorname{supp}\lambda$. As usual, by $\frac{d\mu_t}{d\nu}$ we denote related Radon-Nikodym derivative.

The following theorem is given in [4].

THEOREM 1.4. *Let* λ *be finite Borel measure on* $\mathbb{R}_+$ *and* L *be defined by* (1.14). *Let* μ *and* ν *be* σ*-finite Borel measures on a topological space* X, μ_t *be defined by* (1.16) *and such that* $\mu_t \ll \nu$ *for all* $t \in \operatorname{supp}\lambda$. *Further, let* $\Omega \subseteq X$ *be a* λ*-balanced Borel set and* u *be a non-negative function on* X, *such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right) \frac{d\mu_t}{d\nu}(\mathbf{x}) d\lambda(t) < \infty, \quad \mathbf{x} \in \Omega. \tag{1.17}$$

Suppose $\Phi : I \to \mathbb{R}$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f : \Omega \to \mathbb{R}$ is a Borel measurable function such that $f(\mathrm{x}) \in I$ for all $\mathrm{x} \in \Omega$, and Af is defined by (1.15), then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality

$$\int_{\Omega} u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}) \leqslant \frac{1}{L}\int_{\Omega} v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}) \tag{1.18}$$

holds. For a non-positive concave function Φ, the sign of inequality in (1.18) is reversed.

Notice that the condition on non-negativity of the convex function Φ in Theorem 1.4 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

COROLLARY 1.1. *If in Theorem 1.4 we have $t\Omega = \Omega$ for $\lambda - a.e.$ $t \in \operatorname{supp}\lambda$, then (1.18) holds for all convex functions Φ on an interval $I \subseteq \mathbb{R}$. In that case, for all concave functions Φ relation (1.18) holds with the sign of inequality reversed.*

Now, under the assumptions of the Corollary 1.1, we define the linear functional from inequality (1.18) as:

$$\Delta_4(\Phi) = \frac{1}{L}\int_{\Omega} v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}) - \int_{\Omega} u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}). \tag{1.19}$$

For reader's convenience, we introduce some necessary notation and recall some basic facts about convex functions, log-convex functions (see e.g. [8], [12], [14]) as well as exponentially convex functions (see e.g. [1], [10], [11]).

In 1929, S. N. Bernstein introduced the notion of exponentially convex function in [1]. Later on D. V. Widder in [15] introduced these functions as a sub-class of convex function in a given interval (a,b) (for details see [15], [16]).

The main purpose of this article is to discuss the n-exponential convexity of four Hardy-type and Boas-type linear functionals obtained by taking the positive difference of Hardy-type inequalities and Boas-type inequality defined by (1.7), (1.9), (1.13) and (1.19) respectively.

We continue this section by recalling some notions of our special interest about n-exponential convexity given in [13].

DEFINITION 1.1. A function $\psi : J \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on J if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\tfrac{p_i+p_j}{2}\right) \geqslant 0$$

holds for all choices of $\xi_i \in \mathbb{R}$, $p_i \in J$, $i = 1,\ldots,n$.

A function $\psi : J \to \mathbb{R}$ is n-exponentially convex on J if it is n-exponentially convex in the Jensen sense and continuous on J.

REMARK 1.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 1.1. *Let J be an open interval in $\mathbb{R}$. If ψ is n-exponentially convex in the Jensen sense on J then the matrix $\left[\psi\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly*

$$\det\left[\psi\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0, \quad for\ all\ k \in \mathbb{N}, \quad k \leqslant n.$$

DEFINITION 1.2. Let J be an open interval in $\mathbb{R}$. A function $\psi : J \to \mathbb{R}$ is exponentially convex in the Jensen sense on J if it is n-exponentially convex in the Jensen sense on J for $n \in \mathbb{N}$.

A function $\psi : J \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 1.3. It is known that a function $\eta : J \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$m^2\eta(p) + 2mn\eta\left(\frac{p+q}{2}\right) + n^2\eta(q) \geqslant 0, \tag{1.20}$$

for all $m,n \in \mathbb{R}$ and $p,q \in J$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see for example [14]).

PROPOSITION 1.2. *If Ψ is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1} \leqslant \frac{\Psi(y_2)-\Psi(y_1)}{y_2-y_1}. \tag{1.21}$$

If the function Ψ is concave, the reverse inequality holds.

The paper is organized in the following way: After Introduction, in Section 2, we discuss n-exponential convexity and log-convexity of the linear functionals defined by (1.7), (1.9), (1.13) and (1.19). In Section 3, we give some related examples for the family of convex functions.

2. The main results

First we give some necessary details about the divided differences. It is important to see that for different degree of smoothness of a function divided differences are found to be very interesting.

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0,1,2$, the divided differences of first and second order are defined by:

$$[z_i, z_{i+1}; f] = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} \quad (i = 0,1), \tag{2.1}$$

$$[z_0, z_1, z_2; f] = \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \tag{2.2}$$

The values of the divided differences are independent of the order of the points z_0, z_1, z_2 and may be extended to include the cases when some or all points are equal, that is

$$[z_0, z_0; f] = \lim_{z_1 \to z_0} [z_0, z_1; f] = f'(z_0), \tag{2.3}$$

provided that f' exists.

Now passing through the limit $z_1 \to z_0$ and replacing z_2 by z in (2.2), we have (see [14, p. 16])

$$[z_0, z_0, z; f] = \lim_{z_1 \to z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0) f'(z_0)}{(z - z_0)^2}, \quad z \neq z_0, \tag{2.4}$$

provided that f' exists. Also passing to the limit $z_i \to z$ $(i = 0,1,2)$ in (2.2), we have

$$[z, z, z; f] = \lim_{z_i \to z} [z_0, z_1, z_2; f] = \frac{f''(z)}{2}, \tag{2.5}$$

provided that f'' exists.

One can observe that if for all $z_0, z_1 \in I$, $[z_0, z_1, f] \geqslant 0$, then f is increasing on I and if for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geqslant 0$, then f is convex on I.

Now we will produce n-exponentially convex and exponentially convex functions by applying functionals Δ_i, $i = 1,2,3,4$ on a given family with the same property. In the sequel J and I will be intervals in $\mathbb{R}$. The proofs of our results are similar to the proofs in [13] but for completeness of results and for the reader's convenience we will also give them.

THEOREM 2.1. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1,2,3,4)$ be linear functionals defined by (1.7), (1.9), (1.13) and (1.19). Then the function $p \mapsto \Delta_i(\Phi_p)$ $(i = 1,2,3,4)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $a_i \in \mathbb{R}$, $i = 1,\ldots,n$ and $p_i \in J$, $i = 1,\ldots,n$, we define the function

$$\Upsilon(z) = \sum_{i,j=1}^{n} a_i a_j \Phi_{\frac{p_i + p_j}{2}}(z).$$

Using the assumption that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense, we have

$$[z_0, z_1, z_2; \Upsilon] = \sum_{i,j=1}^{n} a_i a_j [z_0, z_1, z_2; \Phi_{\frac{p_i+p_j}{2}}] \geqslant 0,$$

which shows that Υ is convex on I and therefore we have $\Delta_i(\Upsilon) \geqslant 0$ for $(i = 1,2,3,4)$. Hence

$$\sum_{i,j=1}^{n} a_i a_j \Delta_i(\Phi_{\frac{p_i+p_j}{2}}) \geqslant 0.$$

We conclude that the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2,3,4)$ is n-exponentially convex in Jensen sense on J.

If the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2,3,4)$ is also continuous on J, then $p \mapsto \Delta_i(\Phi_p)$ is n-exponentially convex by definition. □

As a direct consequence of the above theorem, we can give the following corollary.

COROLLARY 2.1. *Let $\Gamma = \{\Phi_p : I \to \mathbb{R}, p \in J\}$ be a family of functions, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1,2,3,4)$ be linear functionals defined by (1.7), (1.9), (1.13) and (1.19). Then $p \mapsto \Delta_i(\Phi_p)$ is exponentially convex in the Jensen sense on J. If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is exponentially convex on J.*

Using analogous arguing as in the proof of [13, Corollary 3.2], we have the following corollary.

COROLLARY 2.2. *Let $\Gamma = \{\Phi_p : I \to \mathbb{R},\ p \in J\}$ be a family, such that the function $p \to [z_0, z_1, z_2; \Phi_p]$ is 2-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1,2,3,4)$ be a linear functionals defined by (1.7), (1.9), (1.13) and (1.19). Then the following statements hold:*

(i) If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is 2-exponentially convex function on J, thus log-convex on J and for $p, q, r \in I$ such that $p < q < r$, we have

$$\Delta_i(\Phi_q)^{r-p} \leqslant \Delta_i(\Phi_p)^{r-q} \Delta_i(\Phi_r)^{q-p}, \quad i = 1,2,3,4. \tag{2.6}$$

(ii) If the function $p \mapsto \Delta_i(\Phi_p)$ is strictly positive and differentiable on J, then for every $p, q, m, n \in J$ such that $p \leqslant m$, $q \leqslant n$, we have

$$\mathscr{B}_{p,q}(f, \Delta_i; \Gamma) \leqslant \mathscr{B}_{m,n}(f, \Delta_i; \Gamma), \quad i = 1,2,3,4 \tag{2.7}$$

where

$$\mathscr{B}_{p,q}(f, \Delta_i; \Gamma) = \begin{cases} \left(\frac{\Delta_i(\Phi_p)}{\Delta_i(\Phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\left(\Delta_i(\Phi_p)\right)}{\Delta_i(\Phi_p)} \right), & p = q, \end{cases} \tag{2.8}$$

for $\Phi_p, \Phi_q \in \Gamma$.

Proof. (i) This can be obtained as a direct consequence of Theorem 2.1 and Remark 1.3.

(ii) Since by (i) the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2,3,4)$ is log-convex on J, that is the function $p \mapsto \log\Delta_i(\Phi_p)$ for $(i = 1,2,3,4)$ is convex on J. Applying Proposition 1.2, we obtain

$$\frac{\log\Delta_i(\Phi_p) - \log\Delta_i(\Phi_q)}{p-q} \leqslant \frac{\log\Delta_i(\Phi_m) - \log\Delta_i(\Phi_n)}{m-n} \tag{2.9}$$

for $p \leqslant m$, $q \leqslant n$, $p \neq q$, $m \neq n$, and we conclude that

$$\mathscr{B}_{p,q}(f,\Delta_i;\Gamma) \leqslant \mathscr{B}_{m,n}(f,\Delta_i;\Gamma), \qquad (i = 1,2,3,4).$$

Cases $p = q$, $m = n$ follows from (2.9) as limiting case. □

REMARK 2.1. Note that the results of Theorem 2.1, Corollary 2.1 and Corollary 2.2 still hold when two of the points z_0, z_1, $z_2 \in I$ coincide for a family of differentiable functions Φ_p such that $p \mapsto [z_0,z_1,z_2;\Phi_p]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense), further, they still hold when all three point coincide for a family of twice differentiable functions with the same property. The proofs are obtained using (2.3), (2.4) and (2.5) respectively and some facts about the exponential convexity.

3. Examples

EXAMPLE 3.1. Consider a family of functions

$$\Gamma_1 = \{g_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\},$$

defined by

$$g_p(t) = \frac{e^{-t\sqrt{p}}}{p}.$$

Since $p \mapsto \frac{d^2 g_p(t)}{dt^2} = e^{-t\sqrt{p}}$ is the Laplace transform of a non-negative function, it is exponentially convex (see [15]). Clearly g_p are convex functions for every $p > 0$. It is obvious that $\Delta_i(g_p)$ for $(i = 1,2,3,4)$ are continuous. It is easy to prove that the function $p \mapsto [z_0,z_1,z_2;g_p]$ is also exponentially convex for arbitrary points $z_0,z_1,z_2 \in I$. For this family of functions, $\mathscr{B}_{p,q}(f,\Delta_i;\Gamma_1)$ becomes

$$\mathscr{B}_{p,q}(f,\Delta_i(g_p);\Gamma_1) = \begin{cases} \left(\frac{\Delta_i(g_p)}{\Delta_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{\Delta_i(id\cdot g_p)}{2\sqrt{p}\Delta_i(g_p)} - \frac{1}{p}\right), & p = q, \end{cases} \tag{3.1}$$

and from (2.7) it follows that the function $\mathscr{B}_{p,q}(f,\Delta_i;\Gamma_1)$ is monotonous in parameters p and q.

EXAMPLE 3.2. Let $\Gamma_2 = \{h_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$, be a family of functions defined by

$$h_p(t) = \begin{cases} \frac{p^{-t}}{(\ln p)^2}, & p \in \mathbb{R}_+ \setminus \{1\}, \\ \frac{t^2}{2}, & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2} h_p(t) = p^{-t}$ is the Laplace transform of a non-negative function (see [15]), it is exponentially convex. Obviously h_p are convex functions for every $p > 0$. It is easy to prove that the function $p \mapsto [z_0, z_1, z_2; h_p]$ is also exponentially convex for arbitrary points $z_0, z_1, z_2 \in I$. Using Corollary 2.1, it follows that $p \mapsto \Delta_i(h_p)$ for $(i = 1,2,3,4)$ are exponentially convex (it is easy to verify that these are continuous) and thus log-convex. From (2.8), we can write

$$\mathscr{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2) = \begin{cases} \left(\frac{\Delta_i(h_p)}{\Delta_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{\Delta_i(id \cdot h_p)}{p\Delta_i(h_p)} - \frac{2}{p \ln p}\right), & p = q \neq 1, \\ \exp\left(-\frac{\Delta_i(id \cdot h_1)}{3\Delta_i(h_1)}\right), & p = q = 1, \end{cases} \tag{3.2}$$

and from (2.7) it follows monotonicity of the functions $\mathscr{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2)$ in parameters p and q for $h_p, h_q \in \Gamma_2$.

EXAMPLE 3.3. Consider a family of functions

$$\Gamma_3 = \{\psi_p : \mathbb{R} \to [0,\infty) : p \in \mathbb{R}\},$$

defined with

$$\psi_p(t) = \begin{cases} \frac{1}{p^2} e^{tp}, & p \in \mathbb{R} \setminus \{0\}, \\ \frac{1}{2} t^2, & p = 0, \end{cases}$$

We have $\frac{d^2}{dt^2}(\psi_p(t)) = e^{tp} > 0$, which shows that ψ_p is convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^2}{dt^2}(\psi_p(t))$ is exponentially convex function by definition. Using the analogous arguments as in Theorem 2.1, we also have that $p \mapsto [z_0, z_1, z_2; \psi_p]$ is exponentially convex (also exponentially convex in J-sense). For the family of the function $\mathscr{B}_{p,q}(f, \Delta_i; \Gamma_3)$ for $(i = 1,2,3,4)$, (2.8) becomes

$$\mathscr{B}_{p,q}(f, \Delta_i(\psi_p); \Gamma_3) = \begin{cases} \left(\frac{\Delta_i(\psi_p)}{\Delta_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\Delta_i(id \cdot \psi_p)}{\Delta_i(\psi_p)} - \frac{2}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{\Delta_i(id \cdot \psi_0)}{3\Delta_i(\psi_0)}\right), & p = q = 0, \end{cases} \tag{3.3}$$

and using (2.7) we can see that these are monotonous functions in parameter p and q for $\psi_p, \psi_q \in \Gamma_3$.

EXAMPLE 3.4. Consider a family of functions

$$\Gamma_4 = \{\phi_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\},$$

defined by

$$\phi_p(t) = \begin{cases} \frac{t^p}{p(p-1)} & p \neq 1,0, \\ -\ln t & p = 0, \\ t\ln t & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2}(\phi_p(t)) = t^{p-2} = e^{(p-2)\ln t} > 0$ is the Laplace transform of a non-negative function (see [15]), it is exponentially convex. Obviously ϕ_p are convex functions for every $t > 0$. It is easy to prove that the function $p \mapsto [z_0, z_1, z_2; \phi_p]$ is also exponentially convex for arbitrary points $z_0, z_1, z_2 \in I$. Using Corollary 2.1 it follows that $p \mapsto \Delta_i(\phi_p)$ for $(i = 1,2,3,4)$ are exponentially convex (it is easy to verify that these are continuous), and thus log-convex. From (2.8), we see that

$$\mathscr{B}_{p,q}(f, \Delta_i(\phi_p); \Gamma_4) = \begin{cases} \left(\frac{\Delta_i(\phi_p)}{\Delta_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{\Delta_i(\phi_p\phi_0)}{\Delta_i(\phi_p)}\right), & p = q \neq 0,1, \\ \exp\left(1 - \frac{\Delta_i(\phi_0^2)}{2\Delta_i(\phi_0)}\right), & p = q = 0, \\ \exp\left(-1 - \frac{\Delta_i(\phi_0\phi_1)}{2\Delta_i(\phi_1)}\right), & p = q = 1, \end{cases} \tag{3.4}$$

for $\phi_p, \phi_q \in \Gamma_4$.

REMARK 3.1. For the case $i = 1$, the means given in (3.4) were already presented in [5] in explicit form.

Acknowledgements. This research is funded by Higher Education Commission of Pakistan. The research of the second and third author was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 117-1170889-0888.

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(Received February 28, 2013)

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