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OPTIMAL BOUNDS FOR TOADER MEAN IN TERMS OF ARITHMETIC AND CONTRAHARMONIC MEANS

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Abstract. We find the greatest value α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 C(a,b)+(1-\alpha_1)A(a,b) < T(a,b) < \beta_1 C(a,b)+(1-\beta_1)A(a,b)$ and $\alpha_2/A(a,b)+(1-\alpha_2)/C(a,b) < 1/T(a,b) < \beta_2/A(a,b)+(1-\beta_2)/C(a,b)$ hold for all $a,b>0$ with $a\neq b$. As applications, we get new bounds for the complete elliptic integral of the second kind. Here, $C(a,b)=(a^2+b^2)/(a+b)$, $A(a,b)=(a+b)/2$, and

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$

denote the contraharmonic, arithmetic, and Toader means of two positive numbers a and b, respectively.

1. Introduction

For $p\in\mathbb{R}$ and $a,b>0$, the contraharmonic mean $C(a,b)$, Toader mean $T(a,b)$ [1] and pth power mean $M_p(a,b)$ are defined by

$$C(a,b)=\frac{a^2+b^2}{a+b},$$

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$

$$=\begin{cases} 2a\mathscr{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a>b,\\ 2b\mathscr{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a<b,\\ a, & a=b\end{cases}$$

and

$$M_p(a,b)=\begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases}$$

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respectively. Here, $\mathscr{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2 t)^{1/2}dt$ ($r\in[0,1]$) is the complete elliptic integral of the second kind.

The Toader mean $T(a,b)$ is well known in mathematical literature for many years, it satisfies

$$T(a,b)=R_E(a^2,b^2)$$

and

$$T(1,r)=\frac{2}{\pi}\mathscr{E}(\sqrt{1-r^2})$$

for all $a,b>0$ and $0<r<1$, where

$$R_E(a,b)=\frac{1}{\pi}\int_0^\infty\frac{[a(t+b)+b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}}dt$$

stands for the symmetric complete elliptic integral of the second kind (see [2–4]), therefore it can't be expressed in terms of the elementary transcendental functions.

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. Many means are special case of $M_p(a,b)$, for example, $M_{-1}(a,b)=H(a,b)=2ab/(a+b)$, $M_0(a,b)=G(a,b)=\sqrt{ab}$, $M_1(a,b)=A(a,b)=(a+b)/2$ and $M_2(a,b)=Q(a,b)=\sqrt{(a^2+b^2)/2}$ are known in the literature as harmonic, geometric, arithmetic and quadratic means, respectively. Recently, the Toader mean has been the subject of intensive research. In particular, many remarkable inequalities for $T(a,b)$ can be found in the literature [5–9].

Vuorinen [10] conjectured that

$$M_{3/2}(a,b)<T(a,b) \tag{1.1}$$

for all $a,b>0$ with $a\neq b$. This conjecture was proved by Qiu and Shen [11], and Barnard, Pearce and Richards [12], respectively.

In [13], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b)<M_{\log 2/\log(\pi/2)}(a,b) \tag{1.2}$$

for all $a,b>0$ with $a\neq b$.

Neuman [2, Corollary 4.3] proved that the double inequality

$$\frac{(a+b)\sqrt{ab}-ab}{AGM(a,b)}<T(a,b)<\frac{4(a+b)\sqrt{ab}+(a-b)^2}{8AGM(a,b)}$$

holds for all $a,b>0$, where $AGM(a,b)$ is the classical arithmetic-geometric mean of a and b, which is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_0=a,\qquad b_0=b,$$
$$a_{n+1}=(a_n+b_n)/2=A(a_n,b_n),\qquad b_{n+1}=\sqrt{a_nb_n}=G(a_n,b_n).$$

In [3, Theorem 4.3], Kazi and Neuman found that the inequality

$$T(a,b) < \sqrt{\frac{\sqrt{2}-1}{8\sqrt{2}}a^2 + \frac{\sqrt{2}+1}{8\sqrt{2}}b^2} + \sqrt{\frac{\sqrt{2}+1}{8\sqrt{2}}a^2 + \frac{\sqrt{2}-1}{8\sqrt{2}}b^2}$$

holds for all $a,b>0$ with $a \neq b$.

In [7], the authors proved that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 Q(a,b) + (1-\beta_1)A(a,b),$$

$$Q^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < T(a,b) < Q^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 1/2$, $\beta_1 \geqslant (4-\pi)/[(\sqrt{2}-1)\pi] = 0.659\ldots$, $\alpha_2 \leqslant 1/2$ and $\beta_2 \geqslant 4 - 2\log\pi/\log 2 = 0.697\ldots$.

It is not difficult to verify that

$$C(a,b) > M_2(a,b) = \sqrt{\frac{a^2+b^2}{2}} \tag{1.3}$$

for all $a,b>0$ with $a \neq b$.

From (1.1)–(1.3) we clearly see that

$$A(a,b) < T(a,b) < C(a,b)$$

for all $a,b>0$ with $a \neq b$.

The main purpose of the paper is to find the greatest value α_1 and α_2, and the least values β_1 and β_2, such that the double inequalities $\alpha_1 C(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 C(a,b) + (1-\beta_1)A(a,b)$ and $\alpha_2/A(a,b) + (1-\alpha_2)/C(a,b) < 1/T(a,b) < \beta_2/A(a,b) + (1-\beta_2)/C(a,b)$ hold for all $a,b>0$ with $a \neq b$. As applications, we present new bounds for the complete elliptic integral of the second kind.

2. Basic knowledge and Lemmas

In order to establish our main results we need some basic knowledge and lemmas, which we present in this section.

For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, the well-known complete elliptic integrals of the first and second kinds are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2} d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \quad \mathcal{K}(1) = +\infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \quad \mathcal{E}(1) = 1, \end{cases}$$

respectively, and the following formulas were presented in [14, Appendix E. pp. 474–475]:

$$\frac{d\mathscr{K}}{dr}=\frac{\mathscr{E}-r'^2\mathscr{K}}{rr'^2},\qquad \frac{d\mathscr{E}}{dr}=\frac{\mathscr{E}-\mathscr{K}}{r},$$

$$\frac{d(\mathscr{E}-r'^2\mathscr{K})}{dr}=r\mathscr{K},\qquad \frac{d(\mathscr{K}-\mathscr{E})}{dr}=\frac{r\mathscr{E}}{r'^2},\quad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}-r'^2\mathscr{K}}{1+r}.$$

LEMMA 2.1. (see [14, Theorem 3.21(1)]) $(\mathscr{E}(r)-r'^2\mathscr{K}(r))/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$.

LEMMA 2.2. $5\mathscr{E}(r)-3r'^2\mathscr{K}(r)$ *is positive and strictly increasing on (0,1).*

Proof. Let $f(r)=5\mathscr{E}(r)-3r'^2\mathscr{K}(r)$ and $g(r)=2\mathscr{E}(r)-2\mathscr{K}(r)+3r^2\mathscr{K}(r)$. Then simple computations lead to

$$f(0)=\pi, \tag{2.1}$$

$$g(0)=0, \tag{2.2}$$

$$f'(r)=\frac{g(r)}{r}, \tag{2.3}$$

$$g'(r)=\frac{r(\mathscr{E}(r)+3r'^2\mathscr{K}(r))}{r'^2}>0 \tag{2.4}$$

for all $r\in(0,1)$.

Therefore, Lemma 2.2 follows easily from (2.1)–(2.4). □

LEMMA 2.3. (see [14, Theorem 1.25]) *For* $-\infty<a<b<\infty$, *let* $f,g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x)\neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}\ \ \textit{and}\ \ \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

3. Main results

THEOREM 3.1. *The double inequality*

$$\alpha_1C(a,b)+(1-\alpha_1)A(a,b)<T(a,b)<\beta_1C(a,b)+(1-\beta_1)A(a,b) \tag{3.1}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\leqslant 1/4$ *and* $\beta_1\geqslant 4/\pi-1=0.2732\cdots$.

Proof. Since $A(a,b)$, $T(a,b)$ and $C(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a>b$. Let $t=b/a\in(0,1)$ and $r=(1-t)/(1+t)\in(0,1)$. Then simple computations leads to

$$\frac{T(a,b)-A(a,b)}{C(a,b)-A(a,b)}=\frac{\frac{2}{\pi}\mathscr{E}'(t)-\frac{1+t}{2}}{\frac{1+t^2}{1+t}-\frac{1+t}{2}}=\frac{\frac{2}{\pi}\mathscr{E}(\frac{2\sqrt{r}}{1+r})-\frac{1}{1+r}}{\frac{r^2}{1+r}}=\frac{\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))-1}{r^2}. \tag{3.2}$$

Let $f_1(r)=\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))-1$, $f_2(r)=r^2$ and

$$f(r)=\frac{f_1(r)}{f_2(r)}=\frac{\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))-1}{r^2}. \tag{3.3}$$

Then simple computations lead to

$$f_1(0)=f_2(0)=0, \tag{3.4}$$

$$f_1'(r)=\frac{2}{\pi}\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r},$$

$$f_2'(r)=2r,$$

$$\frac{f_1'(r)}{f_2'(r)}=\frac{1}{\pi}\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r^2}. \tag{3.5}$$

It follows from Lemmas 2.1 and 2.3 together with (3.3)–(3.5) that $f(r)$ is strictly increasing on $(0,1)$. Moreover,

$$\lim_{r\to0^+}f(r)=\frac{1}{4} \tag{3.6}$$

and

$$\lim_{r\to1^-}f(r)=\frac{4}{\pi}-1. \tag{3.7}$$

Therefore, inequality (3.1) follows from (3.2), (3.3), (3.6) and (3.7) together with the monotonicity of $f(r)$. □

THEOREM 3.2. *The double inequality*

$$\frac{\alpha_2}{A(a,b)}+\frac{1-\alpha_2}{C(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_2}{A(a,b)}+\frac{1-\beta_2}{C(a,b)} \tag{3.8}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\leqslant\pi/2-1=0.5707\cdots$ and $\beta_2\geqslant3/4$.

Proof. Without loss of generality, we assume that $a>b$. Let $r=(a-b)/(a+b)\in(0,1)$. Then

$$\frac{\frac{1}{T(a,b)}-\frac{1}{C(a,b)}}{\frac{1}{A(a,b)}-\frac{1}{C(a,b)}}=\frac{1+r^2-\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))}{\frac{2}{\pi}r^2(2\mathscr{E}(r)-r'^2\mathscr{K}(r))}. \tag{3.9}$$

Let $g_1(r)=1+r^2-\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))$, $g_2(r)=\frac{2}{\pi}r^2(2\mathscr{E}(r)-r'^2\mathscr{K}(r))$ and

$$g(r)=\frac{g_1(r)}{g_2(r)}=\frac{1+r^2-\frac{2}{\pi}(2\mathscr{E}(r)-r'^2\mathscr{K}(r))}{\frac{2}{\pi}r^2(2\mathscr{E}(r)-r'^2\mathscr{K}(r))}. \tag{3.10}$$

Then simple computations lead to

$$g_1(0)=g_2(0)=0, \tag{3.11}$$

$$g_1'(r) = 2r - \frac{2}{\pi}\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{r},$$

$$g_2'(r) = \frac{2r}{\pi}(5\mathscr{E}(r) - 3r'^2\mathscr{K}(r)),$$

$$\frac{g_1'(r)}{g_2'(r)} = \frac{1 - \frac{1}{\pi}\frac{(\mathscr{E}(r) - r'^2\mathscr{K}(r))}{r^2}}{\frac{1}{\pi}(5\mathscr{E}(r) - 3r'^2\mathscr{K}(r))}. \tag{3.12}$$

It follows from Lemmas 2.1 and 2.2 together with (3.12) that $g_1'(r)/g_2'(r)$ is strictly decreasing on $(0,1)$. Then (3.10) and (3.11) lead to the conclusion that $g(r)$ is is strictly decreasing on $(0,1)$. Moreover,

$$\lim_{r\to 0^+} g(r) = \frac{3}{4} \tag{3.13}$$

and

$$\lim_{r\to 1^-} g(r) = \frac{\pi}{2} - 1. \tag{3.14}$$

Therefore, inequality (3.8) follows from (3.9), (3.10), (3.13) and (3.14) together with the monotonicity of $g(r)$. □

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