Journal of
Mathematical
Inequalities
Volume 7, Number 4 (2013), 759–772 doi:10.7153/jmi-07-69

THE APPLICATIONS ON SOME INEQUALITIES OF THE COMPOSITION OF ENTIRE FUNCTIONS

HONG-YAN XU*, JIN TU AND CAI-FENG YI

(*Communicated by V. Gol'dshtein*)

Abstract. The purpose of our paper is to deal with some growth problem of two composite entire functions of finite $[p,q]$-order and some properties of growth of $f\circ g$, f and g. Some results are obtained as follows: Let f,g be two entire functions and have index-pair $[p_1,q_1]$, $[p_2,q_2]$, respectively. Let $f\circ g$ have index-pair $[p_3,q_3]$. Then we have the following conclusions:

(i) if $p_2+1-q_1>0$, then $p_3=p_1+p_2-q_1+1 \Longleftrightarrow q_3=q_2$;

(ii) if $p_2+1-q_2=0$, then $p_3=p_1 \Longleftrightarrow q_3=q_2$;

(iii) if $p_2+1-q_1<0$, then $p_3=q_1 \Longleftrightarrow q_3=q_1+q_2-p_2-1$.

These results are some improvement and generalization of the form theorems given by Gross, Lahiri, Tu.

1. Introduction

This paper will use standard notation of value distribution theory [5, 18, 19]. Let f be a transcendental entire function, we denote the order of f by $\rho(f)$, the lower order of f by $\lambda(f)$, the Nevanlinna characteristic of f by $T(r,f)$ and the maximum modulus of f by $M(r,f)=\max_{|z|=r|}\{|f(z)|\}$.

Let f, g be two entire functions and suppose that g is transcendental. Pólya [13] investigated the relations between $\rho(f\circ g)$ and $\rho(f)$, $\lambda(f)$ and obtained: if $\rho(f\circ g)<\infty$, then $\rho(f)=0$. Gross [4] pointed out the one can also prove the result: if $\lambda(f\circ g)<\infty$, then $\lambda(f)=0$, by using the Pólya's method.

Many authors have investigated the composition of two entire functions with finite order and achieved many great results (see [1–2, 4, 9, 11, 3, 15]). It should be noted that few paper is concerned with the composition of entire functions with infinite order. Schönhage A. [14] and Tu J., Chen Z. X. and Zheng X. M. [17] investigated the composition of two entire functions with infinite order. To state their results, some definitions and notions about iterated order are introduced as follows (see [8, 17]).

DEFINITION 1. The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f)=\limsup_{r\to\infty}\frac{\log_{i+1}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log_i T(r,f)}{\log r},\quad (i\in N).$$

Mathematics subject classification (2010): 30D30, 30D35.
Keywords and phrases: Entire function, composition, infinite order.
* The corresponding author.

Paper JMI-07-69

Similarly, we can define the iterated i lower order $\lambda_i(f)$ of an entire function f by

$$\lambda_i(f) = \liminf_{r\to\infty} \frac{\log_{i+1} M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log_i T(r,f)}{\log r}, \quad (i \in N),$$

where $\log_1 r = \log r$, $\log_{i+1} r = \log(\log_i r)$ $(i \in N)$, for all sufficiently large r.

REMARK 1. We define $\exp_1 r = e^r$, $\exp_{i+1} r = \exp(\exp_i r)$ $(i \in N)$, $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log r$, for all $r \in [0,\infty)$.

REMARK 2. We can get that $\rho_1(f) = \rho(f)$ and $\lambda_1(f) = \lambda(f)$ from $i = 1$ of Definition 1.

DEFINITION 2. (see [8]). The growth index of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ rantional;} \\ \min\{n \in N : \rho_n(f) < \infty\} & \text{if } f \text{ transendental and } \rho_n(f) < \infty \text{ for some } n \in N\,; \\ \infty & \text{if } f \text{ with } \rho_n(f) = \infty \text{ for all } n \in N. \end{cases}$$

Schönhage [14] investigated the growth of composition of two entire functions with finite iterated order and obtained the following results

THEOREM 1. (see [14]). *Let f, g be two entire functions and suppose that g is transcendental. If $\rho_p(f\circ g) < \infty$, $(p \in N)$, then $\rho_p(f) = 0$.*

REMARK 3. We can get that $\lambda_p(f\circ g) < \infty$, $(p \in N)$, then $\lambda_p(f) = 0$, by using the similar method of Theorem 1.

In 1986, Zhou [20] investigated the growth of composition of entire functions of finite order and obtained the following results:

THEOREM 2. (see [20]). *let f, g be entire functions of finite order such that $g(0) = 0$ and $\rho(g) < \lambda(f) \leqslant \rho(f)$, then*

$$\lim_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} = 0.$$

In 2009, Tu, Chen and Zheng [17] investigated the growth of two composite entire functions of finite iterated order and obtained a series results about the relationships among $T(r, f\circ g)$, $T(r,f)$ and $T(r,g)$ as follows which improved Theorem 2.

THEOREM 3. (see [17, Theorem 3.1]). *Let f, g be entire functions of finite iterated order with $i(f) = p$, $i(g) = q$, if $\lambda_p(f) > 0$, then $i(f\circ g) = p+q$ and $\rho_{p+q}(f\circ g) = \rho_q(g)$.*

THEOREM 4. (see [17, Theorem 4.1]). *Let f, g be entire functions of iterated order with $i(f)=p$, $i(g)=q$, $\rho_q(g)<\lambda_p(f)\leqslant\rho_p(f)$, then*

$$\lim_{r\to\infty}\frac{\log_q T(r,f\circ g)}{T(r,f)}=0,\quad \lim_{r\to\infty}\frac{\log_{q+1}M(r,f\circ g)}{\log M(r,f)}=0.$$

In [6, 7], O. P. Juneja and his co-authors introduced the concept of entire functions of $[p,q]$-order and lower $[p,q]$-order, and obtained some theorems about theirs properties. In this paper, we further investigated the growth of composition of two entire functions with infinite order by using the concepts of entire functions of $[p,q]$-order and lower $[p,q]$-order. To state our theorems, we first introduce the concepts of entire functions of $[p,q]$-order and lower $[p,q]$-order (see [6–7, 10]).

DEFINITION 3. If $f(z)$ is a transcendental entire function, the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f)=\limsup_{r\to\infty}\frac{\log_{p+1}M(r,f)}{\log_q r}=\limsup_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r}.$$

Similarly, we can define lower $[p,q]$-order of $f(z)$ by

$$\lambda_{[p,q]}(f)=\liminf_{r\to\infty}\frac{\log_{p+1}M(r,f)}{\log_q r}=\liminf_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r},$$

where p,q are positive integers satisfying $p\geqslant q\geqslant 1$.

REMARK 4. It is easy to see that $0\leqslant\lambda_{[p,q]}(f)\leqslant\rho_{[p,q]}(f)\leqslant\infty$. If $f(z)$ is a polynomial, then $\lambda_{[p,q]}(f)=\rho_{[p,q]}(f)=0$ for any $p\geqslant q\geqslant 1$. By Definition 1, we have that $\rho_{[1,1]}(f)=\rho_1(f)=\rho(f)$, $\lambda_{[1,1]}(f)=\lambda_1(f)=\lambda(f)$, $\rho_{[p+1,1]}(f)=\rho_p(f)$ and $\lambda_{[p+1,1]}(f)=\lambda_p(f)$.

REMARK 5. If $f(z)$ is an entire function satisfying $0<\rho_{[p,q]}(f)<\infty$, then

(i) $\rho_{[p-n,q]}(f)=\infty$ $(n<p)$, $\rho_{[p,q-n]}(f)=0$ $(n<q)$, $\rho_{[p+n,q+n]}(f)=1$ $(n<p)$ for $n=1,2,\ldots$;

(ii) If $[p',q']$ is any pair of integers satisfying $q'=p'+q-p$ and $p'<p$, then $\rho_{[p',q']}(f)=0$ if $0<\rho_{[p,q]}(f)<1$ and $\rho_{[p',q']}(f)=\infty$ if $1<\rho_{[p,q]}(f)<\infty$;

(iii) $\rho_{[p',q']}(f)=\infty$ for $q'-p'>q-p$ and $\rho_{[p',q']}(f)=0$ for $q'-p'<q-p$.

Similarly, we have some analogous properties of $\lambda_{[p,q]}(f)$.

DEFINITION 4. A transcendental entire function $f(z)$ is said to have index-pair $[p,q]$, if $0<\rho_{[p,q]}(f)<\infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.

DEFINITION 5. Let f_1, f_2 be two entire functions such that $\rho_{[p_1,q_1]}(f_1)=\rho_1$, $\rho_{[p_2,q_2]}(f_2)=\rho_2$ and $p_1\leqslant p_2$. Then the following results about their comparative growth can be easily deduced:

(i) If $p_2-p_1>q_2-q_1$, then the growth of f_1 is slower than the growth of f_2;

(ii) If $p_2 - p_1 < q_2 - q_1$, then f_1 grows faster than f_2;

(iii) If $p_2 - p_1 = q_2 - q_1 > 0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2 \geqslant 1$ while the growth of f_1 is faster than the growth of f_2 if $\rho_2 < 1$;

(iv) Let $p_2 - p_1 = q_2 - q_1 = 0$, then f_1, f_2 are of the same index-pair $[p_1, q_1]$. If $\rho_1 > \rho_2$, then f_1 grows faster than f_2, and if $\rho_1 < \rho_2$, then f_1 grows slower than f_2. If $\rho_1 = \rho_2$, Definition 3 does not give any precise estimate about the relative growth of f_1 and f_2.

Now, some results of our paper about the growth of composition of two entire functions of infinite order are stated as follows which generalize some previous results.

Let f, g be two entire functions, and f, g have index-pair $[p_1, q_1]$, $[p_2, q_2]$, respectively. Set

$$A_l = \lambda_{[p_1,q_1]}(f), \quad A = \rho_{[p_1,q_1]}(f), \quad B_l = \lambda_{[p_2,q_2]}(g) \quad \textit{and} \quad B = \rho_{[p_2,q_2]}(g).$$

THEOREM 5. *Let f, g be two entire functions, and f, g have index-pair $[p_1, q_1]$, $[p_2, q_2]$, respectively. If $0 < A_l \leqslant A \leqslant \infty$, then*

(i) $p_2 + 1 - q_1 > 0$, we have $\rho_{[p_1+p_2-q_1+1,q_2]}(f \circ g) = \rho_{[p_2,q_2]}(g) = B$;

(ii) $p_2 + 1 - q_1 = 0$, we have $A_l B \leqslant \rho_{[p_1+p_2-q_1+1,q_2]}(f \circ g) \leqslant AB$;

(iii) $p_2 + 1 - q_1 < 0$ and $q_1 + q_2 - p_2 - 1 \geqslant 1$, we have $A_l \leqslant \rho_{[p_1,q_1+q_2-p_2-1]}(f \circ g) \leqslant A$ for $B > 0$ and $\rho_{[p_1,q_1+q_2-p_2-1]}(f \circ g) \leqslant A$, $\rho_{[p_1,q_1+q_2-p_2]}(f \circ g) \geqslant A_l$ for $B = 0$.

THEOREM 6. *Let f, g be entire functions, and f, g have index-pair $[p_1, q_1]$, $[p_2, q_2]$, respectively. If $0 < A < \infty$ and $0 < B_l \leqslant B < \infty$. We have*

(i) if $p_2 + 1 - q_1 > 0$, then $B_l \leqslant \rho_{[p_1+p_2-q_1+1,q_2]}(f \circ g) \leqslant B$;

(ii) if $p_2 + 1 - q_1 = 0$, then $AB_l \leqslant \rho_{[p_1,q_2]}(f \circ g) \leqslant AB$;

(iii) if $p_2 + 1 - q_1 < 0$ and $q_1 + q_2 - p_2 - 1 \geqslant 1$, then $\rho_{[p_1,q_1+q_2-p_2-1]}(f \circ g) = A$.

REMARK 6. We can easily get Theorem 3 and Theorem 3.2 of [17] when $q_1 = q_2 = 1$ in Theorems 5 and 6.

THEOREM 7. *Let f, g be two entire functions, and f, g have index-pair $[p_1, q_1]$, $[p_2, q_2]$, respectively. We can get the following conclusions:*

(1) $p_2 + 1 - q_1 > 0$, then

(i) $q_1 > q_2$ and $B_l > 0$, we have

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f \circ g)}{\log_{p_1+1} M(r, f)} = \infty;$$

(ii) $q_1 = q_2$ and $0 < A_l, A, B_l, B < \infty$, we have

$$\frac{B_l}{A} \leqslant \lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f \circ g)}{\log_{p_1+1} M(r, f)} \leqslant \frac{B}{A_l};$$

(iii) $q_1 < q_2$ and $A_l > 0$, we have

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f \circ g)}{\log_{p_1+1} M(r, f)} = 0;$$

If $p_2+1-q_1=0$, *then*
(i) $q_1>q_2$ *and* $A_l>0$, $B_l>0$, *we have*

$$\lim_{r\to\infty}\frac{\log_{p_1+1}M(r,f\circ g)}{\log_{p_1+1}M(r,f)}=\infty;$$

(ii) $q_1=q_2$ *and* $0<A_l,A,B_l,B<\infty$, *we have*

$$\frac{A_lB_l}{A}\leqslant\lim_{r\to\infty}\frac{\log_{p_1+1}M(r,f\circ g)}{\log_{p_1+1}M(r,f)}\leqslant\frac{AB}{A_l};$$

(iii) $q_1<q_2$ *and* $0<A_l,A,B<\infty$, *we have*

$$\lim_{r\to\infty}\frac{\log_{p_1+1}M(r,f\circ g)}{\log_{p_1+1}M(r,f)}=0;$$

If $p_2+1-q_1<0$ *and* $0<A_l,A<\infty$, *then we have*

$$\lim_{r\to\infty}\frac{\log_{p_1+1}M(r,f\circ g)}{\log_{p_1+1}M(r,f)}=\infty.$$

REMARK 7. From Theorem 7, we can get a series results of comparative growths of $M(r,f\circ g)$ and $M(r,g)$.

THEOREM 8. *Let* f, g *be entire functions, and* g *have index-pair* $[p_2,q_2]$ *satisfying* $B_l>0$. *If* $f\circ g$ *have index-pair* $[p_1,q_1]$ *and* $0<\lambda_{[p_1,q_1]}(f\circ g)=\chi<\infty$, *then* $\lambda_{[p_1,q_1]}(f)=0$.

THEOREM 9. *Let* f, g *be entire functions, and* g *have index-pair* $[p_2,q_2]$ *satisfying* $B_l>0$. *If* $f\circ g$ *have index-pair* $[p_1,q_1]$ *and* $0<\rho_{[p_1,q_1]}(f\circ g)=\zeta<\infty$, *then* $\rho_{[p_1,q_1]}(f)=0$.

REMARK 8. It is easily to see that Theorems 8 and 9 generalize and improve some results given by Gross [4].

THEOREM 10. *Let* f, g *be two entire functions and* p,q *be two positive integers satisfying* $p\geqslant q\geqslant 1$, *if* $\lambda_{[p,q]}(f\circ g)=\sigma<\lambda_{[p,q]}(g)=\tau<\infty$, *then* $\lambda(f)=0$.

From Theorems 8–10 and Theorem 5, we can easily get the following results.

THEOREM 11. *Let* f, g *be two entire functions and have index-pair* $[p_1,q_1]$, $[p_2,q_2]$, *respectively. Let* $f\circ g$ *have index-pair* $[p_3,q_3]$. *Then we can get the following conclusions:*
(i) if $p_2+1-q_1>0$, *then* $p_3=p_1+p_2-q_1+1\Longleftrightarrow q_3=q_2$;
(ii) if $p_2+1-q_1=0$, *then* $p_3=p_1\Longleftrightarrow q_3=q_2$;
(iii) if $p_2+1-q_1<0$, *then* $p_3=q_1\Longleftrightarrow q_3=q_1+q_2-p_2-1$.

For an entire function

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the maximum term $\mu(r,f_1)$ of f_1 on $|z| = r$ being defined by $\mu(r,f_1) = \max_{n\geqslant 0}\{|a_n|r^n\}$.

Since for $0 < r < R$, we have

$$\mu(r,f_1) \leqslant M(r,f_1) \leqslant \frac{R}{R-r}\mu(R,f_1).$$

Thus, from the above inequality and Definition 3, we can get the definition of the $[p,q]$-order and lower $[p,q]$-order of f_1 as follows.

DEFINITION 6. The $[p,q]$-order and lower $[p,q]$-order of $f_1(z)$ are defined by

$$\rho_{[p,q]}(f_1) = \limsup_{r\to\infty} \frac{\log_{p+1}\mu(r,f_1)}{\log_q r}$$

and

$$\lambda_{[p,q]}(f_1) = \liminf_{r\to\infty} \frac{\log_{p+1}\mu(r,f_1)}{\log_q r},$$

where p,q are positive integers satisfying $p \geqslant q \geqslant 1$.

For Theorems 7, it is a natural question to ask: what will happen when entire function f is replaced by entire function f_1 in Theorems 7? We investigate the above question and obtain some results as follows.

THEOREM 12. *Let f_1, g be two entire functions, and f_1, g have index-pair $[p_1,q_1]$, $[p_2,q_2]$, respectively. Let $L_l = \lambda_{[p_1,q_1]}(f_1)$ and $L = \rho_{[p_1,q_1]}(f_1)$. We can get the following conclusions:*

(1) $p_2 + 1 - q_1 > 0$.

(i) $q_1 > q_2$ and $B_l > 0$, then

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1}\mu(r,f_1\circ g)}{\log_{p_1+1}\mu(r,f_1)} = \infty;$$

(ii) $q_1 = q_2$ and $0 < L, B_l < \infty$, then

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1}\mu(r,f_1\circ g)}{\log_{p_1+1}\mu(r,f_1)} \geqslant \frac{B_l}{L};$$

(2) $p_2 + 1 - q_1 = 0$.

(i) $q_1 > q_2$ and $L_l > 0, B_l > 0$, then

$$\lim_{r\to\infty} \frac{\log_{p_1+1}\mu(r,f_1\circ g)}{\log_{p_1+1}\mu(r,f_1)} = \infty;$$

(ii) $q_1 = q_2$ *and* $0 < L_l, L, B_l < \infty$*, then*

$$\lim_{r\to\infty} \frac{\log_{p_1+1} \mu(r, f_1 \circ g)}{\log_{p_1+1} \mu(r, f_1)} \geqslant \frac{L_l B_l}{L};$$

(3) $p_2 + 1 - q_1 < 0$. *If* $0 < L_l, L < \infty$*, then*

$$\lim_{r\to\infty} \frac{\log_{p_1+1} \mu(r, f_1 \circ g)}{\log_{p_1+1} \mu(r, f_1)} = \infty.$$

2. Some Lemmas

For the proof of our results we need the following lemmas.

LEMMA 1. (see [11]). *Let* f, g *be entire functions. If* $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ *for any* $\varepsilon > 0$*, then*

$$T(r, f \circ g) < (1+\varepsilon) T(M(r,g), f). \tag{1}$$

In particular if $g(0) = 0$*, then*

$$T(r, f \circ g) \leqslant T(M(r,g), f) \tag{2}$$

for all $r > 0$.

LEMMA 2. (see [3]). *Let* f, g *be entire functions with* $g(0) = 0$. *Let* α *satisfy* $0 < \alpha < 1$ *and let* $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. *Then for* $r > 0$,

$$M(M(r,g), f) \geqslant M(r, f \circ g) \geqslant M(c(\alpha) M(\alpha r, g), f). \tag{3}$$

Furthermore if $\alpha = \frac{1}{2}$*, for sufficiently large* r,

$$M(r, f \circ g) \geqslant M\left(\frac{1}{8} M\left(\frac{1}{2} r, g\right), f\right). \tag{4}$$

LEMMA 3. (see [16]). *Let* f *and* g *be entire functions with* $g(0) = 0$. *Let* α *satisfy* $0 < \alpha < 1$ *and let* $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. *Also let* $0 < \delta < 1$. *Then*

$$\mu(r, f \circ g) \geqslant (1-\delta) \mu(c(\alpha) \mu(\alpha\delta r, g), f).$$

And if g *is any entire function with* $\alpha = \delta = \frac{1}{2}$*, for sufficiently large* r,

$$\mu(r, f \circ g) \geqslant \frac{1}{2} \mu\left(\frac{1}{8} \mu\left(\frac{1}{4} r, g\right), f\right).$$

3. Proofs of Theorems 5 and 6

Proof of Theorem 5. From the assumptions of Theorem 5, we have

$$A = \limsup_{r\to\infty} \frac{\log_{p_1} T(r,f)}{\log_{q_1} r}, \quad B = \limsup_{r\to\infty} \frac{\log_{p_2+1} M(r,g)}{\log_{q_2} r}. \tag{5}$$

Then for any $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,f) \leqslant \exp_{p_1}\left\{(A+\varepsilon)\log_{q_1} r\right\}, \quad M(r,g) \leqslant \exp_{p_2+1}\left\{(B+\varepsilon)\log_{q_2} r\right\}. \tag{6}$$

From (6) and Lemma 1, we have

$$\begin{aligned} T(r,f\circ g) &\leqslant 2T(M(r,g),f) \leqslant 2\exp_{p_1}\left\{(A+\varepsilon)\log_{q_1} M(r,g)\right\} \\ &\leqslant 2\exp_{p_1}\left\{(A+\varepsilon)\log_{q_1}\left(\exp_{p_2+1}\left\{(B+\varepsilon)\log_{q_2} r\right\}\right)\right\} \\ &\leqslant 2\exp_{p_1}\left\{(A+\varepsilon)\exp_{p_2+1-q_1}\left\{(B+\varepsilon)\log_{q_2} r\right\}\right\}. \end{aligned} \tag{7}$$

Then

(i) $p_2+1-q_1>0$. From (7), we have

$$\limsup_{r\to\infty} \frac{\log_{p_1+p_2-q_1+1} T(r,f\circ g)}{\log_{q_2} r} \leqslant B. \tag{8}$$

i.e., $\rho_{[p_1+p_2-q_1+1,q_2]}(f\circ g) \leqslant B$;

(ii) $p_2+1-q_1=0$. From (7), we have $\rho_{[p_1,q_2]}(f\circ g)\leqslant AB$;

(iii) $p_2+1-q_1<0$. From (7) and $q_1+q_2-p_2-1\geqslant 1$, we have $\rho_{[p_1,q_1+q_2-p_2-1]}(f\circ g)\leqslant A$.

Now, we consider two cases as follows.

Case 1. $B>0$. Then there exists a sequence $\{r_n\}\to\infty$ such that for any ε $(0<\varepsilon<B)$ and sufficiently large r_n, we have

$$M(r_n,g) \geqslant \exp_{p_2+1}\left\{(B-\varepsilon)\log_{q_2} r_n\right\}. \tag{9}$$

Since $A_l>0$ and by the same reasoning as K. Niino and C. C. Yang (see [12]), for sufficiently larger r_n, from (9), we have

$$\begin{aligned} T(r_n,f\circ g) &\geqslant \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r_n}{4},g\right)+o(1),f\right) \geqslant \frac{1}{3}\log M\left(\frac{1}{9}M\left(\frac{r_n}{4},g\right),f\right) \\ &\geqslant \frac{1}{3}\exp_{p_1}\left\{(A_l-\varepsilon)\log_{q_1}\left(\frac{1}{9}M\left(\frac{r_n}{4},g\right)\right)\right\} \\ &\geqslant \frac{1}{3}\exp_{p_1}\left\{(A_l-\varepsilon)\log_{q_1}\left(\exp_{p_2+1}\left\{(B-\varepsilon)(\log_{q_2} r_n+O(1))\right\}\right)\right\} \\ &\geqslant \frac{1}{3}\exp_{p_1}\left\{(A_l-\varepsilon)\exp_{p_2+1-q_1}\left\{(B-\varepsilon)(\log_{q_2} r_n+O(1))\right\}\right\}. \end{aligned} \tag{10}$$

Then

(i) $p_2+1-q_1>0$. From (10), we have

$$\lim_{r_n\to\infty}\frac{\log_{p_1+p_2-q_1+1}T(r_n,f\circ g)}{\log_{q_2}r_n}\geqslant B. \tag{11}$$

By (8) and (11), we have

$$\lim_{r_n\to\infty}\frac{\log_{p_1+p_2-q_1+1}T(r_n,f\circ g)}{\log_{q_2}r_n}=B;$$

(ii) $p_2+1-q_1=0$. From (10), we have $\rho_{[p_1,q_2]}(f\circ g)\geqslant A_l B$;

(iii) $p_2+1-q_1<0$. From (10) and $q_1+q_2-p_2-1\geqslant 1$, we have $\rho_{[p_1,q_1+q_2-p_2-1]}(f\circ g)\geqslant A_l$.

Case 2. $B=0$. Then we have

$$\limsup_{r\to\infty}\frac{\log_{p_2}M(r,g)}{\log_{q_2}r}=\infty.$$

Hence, there exists a sequence $\{r_n\}\to\infty$ such that for any $\eta>0$, we have

$$\lim_{r_n\to\infty}\frac{\log_{p_2}M(r_n,g)}{\log_{q_2}r_n}\geqslant\eta,\quad i.e.,\quad M(r_n,g)\geqslant\exp_{p_2}\left(\eta\log_{q_2}r_n\right). \tag{12}$$

Then

(i) $p_2+1-q_1>0$. From (12) and (10), we have

$$\lim_{r_n\to\infty}\frac{\log_{p_1+p_2-q_1}T(r_n,f\circ g)}{\log_{q_2}r_n}\geqslant\eta.$$

Since η is arbitrarily large, we have

$$\lim_{r\to\infty}\frac{\log_{p_1+p_2-q_1}T(r,f\circ g)}{\log_{q_2}r}=\infty. \tag{13}$$

By (8) and (13), we get $\rho_{[p_1+p_2-q_1,q_2]}(f\circ g)=\rho_{[p_2,q_2]}(g)=B$;

(ii) $p_2+1-q_1=0$. From (12) and (10), we have $\rho_{[p_1-1,q_2]}(f\circ g)=\infty$. Then we have $\rho_{[p_1,q_2]}(f\circ g)=0=B$;

(iii) $p_2+1-q_1<0$. From (12) and (10), we have $\rho_{[p_1,q_1+q_2-p_2]}(f\circ g)\geqslant A_l$.

Thus, the proof of Theorem 5 is completed. □

Proof of Theorem 6. Since $A>0$, then there exists a sequence $\{R_n\}\to\infty$ such that for any ε $(0<\varepsilon<A)$ and sufficiently large R_n, we have

$$M(R_n,f)\geqslant\exp_{p_1+1}\left\{(A-\varepsilon)\log_{q_1}R_n\right\}. \tag{14}$$

Since $M(r,g)$ is an increasing continuous function, then there exists a sequence $\{r_n\}\to\infty$ satisfying $R_n=\frac{1}{9}M\left(\frac{r_n}{2},g\right)\geqslant\frac{1}{3}\exp_{p_2+1}\left\{(B_l-\varepsilon)\log_{q_2}\frac{r_n}{2}\right\}$ such that for sufficiently large r_n and from Lemma 2, we have

$$\begin{aligned}M(r_n,f\circ g)&\geqslant M\left(\frac{1}{9}M\left(\frac{r_n}{2},g\right),f\right)=M(R_n,f)\\&\geqslant\exp_{p_1+1}\left\{(A-\varepsilon)\exp_{p_2+1-q_1}\left\{(B_l-\varepsilon)(\log_{q_2}r_n+O(1))\right\}\right\}.\end{aligned}\tag{15}$$

Then

(i) $p_2+1-q_1>0$. From (15), we have $\rho_{[p_1+p_2-q_1+1,q_2]}(f\circ g)\geqslant B_l$. And from Lemma 2, we have

$$\begin{aligned}M(r,f\circ g)&\leqslant M(M(r,g),f)\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\log_{q_1}M(r,g)\right\}\\&\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\log_{q_1}\left(\exp_{p_2+1}\left\{(B+\varepsilon)\log_{q_2}r\right\}\right)\right\}\\&\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\exp_{p_2+1-q_1}\left\{(B+\varepsilon)\log_{q_2}r\right\}\right\}.\end{aligned}\tag{16}$$

From (16) and the definition of $[p,q]$-order, we have $\rho_{[p_1+p_2-q_1+1,q_2]}(f\circ g)\leqslant B$;

(ii) $p_2+1-q_1=0$. From (15), we have $\rho_{[p_1,q_2]}(f\circ g)\geqslant AB_l$. And from (16). we have $\rho_{[p_1,q_2]}(f\circ g)\leqslant AB$, then we have $AB_l\leqslant\rho_{[p_1,q_2]}(f\circ g)\leqslant AB$;

(iii) $p_2+1-q_1<0$. From (15), we have $\rho_{[p_1,q_1+q_2-p_2-1]}(f\circ g)\geqslant A$. From (16), we have $\rho_{[p_1,q_1+q_2-p_2-1]}(f\circ g)\leqslant A$. Then we have $\rho_{[p_1,q_1+q_2-p_2-1]}(f\circ g)=A$.

Thus, we can get the conclusions of Theorem 6. □

4. Proofs of Theorems 7 and 12

Proof of Theorem 7. From the assumptions of Theorem 1 and by Lemma 2, for any $\varepsilon>0$, then there exists a positive number r_0 and for all $r\geqslant r_0$, we have

$$\begin{aligned}M(r,f\circ g)&\geqslant M\left(\frac{1}{8}M\left(\frac{r}{2},g\right),f\right)\geqslant\exp_{p_1+1}\left\{(A_l-\varepsilon)\log_{q_1}\left(\frac{1}{8}M\left(\frac{r}{2},g\right)\right)\right\}\\&\geqslant\exp_{p_1+1}\left\{(A_l-\varepsilon)\exp_{p_2+1-q_1}\left\{(B_l-\varepsilon)\log_{q_2}r\right\}+O(1)\right\},\end{aligned}\tag{17}$$

$$\begin{aligned}M(r,f\circ g)&\leqslant M(M(r,g),f)\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\log_{q_1}M(r,g)\right\}\\&\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\exp_{p_2+1-q_1}\left\{(B+\varepsilon)\log_{q_2}r\right\}\right\},\end{aligned}\tag{18}$$

and

$$\exp_{p_1+1}\left\{(A_l-\varepsilon)\log_{q_1}r\right\}\leqslant M(r,f)\leqslant\exp_{p_1+1}\left\{(A+\varepsilon)\log_{q_1}r\right\}.\tag{19}$$

The three cases will be considered as follows.

Case 1. $p_2+1-q_1>0$. From (17)–(29), we have

$$\frac{(B+\varepsilon)\log_{q_2}r}{(A_l-\varepsilon)\log_{q_1}r}\geqslant\frac{\log_{p_1+p_2+2-q_1}M(r,f\circ g)}{\log_{p_1+1}M(r,f)}\geqslant\frac{(B_l-\varepsilon)\log_{q_2}r+O(1)}{(A+\varepsilon)\log_{q_1}r}.\tag{20}$$

(i) $q_1 > q_2$. Since $B_l > 0$ and (20), we have

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} = \infty;$$

(ii) $q_1 = q_2$. Since $0 < A_l, A, B_l, B < \infty$, from (20), we have

$$\frac{B_l}{A} \leqslant \lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} \leqslant \frac{B}{A_l};$$

(iii) $q_1 < q_2$. From (20) and $A_l > 0$, we have

$$\lim_{r\to\infty} \frac{\log_{p_1+p_2+2-q_1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} = 0.$$

Case 2. $p_2 + 1 - q_1 = 0$. From (17)–(19), we have

$$\frac{(A+\varepsilon)(B+\varepsilon)\log_{q_2} r}{(A_l-\varepsilon)\log_{q_1} r} \geqslant \frac{\log_{p_1+1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} \geqslant \frac{(A_l-\varepsilon)(B_l-\varepsilon)\log_{q_2} r + O(1)}{(A+\varepsilon)\log_{q_1} r}. \quad (21)$$

(i) $q_1 > q_2$. Since $A_l > 0$ and $B_l > 0$, from (21), we have

$$\lim_{r\to\infty} \frac{\log_{p_1+1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} = \infty;$$

(ii) $q_1 = q_2$. Since $0 < A_l, A, B_l, B < \infty$, from (21), we have

$$\frac{A_l B_l}{A} \leqslant \lim_{r\to\infty} \frac{\log_{p_1+1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} \leqslant \frac{AB}{A_l};$$

(iii) $q_1 < q_2$. Since $0 < A_l, A, B < \infty$, from (21), we have

$$\lim_{r\to\infty} \frac{\log_{p_1+1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} = 0.$$

Case 3. $p_2 + 1 - q_1 < 0$. Since $p_2 \geqslant q_2 \geqslant 1$, we have $q_1 + q_2 - p_2 - 1 \leqslant q_1$. From (17)–(19) and $0 < A_l, A < \infty$, we have

$$\lim_{r\to\infty} \frac{\log_{p_1+1} M(r, f\circ g)}{\log_{p_1+1} M(r,f)} \geqslant \frac{(A_l-\varepsilon)\log_{q_1+q_2-p_2-1} r}{(A+\varepsilon)\log_{q_1} r} \longrightarrow \infty. \quad (22)$$

Thus, the proof of Theorem 7 is completed. □

Proof of Theorem 12. From Lemma 3, for sufficiently large r, we have

$$\mu(r, f_1\circ g) \geqslant \frac{1}{2}\mu\left(\frac{1}{16}\mu\left(\frac{r}{4}, g\right)\right), f_1\right).$$

Then from the assumptions of Theorem 7, for any $\varepsilon > 0$, there exists a positive number r_0 and for all $r \geqslant r_0$, we have

$$\mu(r, f_1 \circ g) \geqslant \frac{1}{2}\mu\left(\frac{1}{16}\mu\left(\frac{r}{4}, g\right), f_1\right) \geqslant \exp_{p_1+1}\left\{(L_l - \varepsilon)\log_{q_1}\left(\frac{1}{16}\mu\left(\frac{r}{4}, g\right)\right)\right\} \tag{23}$$
$$\geqslant \exp_{p_1+1}\left\{(L_l - \varepsilon)\exp_{p_2+1-q_1}\left\{(B_l - \varepsilon)\log_{q_2} r\right\} + O(1)\right\},$$

and

$$\exp_{p_1+1}\left\{(L_l - \varepsilon)\log_{q_1} r\right\} \leqslant \mu(r, f_1) \leqslant \exp_{p_1+1}\left\{(L + \varepsilon)\log_{q_1} r\right\}. \tag{24}$$

Using the same argument as in Theorem 7, we can get the conclusions of Theorem 12. □

5. Proofs of Theorems 8, 9 and 10

Proof of Theorem 8. By definition and by the same reasoning as K. Niino and C. C. Yang [12], there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n, we have

$$\frac{1}{3}\log M\left(\frac{1}{9}M\left(\frac{r_n}{4}, g\right), f\right) \leqslant T(r_n, f \circ g) \leqslant \exp_{p_1}\left\{(\chi + \varepsilon)\log_{q_1} r_n\right\}. \tag{25}$$

Set $\lambda_{[p_2,q_2]}(g) = B_l > 0$, for any given ε $(0 < \varepsilon < B_l)$ and for sufficiently r_n, we have

$$\frac{1}{9}M\left(\frac{r_n}{4}, g\right) \geqslant \exp_{p_2+1}\left\{\left(B_l - \frac{\varepsilon}{2}\right)\log_{q_2}\frac{r_n}{4}\right\} \geqslant \exp_{p_2+1}\left\{(B_l - \varepsilon)\log_{q_2} r_n\right\}. \tag{26}$$

Set $R_n = \frac{1}{9}M\left(\frac{r_n}{4}, g\right)$,then $r_n \leqslant \exp\left\{\frac{1}{B_l - \varepsilon}\log_{p_2+1} R_n\right\}$, from (25) and (26), we have

$$\log M(R_n, f) \leqslant 3\exp_{p_1}\left\{(\chi + \varepsilon)\exp_{q_2-q_1}\left\{\frac{1}{B_l - \varepsilon}\log_{p_2+1} R_n\right\}\right\}. \tag{27}$$

Since $0 < \varepsilon < \tau$, then for sufficiently large R_n and $p_2 + 1 - q_2 > 0$, we have $q_1 < q_1 + p_2 + 1 - q_2$ and

$$\frac{\log_{p_1+1} M(R_n, f)}{\log_{q_1} R_n} \leqslant \frac{(\chi + \varepsilon)\exp_{q_2-q_1}\left\{\frac{1}{B_l-\varepsilon}\log_{p_2+1} R_n\right\}}{\log_{q_1} R_n} \longrightarrow 0. \tag{28}$$

From (27) and (28), we have

$$\lim_{R_n \to \infty}\frac{\log_{p-1+1} M(R_n, f)}{\log_{q_1} R_n} = 0. \tag{29}$$

Thus, we get that $\lambda_{[p_1,q_1]}(f) = 0$. □

Proof of Theorem 9. We can get the conclusion of Theorem 9 by using the same argument as in Theorem 8. □

Proof of Theorem 10. By definition and the same reasoning as K. Niino and C. C. Yang [12], there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n, we have

$$\frac{1}{3}\log M\left(\frac{1}{9}M\left(\frac{r_n}{4},g\right),f\right) \leqslant T(r_n, f\circ g) \leqslant \exp_p\{(\sigma+\varepsilon)\log_q r_n\} \tag{30}$$

Since $\tau > 0$, for any given ε $(0<\varepsilon<\tau-\sigma)$ and for sufficiently r_n, we have

$$\frac{1}{9}M\left(\frac{r_n}{4},g\right) \geqslant \exp_p\left\{c_1\left(r-\frac{\varepsilon}{2}\right)\log_q\frac{r_n}{4}\right\} \geqslant \exp_{p+1}\{(\tau-\varepsilon)\log_q r_n\}. \tag{31}$$

Set $R_n = \frac{1}{9}M\left(\frac{r_n}{4},g\right)$, then $\log_q r_n \leqslant \frac{1}{\tau-\varepsilon}\log_{p+1} R_n$, from (30) and (31), we have

$$\log M(R_n,f) \leqslant 2\exp_p\left\{\frac{\sigma+\varepsilon}{\tau-\varepsilon}\log_{p+1} R_n\right\}. \tag{32}$$

Since $0<\varepsilon<\tau-\sigma$, then $\frac{\sigma+\varepsilon}{\tau-\varepsilon}<1$, for sufficiently large R_n, we have

$$\frac{\exp_{p-1}\left\{\frac{\sigma+\varepsilon}{\tau-\varepsilon}\log_{p+1} R_n\right\}}{\log R_n} \to 0. \tag{33}$$

From (32) and (33), we have

$$\lim_{R_n\to\infty}\frac{\log\log M(R_n,f)}{\log R_n} = 0. \tag{34}$$

Thus, we get that $\lambda(f) = 0$. □

Acknowledgement. This work was supported by the NNSF of China (11171170 and 61202313), the Natural Science Foundation of Jiang-Xi Province in China (2010GQS0119, 20132BAB211001, 20122BAB211005, 20132BAB211002 and 20114BAB211003).

The authors would like to thank the referees for their helpful suggestions.

REFERENCES

[1] W. BERGWEILER, *On the growth rate of composite meromorphic functions*, Complex Var. Elliptic Equ., **14** (1990), 187–196.

[2] W. BERGWEILER, *Order and lower order of composite meromorphic functions*, Michigam Math. J., **36** (1989), 135–146.

[3] J. CLUNIE, *The Composition of Entire and Meromorphic Functions, Mathematical Essays Dedicated to A. J. Macintyre*, Ohio University Press, 1970, pp. 75–92.

[4] F. GROSS, *Factorization of meromorphic function*, U.S. Government Printing office, Washington D.C., 1972.

[5] W. K. HAYMAN, *Meromorphic functions*, Oxford at the Clarendon Press, 1964.

[6] O. P. JUNEJA, G. P. KAPOOR AND S. K. BAJPAI, *On the* (p,q) *-order and lower* (p,q) *-order of an entire function*, J. Reine Angew. Math., **282** (1976), 53–67.

[7] O. P. JUNEJA, G. P. KAPOOR AND S. K. BAJPAI, *On the* (p,q) *-type and lower* (p,q) *-type of an entire function*, J. Reine Angew. Math., **290** (1977), 180–190.

[8] L. KINNUNEN, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., **22** (4) (1998), 385–405.

[9] L.-W. LIAO AND C.-C. YANG, *On the growth of composite entire functions*, Yokohama Math. J., **46** (1999), 97–107.

[10] J. LIU, J. TU AND L.-Z. SHI, *Linear differential equations with entire coefficients of* $[p,q]$ *-order in the complex plane*, J. Math. Anal. Appl., 372 (1) (2010), 55–67.

[11] K. NILINO AND N. SMITA, *Growth of a composite function of entire functions*, Kodai Math. J., **3** (1980), 374–379.

[12] K. NIINO AND C.-C. YANG, *Some growth relationships on factors of two compositive entire functions, in: Factorization Theory of Meromorphic functions and Related Topics*, Marcel Dekker Inc., New York/Basel, 1982, pp. 95–99.

[13] G. PÓLYA, *On an integral function of an integral function*, J. London. Math. Soc., **1** (1926), 12–15.

[14] A. SCHÖNHAGE, *Über das Wachstum Zusammengesetzter Funktionen*, Math. Z., **73** (1960), 22–44.

[15] A. P. SINGH, *Growth of composite entire functions*, Kodai Math. J., **8** (1985), 99–102.

[16] A. P. SINGH, *On maximum term of composition of entire functions*, Proc. Nat. Acad. India, **59** (1989), 103–115.

[17] J. TU, Z.-X. CHEN AND X.-M. ZHENG, *Composition of entire functions with finite iterated order*, J. Math. Anal. Appl., **353** (2009), 295–304.

[18] L. YANG, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

[19] H.-X. YI AND C.-C. YANG, *Uniqueness theory of meromorphic functions*, Science Press/Kluwer Academic Publishers, Beijing/New York, 1995/2003.

[20] Z.-Z. ZHOU, *Growth of composite entire functions*, Kodai Math. J., **9** (1986), 419–420.

(Received March 29, 2012)

Hong-Yan Xu
Department of Informatics and Engineering
Jingdezhen Ceramic Institute
Jingdezhen, Jiangxi, 333403, China
e-mail: xhyhhh@126.com

Jin Tu
Institute of Mathematics and informatics
Jiangxi Normal University
Nanchang, Jiangxi, 330022, China
e-mail: tujin2008@sina.com

Cai-Feng Yi
Institute of Mathematics and informatics
Jiangxi Normal University
Nanchang, Jiangxi, 330022, China
e-mail: yicaifeng55@163.com