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BOUNDS FOR THE WEIGHTED GINI MEAN DIFFERENCE OF AN EMPIRICAL DISTRIBUTION

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Abstract. In the paper, we obtain various bounds for more general weighted Gini mean difference of an empirical distribution, which extend the results of Cerone and Dragomir in [2].

1. Introduction

To measure the disparity of a probability distribution, the Gini mean difference and its scale invariant version, the Gini index, are most widely used. The special case of an empirical distribution is particularly important.

The Gini mean difference of the sample $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$ is defined by

$$G(\mathbf{a}) = \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - a_j| = \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} |a_i - a_j|$$

and

$$R(\mathbf{a}) = \frac{1}{\bar{a}} G(\mathbf{a})$$

is the Gini index of $\mathbf{a}$, provided the sample mean $\bar{a}$ is not zero [5, p. 257].

The Gini index of $\mathbf{a}$ equals the Gini mean difference of the "scaled down" sample

$$\tilde{a} = \left(\frac{a_1}{\bar{a}}, \cdots, \frac{a_n}{\bar{a}} \right), \quad (\bar{a} \neq 0)$$

That is to say,

$$R(\mathbf{a}) = R(a_1, \cdots, a_n) = \frac{1}{2n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left| \frac{a_i}{\bar{a}} - \frac{a_j}{\bar{a}} \right|.$$

We state several important properties of the Gini index for an empirical distribution of nonnegative data [5, p. 257].

(i) Let $(a_1, \cdots, a_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} a_i > 0$. Then

$$0 = R(\bar{a}, \cdots, \bar{a}) \leqslant R(a_1, \cdots, a_n) \leqslant R\left(0, \cdots, 0, \sum_{i=1}^{n} a_i\right) = 1 - \frac{1}{n} < 1,$$

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$$R(\beta a_1,\cdots,\beta a_n)=R(a_1,\cdots,a_n),\quad \text{for every } \beta>0.$$

and

$$R(a_1+\lambda,\cdots,a_n+\lambda)=\frac{\bar{a}}{\bar{a}+\lambda}R(a_1,\cdots,a_n),\ \text{ for } \lambda>0.$$

(ii) $R(\cdot)$ is a continuous function on $\mathbb{R}^n_+$.

These and other properties have been investigated by many authors. For a survey and references, see [6, 3, 4, 5].

For $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\cdots,p_n)$ a probability sequence, meaning that $p_i>0$ $(i\in\{1,\cdots,n\})$ and $\sum_{i=1}^n p_i=1$, Cerone and Dragomir considered in [1] the weighted Gini mean difference defined by the formula

$$G(\mathbf{p},\mathbf{a})=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j|a_i-a_j|=\sum_{1\leqslant i<j\leqslant n}p_ip_j|a_i-a_j|, \tag{1.1}$$

and proved that

$$\frac{1}{2}K(\mathbf{p},\mathbf{a})\leqslant G(\mathbf{p},\mathbf{a})\leqslant\inf_{\gamma\in\mathbb{R}}\left[\sum_{i=1}^{n}p_i|a_i-\gamma|\right]\leqslant K(\mathbf{p},\mathbf{a}),$$

where $K(\mathbf{p},\mathbf{a})$ is the mean absolute deviation, namely

$$K(\mathbf{p},\mathbf{a}):=\sum_{i=1}^{n}p_i\left|a_i-\sum_{j=1}^{n}p_ja_j\right|.$$

They have also shown that if more information on the sampling data $\mathbf{a}=(a_1,\cdots,a_n)$ is available, i.e., there exist the real numbers a and A such that $a\leqslant a_i\leqslant A$ for each $i\in\{1,\cdots,n\}$, then

$$G(\mathbf{p},\mathbf{a})\leqslant(A-a)\max_{J\subset\{1,\cdots,n\}}[P_J(1-P_J)]\left(\leqslant\frac{1}{4}(A-a)\right), \tag{1.2}$$

where $P_J:=\sum_{j\in J}p_j$. Also, they have shown that

$$G(\mathbf{p},\mathbf{a})\leqslant\sum_{i=1}^{n}p_i\left|a_i-\frac{A+a}{2}\right|\left(\leqslant\frac{1}{2}(A-a)\right). \tag{1.3}$$

Notice that in general the bounds for the weighted Gini mean difference $G(\mathbf{p},\mathbf{a})$ provided by (1.2) and (1.3) cannot be compared to conclude that one is always better than the other [1].

Recently, in [2], Cerone and Dragomir provided various bounds for the more general r-weighted Gini mean difference. For $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbb{R}^n$ and $\mathbf{p}=(p_1,\cdots,p_n)$ a probability sequence, meaning that $p_i>0$ $(i\in\{1,\cdots,n\})$ and $\sum_{i=1}^n p_i=1$, Cerone and Dragomir considered in [2] the r-weighted Gini mean difference, for $r\in[1,\infty)$, defined by the formula

$$G_r(\mathbf{p},\mathbf{a}):=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}p_ip_j|a_i-a_j|^r=\sum_{1\leqslant i<j\leqslant n}p_ip_j|a_i-a_j|^r, \tag{1.4}$$

and proved

$$\frac{1}{2}\max_{(i,j)\in\Delta}\left\{\frac{p_i^r p_j^r + p_i p_j (1-p_i p_j)^{r-1}}{(1-p_i p_j)^{r-1}}|a_i - a_j|^r\right\} \leqslant G_r(\mathbf{p},\mathbf{a}) \leqslant \frac{1}{2}\max_{(i,j)\in\Delta}|a_i - a_j|^r, \quad (1.5)$$

where $\Delta := \{(i,j)|i,j\in\{1,\cdots,n\}\}$.

The main aim of the paper is to continue the work begun in [1, 2] and extends their results for the more general weighted Gini mean difference that has been introduced in [1, 2].

2. Main results

Fixed two positive integers n,m, for $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbb{R}^n$, $\mathbf{b}=(b_1,\cdots,b_m)\in\mathbb{R}^m$ and $\mathbf{p}=(p_1,\cdots,p_n)$, $\mathbf{q}=(q_1,\cdots,q_m)$ two probability sequences, meaning that p_i, $q_j>0$ $(i\in\{1,\cdots,n\},\ j\in\{1,\cdots,m\})$ and $\sum_{i=1}^n p_i = 1$, $\sum_{j=1}^m q_j = 1$, we introduce the following general r-weighted Gini mean difference, for $r\in[1,\infty)$, defined by the formula

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) := \frac{1}{2}\sum_{j=1}^m\sum_{i=1}^n p_i q_j |a_i - b_j|^r. \quad (2.1)$$

Obviously, if taking $m=n$, $p_i=q_i$, $a_i=b_i$, for all $i\in\{1,\cdots,n\}$, then $G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})$ becomes the r-weighted Gini mean difference (1.4). Now, if we define

$$\widehat{\Delta} := \{(i,j)|i\in\{1,\cdots,n\}, j\in\{1,\cdots,m\}\},$$

then we can simply write the relation (2.1) in the form

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) = \frac{1}{2}\sum_{(i,j)\in\widehat{\Delta}} p_i q_j |a_i - b_j|^r, \quad r\geqslant 1.$$

Now we state the following result concerning upper and lower bounds for $G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})$.

THEOREM 2.1. *For any p_i, $q_j\in(0,1)$ with $\sum_{i=1}^n p_i = 1$, $\sum_{j=1}^m q_j = 1$ and $a_i, b_j \in \mathbb{R}$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$, assume that*

$$\sum_{i=1}^n a_i p_i = \sum_{j=1}^m b_j p_j.$$

Then we have the following inequalities

$$\begin{aligned}\frac{1}{2}\max_{(k,l)\in\widehat{\Delta}}&\left\{\frac{p_k^r q_l^r + (1-p_k q_l)^{r-1} p_k q_l}{(1-p_k q_l)^{r-1}}|a_k - b_l|^r\right\}\\ &\leqslant G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2}\max_{(k,l)\in\widehat{\Delta}}|a_k - b_l|^r\end{aligned} \quad (2.2)$$

where $r>1$.

Proof. Since

$$\sum_{(i,j)\in\widehat{\Delta}} p_i q_j (a_i - b_j) = 0,$$

then, for any fixed $(k,l) \in \widehat{\Delta}$, we have

$$p_k q_l (a_k - b_l) = -\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j (a_i - b_j).$$

Taking the modulus in above equation and by Hölder's inequality, we have successively

$$\begin{aligned} & p_k q_l |a_k - b_l| \\ =& \left| \sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j (a_i - b_j) \right| \\ \leqslant& \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j |a_i - b_j|^r \right)^{1/r} \qquad (2.3) \\ =& \left(\sum_{(i,j)\in\widehat{\Delta}} p_i q_j - p_k q_l \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}} p_i q_j |a_i - b_j|^r - p_k q_l |a_k - b_l|^r \right)^{1/r} \\ =& (1 - p_k q_l)^{\frac{r-1}{r}} \left(2G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) - p_k q_l |a_k - b_l|^r\right)^{1/r} \end{aligned}$$

where $\frac{1}{r} + \frac{1}{\alpha} = 1, \alpha > 1$.

Taking the prower r in (2.3), we have

$$p_k^r q_l^r |a_k - b_l|^r \leqslant (1 - p_k q_l)^{r-1} \left(2G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) - p_k q_l |a_k - b_l|^r\right)$$

which implies

$$\frac{1}{2} \cdot \frac{p_k^r q_l^r + (1 - p_k q_l)^{r-1} p_k q_l}{(1 - p_k q_l)^{r-1}} |a_k - b_l|^r \leqslant G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \qquad (2.4)$$

for each $(k,l) \in \widehat{\Delta}$. Taking the maximum over $(k,l) \in \widehat{\Delta}$ in (2.4), we deduce the first inequality in (2.2). The second inequality is obvious on observing that

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} p_i q_j \max_{(i,j)\in\widehat{\Delta}} |a_i - b_j|^r = \frac{1}{2} \max_{(i,j)\in\widehat{\Delta}} |a_i - b_j|^r. \quad \square$$

REMARK 2.1. If taking $m = n$, $p_i = q_i$, $a_i = b_i$, for all $i \in \{1, \cdots, n\}$, then the bounds in (2.2) become the result (1.5) of Cerone and Dragomir.

REMARK 2.2. The case $r = 2$ is of interest, since

$$\begin{aligned} G_2(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) =& \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} p_i q_j |a_i - b_j|^2 \\ =& \frac{1}{2} \left\{ \sum_{i=1}^{n} p_i a_i^2 - 2 \left(\sum_{i=1}^{n} p_i a_i \right) \left(\sum_{j=1}^{m} q_j b_j \right) + \sum_{j=1}^{m} q_i b_j^2 \right\} \end{aligned}$$

for which we can obtain from Theorem 2.1 the following bounds,

$$\frac{1}{2}\max_{(k,l)\in\widehat{\Delta}}\left\{\frac{p_kq_l}{1-p_kq_l}(a_k-b_l)^2\right\}\leqslant G_2(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})\leqslant\frac{1}{2}\max_{(k,l)\in\widehat{\Delta}}(a_k-b_l)^2.$$

REMARK 2.3. It is easy to check the function

$$h_r(t):=\frac{t^r+t(1-t)^{r-1}}{(1-t)^{r-1}},\quad r>1$$

is strictly increasing on $[0,1)$. Therefore

$$\begin{aligned}\min_{(k,l)\in\widehat{\Delta}}\left\{\frac{p_k^rq_l^r+(1-p_kq_l)^{r-1}p_kq_l}{(1-p_kq_l)^{r-1}}\right\}&=\min_{(k,l)\in\widehat{\Delta}}h_r(p_iq_j)\\&\geqslant h_r\left[\min_{(k,l)\in\widehat{\Delta}}p_iq_j\right]\\&=h_r(\underline{p}\cdot\underline{q})\\&=\frac{\underline{p}^r\underline{q}^r+(1-\underline{p}\cdot\underline{q})^{r-1}\underline{p}\cdot\underline{q}}{(1-\underline{p}\cdot\underline{q})^{r-1}}\end{aligned}$$

where $\underline{p}=\min\{p_1,\cdots,p_n\}$, $\underline{q}=\min\{q_1,\cdots,q_m\}$. Then we can obtain a coarse but, perhaps, a more useful lower bound

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})\geqslant\frac{1}{2}\frac{\underline{p}^r\underline{q}^r+(1-\underline{p}\cdot\underline{q})^{r-1}\underline{p}\cdot\underline{q}}{(1-\underline{p}\cdot\underline{q})^{r-1}}\max_{(k,l)\in\widehat{\Delta}}|a_i-b_j|^r.$$

For the case $r=2$, we then have

$$G_2(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})\geqslant\frac{1}{2}\frac{\underline{p}\cdot\underline{q}}{1-\underline{p}\cdot\underline{q}}\max_{(k,l)\in\widehat{\Delta}}(a_i-b_j)^2.$$

THEOREM 2.2. *For any $p_i,\ q_j\in(0,1)$ with $\sum_{i=1}^n p_i=1$, $\sum_{j=1}^m q_j=1$ and $a_i,b_j\in\mathbb{R}$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$, assume that*

$$\sum_{i=1}^n a_ip_i=\sum_{j=1}^m b_jp_j.$$

Then we have the following inequalities

$$\max_{(k,l)\in\widehat{\Delta}}p_kq_l|a_k-b_l|\leqslant G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})\leqslant\frac{1}{2}\max_{(k,l)\in\widehat{\Delta}}|a_k-b_l|\tag{2.5}$$

Proof. As in the proof of Theorem 2.1, for all $(k,l) \in \widehat{\Delta}$,

$$\begin{aligned} p_k q_l |a_k - b_l| &= \left| \sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j (a_i - b_j) \right| \\ &\leqslant \sum_{(i,j)\in\widehat{\Delta}} p_i q_j |a_i - b_j| - p_k q_l |a_k - a_l| \\ &= 2G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) - p_k q_l |a_k - a_l|. \end{aligned} \tag{2.6}$$

which implies the first inequality in (2.5). The second part is obvious. □

REMARK 2.4. If taking $m = n$, $p_i = q_i$, $a_i = b_i$, for all $i \in \{1,\cdots,n\}$, in [2, Theorem 2, p183], Cerone and Dragomir obtained the following inequalities

$$\begin{aligned} &\frac{1}{2}\max_{(k,l)\in\Delta}\left\{ p_k p_l |a_k - a_l| \left(1 + \frac{1}{\max_{(i,j)\in\Delta\setminus\{k,l\}} p_i p_j} \right) \right\} \\ &\leqslant G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2}\max_{(k,l)\in\Delta} |a_k - a_l|. \end{aligned} \tag{2.7}$$

However, we find that the lower bound of (2.7) is wrong. If we take $p_i = 1/n$ for any i, then (2.7) becomes

$$\frac{1}{2}\left(1 + \frac{1}{n^2}\right)\max_{(k,l)\in\Delta} |a_k - a_l| \leqslant G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2}\max_{(k,l)\in\Delta} |a_k - a_l|,$$

which yields a contradiction.

Next we shall give some more general results.

THEOREM 2.3. *For any p_i, $q_j \in (0,1)$ with $\sum_{i=1}^n p_i = 1$, $\sum_{j=1}^m q_j = 1$ and $a_i, b_j \in \mathbb{R}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, assume that*

$$\sum_{i=1}^n a_i p_i = \sum_{j=1}^m b_j p_j.$$

Then we have the following inequalities

$$\begin{aligned} &\frac{1}{2}\max_{\Delta_1\subset\widehat{\Delta}}\left\{ \frac{\left(\sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|\right)^r}{\left(1 - \sum_{(k,l)\in\Delta_1} p_k q_l\right)^{r-1}} + \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|^r \right\} \\ &\leqslant G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2}\max_{(k,l)\in\widehat{\Delta}} |a_k - b_l|^r \end{aligned} \tag{2.8}$$

where $r > 1$.

Proof. Since

$$\sum_{(i,j)\in\widehat{\Delta}} p_i q_j (a_i - b_j) = 0,$$

then, for any fixed $\Delta_1 \subset \widehat{\Delta}$, we have

$$\sum_{(k,l)\in\Delta_1} p_k q_l (a_k - b_l) = -\sum_{(i,j)\in\widehat{\Delta}\setminus\Delta_1} p_i q_j (a_i - b_j).$$

Taking the modulus in above equation and by Hölder's inequality, we have successively

$$\begin{aligned}
&\sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l| \\
&= \left| \sum_{(i,j)\in\widehat{\Delta}\setminus\Delta_1} p_i q_j (a_i - b_j) \right| \\
&\leqslant \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\Delta_1} p_i q_j \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\Delta_1} p_i q_j |a_i - b_j|^r \right)^{1/r} \\
&= \left(\sum_{(i,j)\in\widehat{\Delta}} p_i q_j - \sum_{(k,l)\in\Delta_1} p_k q_l \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}} p_i q_j |a_i - b_j|^r - \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|^r \right)^{1/r} \\
&= \left(1 - \sum_{(k,l)\in\Delta_1} p_k q_l \right)^{\frac{r-1}{r}} \left(2G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) - \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|^r \right)^{1/r}
\end{aligned} \tag{2.9}$$

where $\frac{1}{r} + \frac{1}{\alpha} = 1, \alpha > 1$.

Taking the prower r in (2.9), we have

$$\begin{aligned}
\left(\sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l| \right)^r \leqslant & \left(1 - \sum_{(k,l)\in\Delta_1} p_k q_l \right)^{r-1} \\
& \times \left(2G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) - \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|^r \right)
\end{aligned}$$

which implies

$$\frac{1}{2} \left\{ \frac{\left(\sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l| \right)^r}{\left(1 - \sum_{(k,l)\in\Delta_1} p_k q_l \right)^{r-1}} + \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l|^r \right\} \leqslant G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \tag{2.10}$$

for each $\Delta_1 \subset \widehat{\Delta}$. Taking the maximum over $\Delta_1 \subset \widehat{\Delta}$ in (2.10), we deduce the first inequality in (2.8). The second inequality is obvious. □

THEOREM 2.4. *For any* $p_i,\ q_j \in (0,1)$ *with* $\sum_{i=1}^n p_i = 1,\ \sum_{j=1}^m q_j = 1$ *and* $a_i, b_j \in \mathbb{R},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$, *assume that*

$$\sum_{i=1}^n a_i p_i = \sum_{j=1}^m b_j p_j.$$

Then we have the following inequalities

$$\max_{\Delta_1 \subset \widehat{\Delta}} \sum_{(k,l)\in\Delta_1} p_k q_l |a_k - b_l| \leqslant G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2} \max_{(k,l)\in\widehat{\Delta}} |a_k - b_l|. \tag{2.11}$$

Proof. We omit the proof of the result. □

3. Related results

Let

$$G_r(\mathbf{a},\mathbf{b}) := \frac{1}{2n^2} \sum_{(i,j)\in\widehat{\Delta}} |a_i - b_j|^r,$$

then the following result provides a connection between $G_r(\mathbf{a},\mathbf{b})$ and $G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})$.

THEOREM 3.1. *For any* $p_i,\ q_j \in (0,1)$ *with* $\sum_{i=1}^n p_i = 1,\ \sum_{j=1}^m q_j = 1$ *and* $a_i, b_j \in \mathbb{R},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$, *assume that*

$$\sum_{i=1}^n a_i p_i = \sum_{j=1}^m b_j p_j.$$

Then we have the following inequality

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \sum_{(k,l)\in\widehat{\Delta}} p_k q_l \left(1 + \frac{p_k^r q_l^r}{\left(\sum_{(i,j)\in\widehat{\Delta}} p_i^\alpha q_j^\alpha - p_k^\alpha q_l^\alpha\right)^{r-1}} \right)^{-1} n^2 G_r(\mathbf{a},\mathbf{b}) \tag{3.1}$$

where $\frac{1}{r} + \frac{1}{\alpha} = 1,\ \alpha > 1,\ r > 1$.

Proof. By Hölder's inequality, we have that

$$\begin{aligned}
p_k q_l |a_k - b_l| &= \left| \sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i q_j (a_i - b_j) \right| \\
&\leqslant \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} p_i^{\alpha} q_j^{\alpha} \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}} |a_i - b_j|^r \right)^{1/r} \\
&= \left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{1/\alpha} \left(\sum_{(i,j)\in\widehat{\Delta}} |a_i - b_j|^r - |a_k - b_l|^r \right)^{1/r} \\
&= \left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{1/\alpha} \left(2n^2 G_r(\mathbf{a},\mathbf{b}) - |a_k - b_l|^r \right)^{1/r}
\end{aligned} \tag{3.2}$$

where $\frac{1}{r} + \frac{1}{\alpha} = 1, \alpha > 1$.

Taking the prower r in (3.2), we have

$$p_k^r q_l^r |a_k - b_l|^r \leqslant \left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{r-1} \left(2n^2 G_r(\mathbf{a},\mathbf{b}) - |a_k - b_l|^r \right)$$

which implies

$$|a_k - b_l|^r \leqslant \left(1 + \frac{p_k^r q_l^r}{\left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{r-1}} \right)^{-1} 2n^2 G_r(\mathbf{a},\mathbf{b}) \tag{3.3}$$

for each $(k,l) \in \widehat{\Delta}$. Now, if we multiply (3.3) with $p_k q_l > 0$ and sum over $(k,l) \in \widehat{\Delta}$, then we have

$$G_r(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \sum_{(k,l)\in\widehat{\Delta}} p_k q_l \left(1 + \frac{p_k^r q_l^r}{\left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{r-1}} \right)^{-1} n^2 G_r(\mathbf{a},\mathbf{b}). \quad \square$$

REMARK 3.1. If $r = \alpha = 2$, then

$$\sum_{(k,l)\in\widehat{\Delta}} p_k q_l \left(1 + \frac{p_k^r q_l^r}{\left(\sum_{(i,j)\in\widehat{\Delta}} p_i^{\alpha} q_j^{\alpha} - p_k^{\alpha} q_l^{\alpha} \right)^{r-1}} \right)^{-1} = 1 - \frac{\sum_{(k,l)\in\widehat{\Delta}} p_k^3 q_l^3}{\sum_{(i,j)\in\widehat{\Delta}} p_k^2 q_l^2}$$

which implies

$$G_2(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b}) \leqslant \left(1 - \frac{\sum_{(k,l)\in\widehat{\Delta}} p_k^3 q_l^3}{\sum_{(i,j)\in\widehat{\Delta}} p_k^2 q_l^2} \right) n^2 G_2(\mathbf{a},\mathbf{b}). \tag{3.4}$$

If taking $m=n$, $p_i=q_i$, $a_i=b_i$, for all $i\in\{1,\cdots,n\}$, then the bound in (3.4) becomes Theorem 4. in [2].

The next result compares the Gini mean difference $G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})$ and $G_r(\mathbf{a},\mathbf{b})$.

THEOREM 3.2. *For any $p_i,\ q_j\in(0,1)$ with $\sum_{i=1}^n p_i=1$, $\sum_{j=1}^m q_j=1$ and $a_i,b_j\in\mathbb{R}$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$, assume that*

$$\sum_{i=1}^n a_i p_i=\sum_{j=1}^m b_j p_j.$$

Then we have the following inequality

$$G_1(\mathbf{p},\mathbf{q};\mathbf{a},\mathbf{b})\leqslant\frac{1}{4}\left(\sum_{(i,j)\in\widehat{\Delta}}p_i^\alpha q_j^\alpha\right)^{1/\alpha}\left(2n^2G_r(\mathbf{a},\mathbf{b})\right)^{1/r}\tag{3.5}$$

where $\frac{1}{r}+\frac{1}{\alpha}=1,\alpha>1,r>1$.

Proof. By Hölder's inequality, we have that

$$\begin{aligned}p_kq_l|a_k-b_l|&=\left|\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}}p_iq_j(a_i-b_j)\right|\\&\leqslant\left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}}p_i^\alpha q_j^\alpha\right)^{1/\alpha}\left(\sum_{(i,j)\in\widehat{\Delta}\setminus\{(k,l)\}}|a_i-b_j|^r\right)^{1/r}\\&=\left(\sum_{(i,j)\in\widehat{\Delta}}p_i^\alpha q_j^\alpha-p_k^\alpha q_l^\alpha\right)^{1/\alpha}\left(\sum_{(i,j)\in\widehat{\Delta}}|a_i-b_j|^r-|a_k-b_l|^r\right)^{1/r}\\&=\left(\sum_{(i,j)\in\widehat{\Delta}}p_i^\alpha q_j^\alpha-p_k^\alpha q_l^\alpha\right)^{1/\alpha}\left(2n^2G_r(\mathbf{a},\mathbf{b})-|a_k-b_l|^r\right)^{1/r}\end{aligned}\tag{3.6}$$

where $\frac{1}{r}+\frac{1}{\alpha}=1,\alpha>1$.

Using the elementary inequality

$$(c^r-d^r)^{1/r}(\gamma^\alpha-\delta^\alpha)^{1/\alpha}\leqslant c\gamma-d\delta$$

provided $c\geqslant d$, $\gamma>\delta$ and $\alpha,r>1$ with $\frac{1}{r}+\frac{1}{\alpha}=1$, we can obtain that

$$p_kq_l|a_k-b_l|\leqslant\left(\sum_{(i,j)\in\widehat{\Delta}}p_i^\alpha q_j^\alpha\right)^{1/\alpha}\left(2n^2G_r(\mathbf{a},\mathbf{b})\right)^{1/r}-p_kq_l|a_k-b_l|$$

which yields

$$2p_kq_l|a_k-b_l| \leqslant \left(\sum_{(i,j)\in\widehat{\Delta}} p_i^\alpha q_j^\alpha\right)^{1/\alpha} \left(2n^2G_r(\mathbf{a},\mathbf{b})\right)^{1/r}$$

for each $(k,l)\in\widehat{\Delta}$. Summing in the above inequality over $(k,l)\in\widehat{\Delta}$ we deduce the desired result. □

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