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# GOOD $\lambda$–INEQUALITIES AND REARRANGEMENT INEQUALITIES FOR VECTOR–VALUED MARTINGALES

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*Abstract.* In this paper, we obtain some good $\lambda$-inequalities and rearrangement inequalities for vector-valued martingales, which are closely related to the geometric properties of the underlying Banach space. In particular, our results extend some important inequalities in classical martingale $H_p$ theory, and we establish a relationship between the good $\lambda$-inequality and the rearrangement inequality for some vector-valued martingale function pairs.

## 1. Introduction and preliminaries

As is well-known, the good $\lambda$-inequality and the rearrangement inequality are two classes of important inequalities in martingale $H_p$ theory. They have played a similiar role in the study of the $\Phi$-inequalities of martingales (see [3, 6]). Let $(A,B)$ be a pair of non-negative and nondecreasing processes, where $A$ is adapted and $B$ is predictable with $B_0 = 0$. A sufficient condition for the function pair $(A_\infty, B_\infty)$ to satisfy the good $\lambda$-inequality was given in [5], and a sufficient condition for $(A_\infty, B_\infty)$ to satisfy the rearrangement inequality was given in [6]. One can find that the two sufficient conditions seem quite similar in forms. However, it is not clear that there is a relationship between the good $\lambda$-inequality and the rearrangement inequality. This paper will devote to establish some good $\lambda$-inequalities and rearrangement inequalities for vector-valued martingales and to find the relationship between them.

Let $f = (f_n)_{n\geqslant 0}$ be a scalar-valued martingale having a predictable control $D = (D_n)_{n\geqslant 0}$, good $\lambda$-inequalities for the function pairs $(M(f), S(f)+D_\infty)$ and $(S(f), M(f) + D_\infty)$ were established by Burkholder [3], and their rearrangement inequalities were obtained by Long [6]. These inequalities play an important role in classical martingale $H_p$ theory. The famous Burkholder-Gundy-Davis inequality with respect to $L_\Phi$-norm was proved by use of them [5]. In this paper, we extend these inequalities to vector-valued martingales, and the geometric properties of the underlying Banach space are characterized. In particular, our results show that if the underlying Banach space has some geometric property, then the good $\lambda$-inequality and the rearrangement inequality are equivalent.

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This paper is divided into three sections. Some notations and definitions used in this paper are given in the remainder of this section. Several good $\lambda$-inequalities and rearrangement inequalities for vector-valued martingales are respectively established in Section 2 and 3.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $\{\mathscr{F}_n\}_{n\geqslant 0}$ a non-decreasing sequence of sub-$\sigma$-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\bigcup_n \mathscr{F}_n)$. The expectation operator is denoted by $\mathbb{E}$. Let $(X, \|\cdot\|)$ be a Banach space, and $f = (f_n)_{n\geqslant 0}$ an $X$-valued martingale relative to $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n)_{n\geqslant 0})$, we denote its martingale difference by $df_i = f_i - f_{i-1}$ ( $i \geqslant 0$, with convention $df_0 = 0$, ), denote its maximal function and $q$-variation($0 < q < \infty$) by

$$M_n(f) = \sup_{0\leqslant i\leqslant n} \| f_i \|, \quad M(f) = \sup_{n\geqslant 0} \| f_n \|;$$

$$S_n^{(q)}(f) = (\sum_{i=0}^{n} \| df_i \|^q)^{\frac{1}{q}}, \quad S^{(q)}(f) = (\sum_{i=0}^{\infty} \| df_i \|^q)^{\frac{1}{q}}.$$

For more knowledge about the theory of martingale and of Banach space geometry, we refer to [4, 7, 8, 9].

Let $(f, g)$ be a pair of non-negative measurable functions on $(\Omega, \mathscr{F}, \mathbb{P})$, it is called to satisfy the good $\lambda$-inequality, if there is $\alpha > 1$, and for all $\beta > 0$ small enough, there exist constants $\varepsilon_\beta$ satisfying $\lim_{\beta\to 0} \varepsilon_\beta = 0$, such that

$$\mathbb{P}(f > \alpha\lambda, g \leqslant \beta\lambda) \leqslant \varepsilon_\beta \mathbb{P}(f > \lambda), \quad \lambda > 0.$$

Let $F$ be a measurable function on $\Omega$, and its non-increasing rearrangement function $F^*(t)$ is defined as

$$F^*(t) = \inf\{s > 0 : \mathbb{P}(x : |F(x)| > s) \leqslant t\}, \quad t \geqslant 0.$$

For a pair of measurable functions $(F, G)$, it is called to satisfy the rearrangement inequality, if there is a constant $C > 0$ such that

$$F^*(t) \leqslant F^*(2t) + CG^*\left(\frac{t}{2}\right), \quad \forall t > 0.$$

Throughout this paper, we use $C$ or $C_p$ (depending only on $p$) to denote some constant and may be different at each occurrence.

## 2. Good $\lambda$-inequalities for vector-valued martingales

LEMMA 2.1. [4] *Let $X$ be a Banach space, $1 < p \leqslant 2$. If $X$ is isomorphic to a $p$-uniformly smooth space, then for any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ with $S^{(p)}(f) \in L_p$, there exists a constant $C_p > 0$ such that*

$$\mathbb{E}(M(f) \mid \mathscr{F}_0) \leqslant C_p \mathbb{E}(S^{(p)}(f) \mid \mathscr{F}_0).$$

LEMMA 2.2. (Davis decomposition) *For each $X$-valued martingale $f=(f_n)_{n\geqslant 0}$, there exists a decomposition of $f$: $f=g+h$, where $g=(g_n)_{n\geqslant 0}$ and $h=(h_n)_{n\geqslant 0}$ are martingales and satisfy*

$$\| dg_n \| \leqslant 4M_{n-1}(df), \quad \mathbb{E}(\sum_{i=0}^{\infty} \| dh_i \|) \leqslant 4\mathbb{E}(M(df)),$$

*where $M_n(df)=\sup_{0\leqslant i\leqslant n} \| df_i \|$, $M(df)=\sup_{n\geqslant 0} \| df_n \|$*

The proof of Lemma 2.2 is similar to that for scalar-valued martingales. A proof can also be found in [4].

LEMMA 2.3. [4] *Let $X$ be a Banach space, $1<p\leqslant 2$. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a $p$-uniformly smooth space;*

*(ii) For any $X$-valued martingale $f=(f_n)_{n\geqslant 0}$, there exists a constant $C>0$ such that*

$$\| M(f) \|_1 \leqslant C \| S^{(p)}(f) \|_1 .$$

LEMMA 2.4. [3, 5] *Let $\Phi$ be a nondecreasing continuous function on $[0,\infty)$ with $\Phi(0)=0$, and satisfies the growth condition: $\Phi(2\lambda)\leqslant C\Phi(\lambda), \forall \lambda>0$. If a pair of non-negative functions $(f,g)$ satisfies the good $\lambda$-inequality, then for $\beta>0$ small enough, we have*

$$\mathbb{E}(\Phi(f)) \leqslant C_{\alpha,\beta}\mathbb{E}(\Phi(g)).$$

An $X$-valued martingale $f=(f_n)_{n\geqslant 0}$ having a predictable control $D=(D_n)_{n\geqslant 0}$ means that $D$ is non-negative, adapted and nondecreasing, and such that $\| df_n \| \leqslant D_{n-1}, \forall n$.

THEOREM 2.5. *Let $X$ be a Banach space, $1<p\leqslant 2$. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a $p$-uniformly smooth space;*

*(ii) For any $X$-valued martingale $f=(f_n)_{n\geqslant 0}$ with $S^{(p)}(f)\in L_p$ having a predictable control $D=(D_n)_{n\geqslant 0}$, the function pair $(M(f), S^{(p)}(f)+D_\infty)$ satisfies the good $\lambda$-inequality.*

*Proof.* (i)$\Rightarrow$(ii). Let $\rho_n = S_n^{(p)}(f)+D_n$, then

$$\begin{aligned} S_n^{(p)}(f) &= (\sum_{i=0}^{n-1} \| df_i \|^p + \| df_n \|^p)^{\frac{1}{p}} \\ &\leqslant S_{n-1}^{(p)}(f) + D_{n-1} = \rho_{n-1}. \end{aligned}$$

For $\beta > 0$ and $\lambda > 0$, define stopping time

$$\tau = \inf\{n : \rho_n > \beta\lambda\}.$$

Now we consider the stopping martingale $f^{(\tau)} = (f_n^{(\tau)})_{n\geqslant 0} = (f_{n\wedge\tau})_{n\geqslant 0}$, define another stopping time

$$T = \inf\{n : \| f_n^{(\tau)} \| > \lambda\}.$$

For $\alpha > 1$, we have

$$\begin{aligned}\mathbb{P}(M(f) > \alpha\lambda) &\leqslant \mathbb{P}(M(f) > \alpha\lambda, \tau = \infty) + \mathbb{P}(\tau < \infty) \\ &\leqslant \mathbb{P}(M(f^{(\tau)}) > \alpha\lambda) + \mathbb{P}(\tau < \infty) \\ &\leqslant \mathbb{P}(M(f^{(\tau)}) - M_{T-1}(f^{(\tau)}) > (\alpha - 1)\lambda) + \mathbb{P}(\tau < \infty).\end{aligned} \tag{2.1}$$

Consider a new family of $\sigma$-fields $\{\mathscr{F}'_n\}_{n\geqslant 0}$ with $\mathscr{F}'_n = \mathscr{F}_{n+T}$ and a new process $g = (g'_n)_{n\geqslant 0}$ with $g'_n = f_{n+T}^{(\tau)} - f_{T-1}^{(\tau)}$. It is clear that $g = (g'_n)_{n\geqslant 0}$ is an $X$-valued martingale with respect to $(\Omega, \Sigma, \mathbb{P}, (\mathscr{F}'_n)_{n\geqslant 0})$. Noticing that

$$M(f^{(\tau)}) - M_{T-1}(f^{(\tau)}) \leqslant \sup_{m\geqslant T} \| f_m^{(\tau)} - f_{T-1}^{(\tau)} \| = M(g')$$

and

$$\begin{aligned}S^{(p)}(g') = S^{(p)}(f^{(\tau)} - f_{T-1}^{(\tau)}) &\leqslant S^{(p)}(f^{(\tau)})\chi(T < \infty) \\ &= S_\tau^{(p)}(f)\chi(T < \infty) \leqslant \rho_{\tau-1}\chi(T < \infty) \leqslant \beta\lambda\chi(T < \infty),\end{aligned}$$

where $\chi(A)$ denotes the characteristic function of the set $A$, then by Lemma 2.1 we get

$$\begin{aligned}&\mathbb{P}(M(f^{(\tau)}) - M_{T-1}(f^{(\tau)}) > (\alpha - 1)\lambda) \\ &\leqslant \mathbb{P}(M(g') > (\alpha - 1)\lambda) \leqslant \frac{1}{(\alpha - 1)\lambda}\mathbb{E}[\mathbb{E}(M(g') \mid \mathscr{F}_T)] \\ &\leqslant \frac{C}{(\alpha - 1)\lambda}\mathbb{E}[\mathbb{E}(S^{(p)}(g') \mid \mathscr{F}_T)] \leqslant \frac{C\beta}{\alpha - 1}\mathbb{P}(T < \infty) \\ &= \frac{C\beta}{\alpha - 1}\mathbb{P}(M(f^{(\tau)}) > \lambda) \leqslant \frac{C\beta}{\alpha - 1}\mathbb{P}(M(f) > \lambda).\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) we obtain

$$\mathbb{P}(M(f) > \alpha\lambda) \leqslant \frac{C\beta}{\alpha - 1}\mathbb{P}(M(f) > \lambda) + \mathbb{P}(S^{(p)}(f) + D_\infty > \beta\lambda),$$

which implies that $(M(f), S^{(p)}(f) + D_\infty)$ satisfies the good $\lambda$-inequality.

(ii)$\Rightarrow$(i). For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$, by Lemma 2.2, there exists a decomposition of $f$: $f = g + h$, and $(M(g), S^{(p)}(g) + M(df))$ satisfies the good $\lambda$-inequality. Since

$$M(df) \leqslant \min\{2M(f), S^{(p)}(f)\}, \quad \max\{M(h), S^{(p)}(h)\} \leqslant \sum_{i=0}^{\infty} \| dh_i \|$$

and $S^{(p)}(g) \leqslant S^{(p)}(f) + S^{(p)}(h)$, take $\Phi(t) = t$, then by Lemma 2.2 and 2.4 we have

$$\begin{aligned}
\| M(f) \|_1 &\leqslant \| M(g) \|_1 + \| M(h) \|_1 \\
&\leqslant C(\| S^{(p)}(g) + M(df) \|_1 + \| \sum_{i=1}^{\infty} \| dh_i \| \|_1) \\
&\leqslant C(\| S^{(p)}(f) + S^{(p)}(h) \|_1 + \| M(df) \|_1 + \| \sum_{i=1}^{\infty} \| dh_i \| \|_1) \\
&\leqslant C \| S^{(p)}(f) \|_1 .
\end{aligned}$$

It follows from Lemma 2.3 that $X$ is isomorphic to a $p$-uniformly smooth space.

The proof is completed. □

LEMMA 2.6. [4] *Let $X$ be a Banach space, $2 \leqslant q < \infty$. If $X$ is isomorphic to a $q$-uniformly convex space, then for any $X$-valued $q$-integrable martingale $f = (f_n)_{n\geqslant 0}$ there exists a constant $C_q > 0$ such that*

$$\mathbb{E}(S^{(q)}(f) \mid \mathscr{F}_0) \leqslant C_q \mathbb{E}(M(f) \mid \mathscr{F}_0).$$

LEMMA 2.7. [4] *Let $X$ be a Banach space, $2 \leqslant q < \infty$. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a $q$-uniformly convex space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$, there exists a constant $C > 0$ such that*

$$\| S^{(q)}(f) \|_1 \leqslant C \| M(f) \|_1 .$$

Now let $\rho_n = M_n(f) + D_n$, define stopping times as in Theorem 2.5:

$$\tau = \inf\{n : \rho_n > \beta\lambda\}, \quad T = \inf\{n : \| f_n^{(\tau)} \| > \lambda\},$$

and notice that

$$S^{(2)}(f^{(\tau)}) - S^{(2)}_{T-1}(f^{(\tau)}) \leqslant (S^{(2)}(f^{(\tau)})^2 - S^{(2)}_{T-1}(f^{(\tau)})^2)^{\frac{1}{2}} = S^{(2)}(f^{(\tau)} - f^{(\tau)}_{T-1}),$$

$$\sup_{n\geqslant T} \| f_n^{(\tau)} - f_{T-1}^{(\tau)} \| \leqslant 2M(f^{(\tau)}\chi(T < \infty) \leqslant 2\beta\lambda\chi(T < \infty),$$

then with the aid of Lemma 2.6 and 2.7 we can show

THEOREM 2.8. *Let $X$ be a Banach space. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a 2-uniformly convex space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ having a predictable control $D = (D_n)_{n\geqslant 0}$, the function pair $(S^{(2)}(f), M(f) + D_\infty)$ satisfies the good $\lambda$-inequality.*

Since its proof is similar to that of Theorem 2.5, we omit the details.

According to Kwapien's theorem (see [4]), we obtain

COROLLARY 2.9. *Let $X$ be a Banach space. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a Hilbert space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ having a predictable control $D = (D_n)_{n\geqslant 0}$, both $(S^{(2)}(f), M(f)+D_\infty)$ and $(M(f), S^{(2)}(f)+D_\infty)$ satisfy the good $\lambda$-inequality.*

## 3. Rearrangement inequalities for vector-valued martingales

LEMMA 3.1. [6] *Let $(A,B)$ be a pair of non-negative and nondecreasing processes. Assume that $A$ is adapted, $B$ is predictable and $B_0 = 0$, and that there exist some constants $C > 0$ and $q > 0$ such that for any stopping times $T$ and $\tau$:*

$$\mathbb{E}((A_T - A_{T\wedge(\tau-1)})^q) \leqslant C^q \mathbb{E}(B_T^q \chi(\tau < \infty)). \tag{3.1}$$

*Then*

$$A_\infty^*(t) \leqslant 4^{\frac{1}{q}} C B_\infty^* \frac{t}{2} + A_\infty^*(2t).$$

LEMMA 3.2. [1] *Let $(F,G)$ be a pair of non-negative measurable functions on $(\Omega, \mathscr{F}, \mathbb{P})$. If $(F,G)$ satisfies the rearrangement inequality :*

$$F^*(t) \leqslant F^*(2t) + CG^*\left(\frac{t}{2}\right), \quad \forall t > 0.$$

*Then with the same $C$, we have*

$$F^*(t) \leqslant 2CG^*\left(\frac{t}{2}\right) + \frac{C}{log2}\int_t^\infty \frac{G^*(s)}{s} ds, \quad \forall t > 0.$$

LEMMA 3.3. [2] (Hardy's inequality) *If $1 \leqslant q < \infty$, $r > 0$, and $f$ is a non-negative function defined on $(0,\infty)$, then*

$$\left(\int_0^\infty \left(\int_t^\infty f(u)du\right)^q t^r \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant \frac{q}{r}\left(\int_0^\infty (tf(t))^q t^r \frac{dt}{t}\right)^{\frac{1}{q}}.$$

LEMMA 3.4. *Let $(F,G)$ be a pair of non-negative measurable functions on $(\Omega, \mathscr{F}, \mathbb{P})$. If $(F,G)$ satisfies the rearrangement inequality. Then for $1 \leqslant q < \infty$, we have*

$$\| F \|_q \leqslant C \| G \|_q .$$

*Proof.* It is a straightforward result of Lemma 3.2 and 3.3. □

THEOREM 3.5. *Let $X$ be a Banach space, $1 < p \leqslant 2$. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a $p$-uniformly smooth space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ having a predictable control $D = (D_n)_{n\geqslant 0}$, the function pair $(M(f), S^{(p)}(f) + D_\infty)$ satisfies the rearrangement inequality.*

*Proof.* (i)$\Rightarrow$(ii). For any stopping times $T$ and $\tau$, we only need to show that (3.1) holds for $A = (A_n)_{n\geqslant 0} = (M_n(f))_{n\geqslant 0}$ and $B = (B_n)_{n\geqslant 0} = (S^{(p)}_{n-1}(f) + D_{n-1})_{n\geqslant 0}$. Since $X$ is isomorphic to a $p$-uniformly smooth space, then by Lemma 2.1 we have

$$\begin{aligned}
\mathbb{E}(A_T - A_{T\wedge(\tau-1)}) &= \mathbb{E}(M_T(f) - M_{T\wedge(\tau-1)}(f)) \\
&\leqslant \mathbb{E}(M(f^{(T)} - f^{(T)}_{\tau-1})) \\
&\leqslant C\mathbb{E}(S^{(p)}(f^{(T)} - f^{(T)}_{\tau-1})) \\
&\leqslant C\mathbb{E}(S^{(p)}(f^{(T)})\chi(\tau < \infty)) \\
&\leqslant C\mathbb{E}(B_T\chi(\tau < \infty)).
\end{aligned}$$

Hence, $(M(f), S^{(p)}(f) + D_\infty)$ satisfies the rearrangement inequality.

(ii)$\Rightarrow$(i) Similar to that of Theorem 2.5. Here we use Lemma 3.4 instead of Lemma 2.4. The proof is completed. $\square$

The following theorem can be proved in a similar way. We omit the proof.

THEOREM 3.6. *Let $X$ be a Banach space. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a $2$-uniformly convex space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ having a predictable control $D = (D_n)_{n\geqslant 0}$, the function pair $(S^{(2)}(f), M(f) + D_\infty)$ satisfies the rearrangement inequality.*

By Kwapien's theorem, Theorem 3.5 and 3.6, we obtain

COROLLARY 3.7. *Let $X$ be a Banach space. Then the following statements are equivalent:*

*(i) $X$ is isomorphic to a Hilbert space;*

*(ii) For any $X$-valued martingale $f = (f_n)_{n\geqslant 0}$ having a predictable control $D = (D_n)_{n\geqslant 0}$, both $(S^{(2)}(f), M(f) + D_\infty)$ and $(M(f), S^{(2)}(f) + D_\infty)$ satisfy the rearrangement inequality.*

REMARK 3.8. By combining the theorems in Section 2 with the corresponding ones in this section, we see that the good $\lambda$-inequality and the corresponding rearrangement inequality are equivalent.

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## REFERENCES

[1] R. J. BAGBY, D. S. KURTZ, *A rearranged good $\lambda$-inequality*, Trans. Amer.Math. Soc., **293** (1986), 71–81.
[2] C. BENNET, R. SHARPLY, *Interpolation of operators*, Academic Press, Boston, 1988.
[3] D. L. BURKHOLDER, *Distribution function inequalities for martingales*, Ann. Probab., **1** (1973), 19–42.
[4] P. D. LIU, *Martingales and geometry of Banach spaces*, Science Press, Beijing, 2007.
[5] R. L. LONG, *Martingale spaces and inequalities*, Peking Univ Press, Beijing, 1993.
[6] R. L. LONG, *Rearrangement techniques in martingale setting*, Illinois J. Math., **35** (1991), 506–521.
[7] G. PISIER, *Martingales with values in uniformly convex spaces*, Israel J. Math., **20** (1975), 326–350.
[8] F. WEISZ, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Math., Vol. 1568, 1994.
[9] W. A. WOYCZYOSKI, *Geometry and martingales in Banach spaces*, Lecture Notes in Math., Vol. 472, 1975.

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