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HOW TO SOLVE THREE FUNDAMENTAL LINEAR MATRIX INEQUALITIES IN THE LÖWNER PARTIAL ORDERING

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Abstract. This paper shows how to derive analytical solutions of the three fundamental linear matrix inequalities

$$AXB \succcurlyeq C\,(\succ C,\, \preccurlyeq C,\, \prec C),$$
$$AXA^* \succcurlyeq B\,(\succ B,\, \preccurlyeq B,\, \prec B),$$
$$AX + (AX)^* \succcurlyeq B\,(\succ B,\, \preccurlyeq B,\, \prec B)$$

in the Löwner partial ordering by using ranks, inertias and generalized inverses of matrices.

1. Introduction

Throughout this paper,

- $\mathbb{C}^{m\times n}$, $\mathbb{C}_{\mathrm{H}}^{m}$ and $\mathbb{C}_{\mathrm{SH}}^{m}$ stand for the sets of all $m\times n$ complex matrices, all $m\times m$ Hermitian complex matrices and all $m\times m$ skew-Hermitian complex matrices, respectively.
- The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A\in\mathbb{C}^{m\times n}$, respectively.
- $[A, B]$ denotes a row block matrix consisting of A and B.
- $i_+(A)$ and $i_-(A)$, called the partial inertia of $A\in\mathbb{C}_{\mathrm{H}}^{m}$, are defined to be the numbers of the positive and negative eigenvalues of A counted with multiplicities, respectively, where $r(A) = i_+(A) + i_-(A)$.
- $A \succcurlyeq 0$ $(A \succ 0)$ means that A is Hermitian positive semi-definite (positive definite).
- Two $A, B\in\mathbb{C}_{\mathrm{H}}^{m}$ are said to satisfy the inequality $A \succcurlyeq B$ $(A \succ B)$ in the Löwner partial ordering if $A-B$ is Hermitian positive semi-definite (positive definite).

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- A positive semi-definite matrix A of order m is said to be a contraction if all its eigenvalues are less than or equal to 1, i.e., $0 \preccurlyeq A \preccurlyeq I_m$, to be a strict contraction if all its eigenvalues are less than 1, i.e., $0 \preccurlyeq A \prec I_m$.

- The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. The symbols E_A and F_A stand for $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$, their ranks are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. A well-known property of the Moore–Penrose inverse is $(A^{\dagger})^* = (A^*)^{\dagger}$. In particular $AA^{\dagger} = A^{\dagger}A$ if $A = A^*$. We shall repeatedly use them in the latter part of this paper. One of the most important applications of generalized inverses is to derive some closed-form formulas for calculating ranks and inertias of matrices, as well as general solutions of matrix equations; see Lemmas 2.1–2.9 below. Results on the Moore–Penrose inverse can be found, e.g., in [3, 4, 12].

- A matrix-valued function for complex matrices is a map between two complex matrix spaces, which can generally be written as $Y = \phi(X)$ for $Y \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{p\times q}$, or briefly, $\phi : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q}$.

The Löwner partial ordering for matrices, as a natural extension of inequalities for real numbers, is one of the most useful concepts in matrix theory for characterizing relations between two complex Hermitian (real symmetric) matrices of the same size, while a main object of study in core matrix theory is to compare Hermitian matrices in the Löwner partial ordering and to establish various possible matrix inequalities. This subject was extensively studied by many authors, and numerous matrix inequalities in the Löwner partial ordering were established in the literature. In the investigation of the Löwner partial ordering between two Hermitian matrices, a challenging task is to solve matrix inequalities that involve unknown matrices. This topic can generally be stated as follows:

PROBLEM. For a given matrix-valued function $\phi(X)$ that satisfies $\phi(X) = \phi^*(X)$, where X is a variable matrix, establish necessary and sufficient conditions for the matrix inequalities

$$\phi(X) \succcurlyeq 0, \ \ \phi(X) \succ 0, \ \ \phi(X) \preccurlyeq 0, \ \ \phi(X) \prec 0 \tag{1.1}$$

to hold, respectively, and find solutions X of the matrix inequalities.

As usual, linear matrix-valued functions as common representatives of various matrix-valued functions are extensively studied from theoretical and applied points of view. When $\phi(X)$ in (1.1) is a linear matrix-valued function, it is usually called a linear matrix inequality (LMI) in the literature. A systematic work on LMIs and their applications in system and control theory can be found, e.g., in [5, 20]. LMIs in the Löwner partial ordering are usually taken as convex constraints to unknown matrices and vectors in mathematical programming and optimization theory. This paper aims at solving the following three groups of LMIs:

$$AXB \succcurlyeq C\,(\succ C, \ \preccurlyeq C, \ \prec C), \tag{1.2}$$

$$AXA^* \succcurlyeq B\,(\succ B, \ \preccurlyeq B, \ \prec B), \tag{1.3}$$

$$AX + (AX)^* \succcurlyeq B\, (\succ B,\ \preccurlyeq B,\ \prec B). \tag{1.4}$$

They are the simplest cases of LMIs and are the starting point of many advanced studies on various complicated LMIs.

Recall that any Hermitian positive semi-definite (positive definite) matrix M can be written as $M = UU^*$ for certain (nonsingular) matrix U. Hence, the mechanism of a matrix inequality in the Löwner partial ordering can be explained by certain matrix equation that involves an unknown quadratic term. In fact, any matrix inequality $\phi(X) \succcurlyeq 0$ $(\phi(X) \succ 0)$ can equivalently be relaxed to

$$\phi(X) - UU^* = 0 \tag{1.5}$$

for certain (nonsingular) matrix U. Due to the non-commutativity of matrix algebra, there are no general methods for finding analytical solutions of quadratic matrix equations, so that it is hard to solve for the unknown matrices X and U from the equation in (1.5) for a general $\phi(X)$. However, for the three fundamental LMIs in (1.2)–(1.4), we are able to obtain their solutions in closed-form by using the relaxed matrix equation in (1.5), and ordinary operations of the given matrices and their generalized inverses.

Matrix equations and matrix inequalities in the Löwner partial ordering have been main objects of study in matrix theory and their applications. Many new theories and methods were developed in the investigations of matrix equations and inequalities. In particular, the concept of generalized inverses of matrices was introduced when Penrose considered general solutions of the matrix equations $AX = B$ and $AXB = C$, cf. [19]. The equalities of (1.2)–(1.4) correspond to the three matrix equations

$$AXB = C, \ \ AXA^* = B, \ \ AX + (AX)^* = B, \tag{1.6}$$

respectively, which were extensively studied from theoretical and practical points of view, while the three matrix-valued functions

$$\phi_1(X) = C - AXB, \tag{1.7}$$

$$\phi_2(X) = B - AXA^*, \tag{1.8}$$

$$\phi_3(X) = B - AX - (AX)^* \tag{1.9}$$

associated with (1.2)–(1.4) were recently considered in [15, 16, 22, 24, 25, 30, 31]. Because (1.2)–(1.4) and (1.6)–(1.9) are some simplest cases of matrix equations, matrix inequalities and matrix-valued functions, they have been attractive objects of study in matrix theory and applications. In fact, it is remarkable that simply knowing when the LMIs in (1.2)–(1.4) are feasible gives some deep insights into the relations between both sides of the LMIs.

This paper is organized as follows. In Section 2, we give a group of known results on matrix equations, as well as some expansion formulas for calculating (extremal) ranks and inertias of matrices. In Section 3, we solve for the inequality in (1.2), and discuss various algebraic properties of the LMI and its solutions. In particular, we give a group of closed-form formulas for calculating the extremal ranks and inertias of $D - AXB$ subject to $AXB \succcurlyeq C$, and use the formulas to establish necessary and sufficient conditions for the two-sided matrix inequality $D \succcurlyeq AXB \succcurlyeq C$ to be solvable. In Sections 4 and 5, we establish necessary and sufficient conditions for the LMIs in (1.3) and (1.4) to be feasible, respectively, and derive general solutions in closed-forms of these LMIs.

2. Preliminaries

In this section, we present some known or new results on solving matrix equations, as well as formulas for calculating ranks and inertias of matrices, which will be used in the latter part of this paper.

LEMMA 2.1. ([14]) *Let $A, B \in \mathbb{C}^{m\times n}$ be given. Then, the following hold.*

(a) *The matrix equation*

$$AX = B \tag{2.1}$$

has a Hermitian solution $X \in \mathbb{C}_{\mathrm{H}}^{n}$ if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution X of (2.1) can be written in the following parametric form

$$X = A^{\dagger}B + (A^{\dagger}B)^* - A^{\dagger}BA^{\dagger}A + F_AWF_A, \tag{2.2}$$

where $W \in \mathbb{C}_{\mathrm{H}}^{n}$ is arbitrary.

(b) *The matrix equation*

$$AXX^* = B \tag{2.3}$$

has a solution XX^* if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, $AB^* \succcurlyeq 0$ and $r(AB^*) = r(B)$. In this case, the general solution XX^* of (2.3) can be written in the following parametric form

$$XX^* = B^*(AB^*)^{\dagger}B + F_AWW^*F_A, \tag{2.4}$$

where $W \in \mathbb{C}^{n\times n}$ is arbitrary.

LEMMA 2.2. ([19]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$ be given. Then, the matrix equation*

$$AXB = C \tag{2.5}$$

has a solution $X \in \mathbb{C}^{n\times q}$ if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$, or equivalently, $E_AC = 0$ and $CF_B = 0$. In this case, the general solution X of (2.5) can be written in the following parametric forms

$$X = A^{\dagger}CB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{2.6}$$

$$X = A^{\dagger}CB^{\dagger} + F_AU_1 + U_2E_B, \tag{2.7}$$

respectively, where $W, U_1, U_2 \in \mathbb{C}^{n\times p}$ are arbitrary.

LEMMA 2.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_{\mathrm{H}}^{m}$ be given. Then, the following hold.*

(a) [10] *The matrix equation*

$$AXA^* = B \tag{2.8}$$

has a solution $X \in \mathbb{C}^n_{\mathrm{H}}$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *or equivalently,* $AA^\dagger B = B$. *In this case, the general Hermitian solution* X *of* (2.8) *can be written in the following parametric forms*

$$X = A^\dagger B(A^\dagger)^* + U - A^\dagger AUA^\dagger A, \tag{2.9}$$

$$X = A^\dagger B(A^\dagger)^* + F_A V + V^* F_A, \tag{2.10}$$

respectively, where $U \in \mathbb{C}^n_{\mathrm{H}}$ *and* $V \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(b) [10, 14] *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXX^*A^* = B \tag{2.11}$$

if and only if $B \succcurlyeq 0$ *and* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. *In this case, the general solution* XX^* *of* (2.11) *can be written as*

$$XX^* = A^\dagger B(A^\dagger)^* + F_A VB(A^\dagger)^* + A^\dagger BV^* F_A + F_A WW^* F_A, \tag{2.12}$$

where $V \in \mathbb{C}^{n\times m}$ *and* $W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

(c) [1] *Under* $A, B \in \mathbb{C}^{m\times m}$, *there exists an* $X \in \mathbb{C}^{m\times m}$ *such that*

$$AXX^*A^* = B \tag{2.13}$$

if and only if $B \succcurlyeq 0$ *and* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. *In this case, the general solution* XX^* *of* (2.13) *can be written as*

$$XX^* = (A^\dagger B^{\frac{1}{2}} + F_A V)(A^\dagger B^{\frac{1}{2}} + F_A V)^*, \tag{2.14}$$

where $V \in \mathbb{C}^{m\times m}$ *is arbitrary.*

LEMMA 2.4. ([31]) *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^m_{\mathrm{H}}$ *be given. Then, the following hold.*

(a) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* = B \tag{2.15}$$

if and only if $E_A BE_A = 0$. *In this case, the general solution* X *of* (2.15) *can be written in the following parametric form*

$$X = \frac{1}{2}A^\dagger B(2I_m - AA^\dagger) + VA^* + F_A W, \tag{2.16}$$

where both $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* = BB^* \tag{2.17}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$*. In this case, the general solution* X *of* (2.17) *can be written as*

$$X = \frac{1}{2}A^{\dagger}BB^* + VA^* + F_AW, \tag{2.18}$$

where both $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

LEMMA 2.5. *Let* $A_1 \in \mathbb{C}^{m\times p}$, $B_1 \in \mathbb{C}^{q\times n}$, $A_2 \in \mathbb{C}^{m\times r}$, $B_2 \in \mathbb{C}^{s\times n}$ *and* $C \in \mathbb{C}^{m\times n}$ *be given. Then, the following hold.*

(a) [18] *There exist* $X \in \mathbb{C}^{p\times q}$ *and* $Y \in \mathbb{C}^{r\times s}$ *such that*

$$A_1XB_1 + A_2YB_2 = C \tag{2.19}$$

if and only if the following four rank equalities

$$r[C, A_1, A_2] = r[A_1, A_2], \ \ r\begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2.20}$$

$$r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \ \ r\begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1) \tag{2.21}$$

hold, or equivalently,

$$[A_1, A_2][A_1, A_2]^{\dagger}C = C, \ \ C\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^{\dagger}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C, \ \ E_{A_1}CF_{B_2} = 0, \ \ E_{A_2}CF_{B_1} = 0. \tag{2.22}$$

(b) [21] *Under* (2.20) *and* (2.21)*, the general solutions of* (2.19) *can be decomposed as*

$$X = X_0 + X_1X_2 + X_3 \ \ \text{and} \ \ Y = Y_0 - Y_1Y_2 + Y_3, \tag{2.23}$$

where X_0 *and* Y_0 *are a pair of special solutions of* (2.19)*,* X_1, X_2, X_3 *and* Y_1, Y_2, Y_3 *are the general solutions of the following four homogeneous matrix equations*

$$A_1X_1 + A_2Y_1 = 0, \ \ X_2B_1 + Y_2B_2 = 0, \ \ A_1X_3B_1 = 0, \ \ A_2Y_3B_2 = 0. \tag{2.24}$$

By using generalized inverses of matrices, (2.23) *can be written in the following parametric forms*

$$X = X_0 + [I_p, 0]F_GWE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} + F_{A_1}W_1 + W_2E_{B_1}, \tag{2.25}$$

$$Y = Y_0 - [0, I_r]F_GWE_H\begin{bmatrix} 0 \\ I_s \end{bmatrix} + F_{A_2}W_3 + W_4E_{B_2}, \tag{2.26}$$

where $G = [A_1, A_2]$, $H = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$*, and the five matrices* W, W_1, W_2, W_3 *and* W_4 *are arbitrary.*

Lemmas 2.1–2.5 show that general solutions of some simple matrix equations can be written as analytical forms composed by the given matrices and their generalized inverses, as well as arbitrary matrices. These analytical formulas can be easily used to establish various algebraic properties of the solutions of the equations, such as, their ranks, ranges, uniqueness, definiteness, etc.

Ranks and inertias of matrices are both basic concepts and useful quantitative tools in matrix theory. The following two lemmas are obvious from the definitions of rank and inertia, which will be used in the latter part of this paper for solving the previous problems.

LEMMA 2.6. *Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}_{\mathrm{H}}^{m}$. Then, the following hold.*

(a) *A is nonsingular if and only if $r(A) = m$.*

(b) *$B = 0$ if and only if $r(B) = 0$.*

(c) *$C \succ 0$ $(C \prec 0)$ if and only if $i_{+}(C) = m$ $(i_{-}(C) = m)$,*

(d) *$C \succcurlyeq 0$ $(C \preccurlyeq 0)$ if and only if $i_{-}(C) = 0$ $(i_{+}(C) = 0)$.*

LEMMA 2.7. *Let $\mathscr{S}$ be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let $\mathscr{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_{\mathrm{H}}^{m}$. Then, the following hold.*

(a) *Under $m = n$, $\mathscr{S}$ has a nonsingular matrix if and only if $\max_{X\in\mathscr{S}} r(X) = m$.*

(b) *Under $m = n$, all $X \in \mathscr{S}$ are nonsingular if and only if $\min_{X\in\mathscr{S}} r(X) = m$.*

(c) *$0 \in \mathscr{S}$ if and only if $\min_{X\in\mathscr{S}} r(X) = 0$.*

(d) *$\mathscr{S} = \{0\}$ if and only if $\max_{X\in\mathscr{S}} r(X) = 0$.*

(e) *$\mathscr{H}$ has a matrix $X \succ 0$ $(X \prec 0)$ if and only if*

$$\max_{X\in\mathscr{H}} i_{+}(X) = m \ \left(\max_{X\in\mathscr{H}} i_{-}(X) = m\right).$$

(f) *All $X \in \mathscr{H}$ satisfy $X \succ 0$ $(X \prec 0)$ if and only if*

$$\min_{X\in\mathscr{H}} i_{+}(X) = m \ \left(\min_{X\in\mathscr{H}} i_{-}(X) = m\right).$$

(g) *$\mathscr{H}$ has a matrix $X \succcurlyeq 0$ $(X \preccurlyeq 0)$ if and only if*

$$\min_{X\in\mathscr{H}} i_{-}(X) = 0 \ \left(\min_{X\in\mathscr{H}} i_{+}(X) = 0\right).$$

(h) *All $X \in \mathscr{H}$ satisfy $X \succcurlyeq 0$ $(X \preccurlyeq 0)$ if and only if*

$$\max_{X\in\mathscr{H}} i_{-}(X) = 0 \ \left(\max_{X\in\mathscr{H}} i_{+}(X) = 0\right).$$

The question of whether a given matrix function is positive semi-definite or positive definite everywhere is ubiquitous in mathematics and applications. Lemma 2.7(e)–(h) show that if certain explicit formulas for calculating the maximal and minimal inertias of a given Hermitian matrix function are established, we can use them, as demonstrated in Sections 2, 3 and 5 below, to derive necessary and sufficient conditions for the Hermitian matrix function to be definite or semi-definite.

In order to simplify various matrix expressions involving generalized inverses of matrices and arbitrary matrices, we need some useful expansion formulas for calculating ranks and inertias of matrices.

LEMMA 2.8. ([17]) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then, the following rank expansion formulas hold*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2.27}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{2.28}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C), \tag{2.29}$$

$$r\begin{bmatrix} AA^* & B \\ B^* & 0 \end{bmatrix} = r[A,\ B] + r(B), \tag{2.30}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_A B \\ CF_A & D - CA^{\dagger}B \end{bmatrix}. \tag{2.31}$$

If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, *then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B). \tag{2.32}$$

LEMMA 2.9. ([24]) *Let* $A \in \mathbb{C}^{m}_{\mathrm{H}}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{n}_{\mathrm{H}}$, *and define*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the partial inertias of M_1 *and* M_2 *can be expanded as*

$$i_{\pm}(M_1) = r(B) + i_{\pm}(E_B A E_B), \tag{2.33}$$

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}. \tag{2.34}$$

In particular, the following hold.

(a) *If* $A \succcurlyeq 0$, *then*

$$i_{+}(M_1) = r[A, B], \ \ i_{-}(M_1) = r(B). \tag{2.35}$$

(b) *If* $A \preccurlyeq 0$, *then*

$$i_+(M_1) = r(B), \;\; i_-(M_1) = r[A, B]. \tag{2.36}$$

(c) *If* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *then*

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D - B^*A^{\dagger}B). \tag{2.37}$$

(d) $i_{\pm}(M_1) = m \Leftrightarrow i_{\mp}(E_BAE_B) = 0$ *and* $r(E_BAE_B) = r(E_B)$.

(e) $M_2 \succcurlyeq 0 \;\Leftrightarrow\; A \succcurlyeq 0,\; \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $D - B^*A^{\dagger}B \succcurlyeq 0 \;\Leftrightarrow\; D \succcurlyeq 0,\; \mathscr{R}(B^*) \subseteq \mathscr{R}(D)$ *and* $A - BC^{\dagger}B^* \succcurlyeq 0$.

(f) $M_2 \succ 0 \;\Leftrightarrow A \succ 0$ *and* $D - B^*A^{-1}B \succ 0 \;\Leftrightarrow\; D \succ 0$ *and* $A - BD^{-1}B^* \succ 0$.

(g) *Under* $A \succcurlyeq 0$ *and* $A_1 \succcurlyeq 0$, *the inequality* $A \succcurlyeq A_1$ *holds if and only if* $\mathscr{R}(A_1) \subseteq \mathscr{R}(A)$ *and* $A_1 - A_1A^{\dagger}A_1 \succcurlyeq 0$.

LEMMA 2.10. *Let* $A, B \in \mathbb{C}_{\mathrm{H}}^{m}$ *and* $P \in \mathbb{C}^{m\times n}$. *Then the following hold.*

(a) *If* $A \succcurlyeq B$, *then* $P^*AP \succcurlyeq P^*BP$.

(b) $A \succcurlyeq 0$ *if and only if* $A^{\dagger} \succcurlyeq 0$.

(c) *If* $I_m - A \succcurlyeq 0$, *then* $I_m - PP^{\dagger}APP^{\dagger} \succcurlyeq 0$.

(d) *If* $I_m - A \succ 0$, *then* $I_m - PP^{\dagger}APP^{\dagger} \succ 0$.

Proof. Result (a) is obvious from the definition of the positive semi-definiteness of Hermitian matrix. Result (b) is obvious from similarity decomposition of A and the definition of the Moore–Penrose inverse of a matrix. If A is Hermitian, then we can find by Lemma 2.9(c), $*$-congruence transformation and (2.32) that

$$\begin{aligned} i_{\pm}(I_m - PP^{\dagger}APP^{\dagger}) &= i_{\pm}\begin{bmatrix} A & APP^{\dagger} \\ PP^{\dagger}A & I_m \end{bmatrix} - i_{\pm}(A) \\ &= i_{\pm}\begin{bmatrix} A - APP^{\dagger}A & 0 \\ 0 & I_m \end{bmatrix} - i_{\pm}(A) \\ &= i_{\pm}(I_m) + i_{\pm}(A - APP^{\dagger}A) - i_{\pm}(A) \\ &= i_{\pm}(I_m) + i_{\pm}\begin{bmatrix} P^*P & P^*A \\ AP & A \end{bmatrix} - i_{\pm}(PP^*) - i_{\pm}(A) \\ &= i_{\pm}(I_m) + i_{\pm}\begin{bmatrix} P^*P - P^*AP & 0 \\ 0 & A \end{bmatrix} - i_{\pm}(PP^*) - i_{\pm}(A) \\ &= i_{\pm}(I_m) + i_{\pm}[P^*(I_m - A)P] - i_{\pm}(PP^*), \end{aligned}$$

namely

$$i_+(I_m - PP^{\dagger}APP^{\dagger}) = m - r(P) + i_+[P^*(I_m - A)P], \tag{2.38}$$

$$i_{-}(I_m - PP^{\dagger}APP^{\dagger}) = i_{-}[P^*(I_m - A)P]. \tag{2.39}$$

If $A \preccurlyeq I_m$, then (2.39) reduces to

$$i_{-}(I_m - PP^{\dagger}APP^{\dagger}) = i_{-}[P^*(I_m - A)P] = 0.$$

Hence, (c) follows by Lemma 2.6(d). If $A \prec I_m$, then $P^*(I_m - A)P \succcurlyeq 0$ and $i_{+}[P^*(I_m - A)P] = r[P^*(I_m - A)P] = r(P)$. Thus, (2.38) reduces to

$$i_{+}(I_m - PP^{\dagger}APP^{\dagger}) = m - r(P) + r(P) = m.$$

Hence, (d) follows by Lemma 2.6(c). □

In two earlier papers [22] and [30], an expansion formula for the rank of $A - BXC$ was established as follows

$$r(A - BXC) = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A, B] - r(M) + r[E_{T_1}(X + TM^{\dagger}S)F_{S_1}], \tag{2.40}$$

where

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \ T = [0, \ I_n], \ S = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \ T_1 = TF_M, \ S_1 = E_M S,$$

and the following result was established.

LEMMA 2.11. ([22, 30]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then the global maximal and minimal ranks of $A - BXC$ with respect to $X \in \mathbb{C}^{k\times l}$ are given by*

$$\max_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = \min\left\{ r[A, B], \ \ r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{2.41}$$

$$\min_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = r[A, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.42}$$

In particular,

$$\max_{X\in\mathbb{C}^{k\times n}} r(A - BX) = \min\{r[A, B], \ \ n\}, \quad \min_{X\in\mathbb{C}^{k\times n}} r(A - BX) = r[A, B] - r(B). \tag{2.43}$$

The matrices X that satisfy (2.41)–(2.43), namely, the global maximizers and minimizers of the objective rank function, are not necessarily unique and their general expressions can also be derived from the term $E_{T_1}(X + TM^{\dagger}S)F_{S_1}$ in (2.40); see [22, 30].

LEMMA 2.12. ([16, 25]) *Let $A \in \mathbb{C}^{m}_{\mathrm{H}}$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then, the global maximal and minimal ranks and partial inertias of $A - BXC - (BXC)^*$ are given by*

$$\max_{X\in\mathbb{C}^{n\times p}} r[A - BXC - (BXC)^*] = \min\left\{ r[A, B, C^*], \ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \ r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{2.44}$$

$$\min_{X\in\mathbb{C}^{n\times p}} r[A-BXC-(BXC)^*] = 2r[A,B,C^*] + \max\{s_+ + s_-,\ t_+ + t_-,\ s_+ + t_-,\ s_- + t_+\}, \tag{2.45}$$

$$\max_{X\in\mathbb{C}^{n\times p}} i_{\pm}[A-BXC-(BXC)^*] = \min\left\{ i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},\ i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \tag{2.46}$$

$$\min_{X\in\mathbb{C}^{n\times p}} i_{\pm}[A-BXC-(BXC)^*] = r[A,B,C^*] + \max\{s_{\pm},\ t_{\pm}\}, \tag{2.47}$$

where

$$s_{\pm} = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r\begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix},\ t_{\pm} = i_{\pm}\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}.$$

In particular,

$$\max_{X\in\mathbb{C}^{n\times m}} r[A-BX-(BX)^*] = \min\left\{ m,\ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \tag{2.48}$$

$$\min_{X\in\mathbb{C}^{n\times m}} r[A-BX-(BX)^*] = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B), \tag{2.49}$$

$$\max_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A-BX-(BX)^*] = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{2.50}$$

$$\min_{X\in\mathbb{C}^{n\times m}} i_{\pm}[A-BX-(BX)^*] = i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(B). \tag{2.51}$$

Eqs. (2.44)–(2.51) were derived from some expansion formulas for the rank and inertia of $A-BXC-(BXC)^*$ in [16, 25], while the matrices X that satisfy (2.44)–(2.51) were also given in [16, 25] by using certain simultaneous decomposition of the three given matrices and their generalized inverses.

We also need the following results on the ranks and inertias of the quadratic matrix-valued functions

$$A \pm (BX+C)(BX+C)^* = A \pm (BXX^*B^* + BXC^* + CX^*B^* + CC^*)$$

and their consequences.

LEMMA 2.13. ([27]) *Let $A\in\mathbb{C}_{\mathrm{H}}^{m}$ and $B\in\mathbb{C}^{m\times k}$ and $C\in\mathbb{C}^{m\times n}$ be given, and let*

$$G_1 = \begin{bmatrix} A+CC^* & B \\ B^* & 0 \end{bmatrix},\ G_2 = \begin{bmatrix} A-CC^* & B \\ B^* & 0 \end{bmatrix},\ G_3 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \end{bmatrix}.$$

Then, the following hold.

(a) *The maximal and minimal ranks and partial inertias of $\phi_1(X) = A + (BX + C)(BX+C)^*$ are given by*

$$\max_{X\in\mathbb{C}^{k\times n}} r[\phi_1(X)] = \min\{r[A,B,C],\ r(G_1),\ r(A)+n\}, \tag{2.52}$$

$$\min_{X \in \mathbb{C}^{k\times n}} r[\phi_1(X)] = 2r[A, B, C] + \max\{h_1, \ h_2, \ h_3, \ h_4\}, \tag{2.53}$$

$$\max_{X \in \mathbb{C}^{k\times n}} i_+[\phi_1(X)] = \min\{i_+(G_1), \ i_+(A) + n\}, \tag{2.54}$$

$$\max_{X \in \mathbb{C}^{k\times n}} i_-[\phi_1(X)] = \min\{i_-(G_1), \ i_-(A)\}, \tag{2.55}$$

$$\min_{X \in \mathbb{C}^{k\times n}} i_+[\phi_1(X)] = r[A, B, C] + \max\{i_+(G_1) - r(G_3), \ i_+(A) - r[A, B]\}, \tag{2.56}$$

$$\min_{X \in \mathbb{C}^{k\times n}} i_-[\phi_1(X)] = r[A, B, C] + \max\{i_-(G_1) - r(G_3), \ i_-(A) - r[A, B] - n\}, \tag{2.57}$$

where

$$h_1 = r(G_1) - 2r(G_3), \quad h_2 = r(A) - 2r[A, B] - n,$$
$$h_3 = i_-(G_1) - r(G_3) + i_+(A) - r[A, B],$$
$$h_4 = i_+(G_1) - r(G_3) + i_-(A) - r[A, B] - n.$$

(b) *The maximal and minimal ranks and partial inertias of* $\phi_2(X) = A - (BX + C)(BX + C)^*$ *are given by*

$$\max_{X \in \mathbb{C}^{k\times n}} r[\phi_2(X)] = \min\{r[A, B, C], \ r(G_2), \ r(A) + n\}, \tag{2.58}$$

$$\min_{X \in \mathbb{C}^{k\times n}} r[\phi_2(X)] = 2r[A, B, C] + \max\{h_5, \ h_6, \ h_7, \ h_8\}, \tag{2.59}$$

$$\max_{X \in \mathbb{C}^{k\times n}} i_+[\phi_2(X)] = \min\{i_+(G_2), \ i_+(A)\}, \tag{2.60}$$

$$\max_{X \in \mathbb{C}^{k\times n}} i_-[\phi_2(X)] = \min\{i_-(G_2), \ i_-(A) + n\}, \tag{2.61}$$

$$\min_{X \in \mathbb{C}^{k\times n}} i_+[\phi_2(X)] = r[A, B, C] + \max\{i_+(G_2) - r(G_3), \ i_+(A) - r[A, B] - n\}, \tag{2.62}$$

$$\min_{X \in \mathbb{C}^{k\times n}} i_-[\phi_2(X)] = r[A, B, C] + \max\{i_-(G_2) - r(G_3), \ i_-(A) - r[A, B]\}, \tag{2.63}$$

where

$$h_5 = r(G_2) - 2r(G_3), \quad h_6 = r(A) - 2r[A, B] - n,$$
$$h_7 = i_+(G_2) - r(G_3) + i_-(A) - r[A, B],$$
$$h_8 = i_-(G_2) - r(G_3) + i_+(A) - r[A, B] - n.$$

The matrices X *that satisfy* (2.52)–(2.63) *can be derived from those satisfying* (2.44)–(2.47).

When $C = 0$, Lemma 2.13 reduces to the following result.

COROLLARY 2.14. ([26]) *Let* $A \in \mathbb{C}_{\mathrm{H}}^{m}$ *and* $B \in \mathbb{C}^{m\times n}$ *be given, and let* $M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$. *Then, the following hold.*

(a) *The maximal and minimal ranks and partial inertias of* $A \pm BXX^*B^*$ *are given by*

$$\max_{X\in\mathbb{C}^{n\times n}} r(A+BXX^*B^*) = r[A, B], \tag{2.64}$$
$$\min_{X\in\mathbb{C}^{n\times n}} r(A+BXX^*B^*) = i_+(A) + r[A, B] - i_+(M), \tag{2.65}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_+(A+BXX^*B^*) = i_+(M), \tag{2.66}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_-(A+BXX^*B^*) = i_-(A), \tag{2.67}$$
$$\min_{X\in\mathbb{C}^{n\times n}} i_+(A+BXX^*B^*) = i_+(A), \tag{2.68}$$
$$\min_{X\in\mathbb{C}^{n\times n}} i_-(A+BXX^*B^*) = r[A, B] - i_+(M), \tag{2.69}$$

and

$$\max_{X\in\mathbb{C}^{n\times n}} r(A-BXX^*B^*) = r[A, B], \tag{2.70}$$
$$\min_{X\in\mathbb{C}^{n\times n}} r(A-BXX^*B^*) = i_-(A) + r[A, B] - i_-(M), \tag{2.71}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_+(A-BXX^*B^*) = i_+(A), \tag{2.72}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_-(A-BXX^*B^*) = i_-(M), \tag{2.73}$$
$$\min_{X\in\mathbb{C}^{n\times n}} i_+(A-BXX^*B^*) = r[A, B] - i_-(M), \tag{2.74}$$
$$\min_{X\in\mathbb{C}^{n\times n}} i_-(A-BXX^*B^*) = i_-(A). \tag{2.75}$$

(b) *If* $A \succcurlyeq 0$, *then*

$$\max_{X\in\mathbb{C}^{n\times n}} r(A+BXX^*B^*) = r[A, B], \tag{2.76}$$
$$\min_{X\in\mathbb{C}^{n\times n}} r(A+BXX^*B^*) = r(A), \tag{2.77}$$

and

$$\max_{X\in\mathbb{C}^{n\times n}} r(A-BXX^*B^*) = r[A, B], \tag{2.78}$$
$$\min_{X\in\mathbb{C}^{n\times n}} r(A-BXX^*B^*) = r[A, B] - r(B), \tag{2.79}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_+(A-BXX^*B^*) = r(A), \tag{2.80}$$
$$\max_{X\in\mathbb{C}^{n\times n}} i_-(A-BXX^*B^*) = r(B), \tag{2.81}$$

$$\min_{X \in \mathbb{C}^{n\times n}} i_{+}(A - BXX^{*}B^{*}) = r[A,\, B] - r(B), \tag{2.82}$$

$$\min_{X \in \mathbb{C}^{n\times n}} i_{-}(A - BXX^{*}B^{*}) = 0. \tag{2.83}$$

The matrices XX^* that satisfy (2.64)–(2.83) *can be derived from those satisfying* (2.44)–(2.47).

3. General solution of $AXB \succcurlyeq C\,(\succ C,\, \preccurlyeq C,\, \prec C)$ and its properties

A necessary condition for (1.2) to hold is $AXB = (AXB)^*$. In such a case, the matrix X satisfying $AXB = (AXB)^*$ is called a symmetrizer of AXB; see [2]. In this section, we derive an analytical presentation for the general solution of the LMI in (1.2) by using the given matrices and their generalized inverses, and establish various algebraic properties of the LMI.

THEOREM 3.1. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m}_{\mathrm{H}}$ be given, and define*

$$M = [E_A,\, F_B], \quad N = \begin{bmatrix} C & CA & 0 \\ C & C & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix}. \tag{3.1}$$

(a) *Then, the following statements are equivalent.*

(i) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB \succcurlyeq C. \tag{3.2}$$

(ii) $M^*CM \preccurlyeq 0$ *and* $\mathscr{R}(M^*CM) = \mathscr{R}(M^*C)$.

(iii) $i_{-}(M^*CM) = r(M^*C)$.

(iv) $i_{-}(N) = r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix}$.

In this case, the general solution $X \in \mathbb{C}^{p\times q}$ of (3.2) *and the corresponding AXB can be written in the following parametric forms*

$$X = A^{\dagger}CB^{\dagger} - A^{\dagger}CM(M^*CM)^{\dagger}M^*CB^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.3}$$

$$AXB = C - CM(M^*CM)^{\dagger}M^*C + E_MUU^*E_M, \tag{3.4}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB \succ C \tag{3.5}$$

if and only if

$$M^*CM \preccurlyeq 0 \ \ and \ \ r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = m + r[A, B^*]. \tag{3.6}$$

In this case, the general solution X of (3.5) can be written as (3.3), in which $U \in \mathbb{C}^{m\times m}$ is any matrix such that $r[CM, \ E_M U] = m$, and $W \in \mathbb{C}^{p\times q}$ is arbitrary.

Proof. Inequality (1.2) is obviously equivalent to the following linear-quadratic matrix equation

$$AXB = C + YY^*. \tag{3.7}$$

By Lemma 2.2, this equation is solvable for X if and only if

$$E_A(C + YY^*) = 0 \ \text{ and } \ (C + YY^*)F_B = 0, \tag{3.8}$$

that is,

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} YY^* = -\begin{bmatrix} E_A C \\ F_B C \end{bmatrix}. \tag{3.9}$$

By Lemma 2.1(b), this quadratic matrix equation is solvable for YY^* if and only if

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} C[E_A, F_B] \preccurlyeq 0 \ \text{ and } \ r\left(\begin{bmatrix} E_A \\ F_B \end{bmatrix} C[E_A, F_B]\right) = r(C[E_A, F_B]),$$

that is,

$$M^*CM \preccurlyeq 0 \ \text{ and } \ r(M^*CM) = r(M^*C). \tag{3.10}$$

Note from $i_-(M^*CM) \leqslant r(M^*CM) \leqslant r(M^*C)$ that

$$M^*CM \preccurlyeq 0 \ \Leftrightarrow \ i_-(M^*CM) = r(M^*CM), \tag{3.11}$$

$$r(M^*CM) = r(CM) \Leftrightarrow \mathscr{R}(M^*CM) = \mathscr{R}(M^*C). \tag{3.12}$$

So that (3.10) is equivalent to (ii) and (iii) in (a), respectively. It is easy to see from (2.33) that

$$i_{\pm}(M^*CM) = i_{\pm}\left(\begin{bmatrix} E_A \\ F_B \end{bmatrix} C[E_A, F_B]\right) = i_{\pm}(N) - r(A) - r(B), \tag{3.13}$$

$$r(M^*CM) = r(N) - 2r(A) - 2r(B), \tag{3.14}$$

$$r(CM) = r(C[E_A, F_B]) = r\begin{bmatrix} C & C \\ A^* & 0 \\ 0 & B \end{bmatrix} - r(A) - r(B)$$

$$= r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(A) - r(B), \tag{3.15}$$

$$r(M) = r[E_A, F_B] = r\begin{bmatrix} A & 0 & I_m \\ 0 & B^* & I_m \end{bmatrix} - r(A) - r(B)$$

$$= m + r[A, B^*] - r(A) - r(B). \tag{3.16}$$

So that (iii) and (iv) in (a) are equivalent. Under the conditions in (a), the general solution YY^* of (3.9) can be written as

$$YY^* = -CM(M^*CM)^\dagger M^*C + E_M UU^* E_M, \tag{3.17}$$

where $U \in \mathbb{C}^{m\times m}$ is arbitrary. Substituting the YY^* into (3.7) gives

$$AXB = C - CM(M^*CM)^\dagger M^*C + E_M UU^* E_M. \tag{3.18}$$

By Lemma 2.2, the general solution X of (3.18) is

$$X = A^\dagger CB^\dagger - A^\dagger CM(M^*CM)^\dagger M^*CB^\dagger + A^\dagger E_M UU^* E_M B^\dagger + W - A^\dagger AWBB^\dagger,$$

establishing (3.3) and (3.4).

It can be seen from (3.18) that (3.5) holds if and only if

$$-CM(M^*CM)^\dagger M^*C + E_M UU^* E_M \succ 0 \tag{3.19}$$

for some U. Under the conditions in (a), we have

$$\begin{aligned} r[-CM(M^*CM)^\dagger M^*C + E_M UU^* E_M] &= r[-CM(M^*CM)^\dagger M^*C,\, E_M UU^* E_M] \\ &= r[CM,\, E_M U]. \end{aligned}$$

Hence,

$$\begin{aligned} \max_U r[-CM(M^*CM)^\dagger M^*C + E_M UU^* E_M] &= r[CM,\, E_M] \\ &= r(MM^\dagger CM) + r(E_M) = r(CM) + m - r(M) \\ &= r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r[A,\, B^*]. \end{aligned} \tag{3.20}$$

Thus, (3.19) is equivalent to (3.6). □

The following result can be shown similarly.

COROLLARY 3.2. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, M and N be as given in* (3.1)*.*

(a) *Then, the following statements are equivalent.*

(i) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB \preccurlyeq C. \tag{3.21}$$

(ii) $M^*CM \succcurlyeq 0$ *and* $\mathscr{R}(M^*CM) = \mathscr{R}(M^*C)$.

(iii) $i_+(M^*CM) = r(M^*C)$.

(iv) $i_+(N) = r\begin{bmatrix} C & A & 0 \\ C & 0 & B^* \end{bmatrix}$.

In this case, the general solution $X \in \mathbb{C}^{p\times q}$ of (3.21) *and the corresponding AXB can be written in the following parametric forms*

$$X = A^{\dagger}CB^{\dagger} - A^{\dagger}CM(M^{*}CM)^{\dagger}M^{*}CB^{\dagger} - A^{\dagger}E_{M}UU^{*}E_{M}B^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.22}$$

$$AXB = C - CM(M^{*}CM)^{\dagger}M^{*}C - E_{M}UU^{*}E_{M}, \tag{3.23}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB \prec C \tag{3.24}$$

if and only if

$$M^{*}CM \succcurlyeq 0 \ \ and \ \ r\begin{bmatrix} A & 0 & C \\ 0 & B^{*} & C \end{bmatrix} = m + r[A, B^{*}]. \tag{3.25}$$

In this case, the general solution X of (3.24) *can be written as* (3.22)*, in which U is any matrix such that $r[CM, \ E_{M}U] = m$, and $W \in \mathbb{C}^{p\times q}$ is arbitrary.*

We next establish some algebraic properties of the fixed parts in (3.3) and (3.22).

COROLLARY 3.3. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, M and N be as given in* (3.1)*, and define*

$$\widehat{X} = A^{\dagger}CB^{\dagger} - A^{\dagger}CM(M^{*}CM)^{\dagger}M^{*}CB^{\dagger}. \tag{3.26}$$

(a) *Under the condition that* (3.2) *has a solution, the $\widehat{X}$ in* (3.26) *satisfies $A\widehat{X}B \succcurlyeq C$, and*

$$i_{\pm}(A\widehat{X}B) = r(A) + r(B) + i_{\pm}(C) - i_{\pm}(N), \tag{3.27}$$

$$r(\widehat{X}) = r(A\widehat{X}B) = 2r(A) + 2r(B) + r(C) - r(N), \tag{3.28}$$

$$i_{+}(A\widehat{X}B - C) = r(A\widehat{X}B - C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^{*} & C \end{bmatrix} - r(N). \tag{3.29}$$

(b) *Under the condition that* (3.21) *has a solution, the $\widehat{X}$ in* (3.26) *satisfies $A\widehat{X}B \preccurlyeq C$, and*

$$i_{\pm}(A\widehat{X}B) = r(A) + r(B) + i_{\pm}(C) - i_{\pm}(N), \tag{3.30}$$

$$r(\widehat{X}) = r(A\widehat{X}B) = 2r(A) + 2r(B) + r(C) - r(N), \tag{3.31}$$

$$r(A\widehat{X}B - C) = i_{+}(A\widehat{X}B - C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^{*} & C \end{bmatrix} - r(N). \tag{3.32}$$

Proof. Under the condition that (3.2) has a solution, set $U = W = 0$ in (3.3). Then we see that the $\widehat{X}$ in (3.26) is a solution of $AXB \succcurlyeq C$. Also note from (3.18) that

$$A\widehat{X}B = (A\widehat{X}B)^{*} = C - CM(M^{*}CM)^{\dagger}M^{*}C, \tag{3.33}$$

$$A\widehat{X}B - C = (A\widehat{X}B - C)^* = -CM(M^*CM)^\dagger M^*C \succcurlyeq 0. \tag{3.34}$$

In this case, applying (2.37) and (2.32) to (3.33) and (3.34), we obtain

$$\begin{aligned} r(\widehat{X}) &= r[A^\dagger CB^\dagger - A^\dagger CM(M^*CM)^\dagger M^*CB^\dagger] \\ &= r\begin{bmatrix} M^*CM & M^*CB^\dagger \\ A^\dagger CM & A^\dagger CB^\dagger \end{bmatrix} - r(M^*CM), \end{aligned} \tag{3.35}$$

$$\begin{aligned} i_\pm(A\widehat{X}B) &= i_\pm[C - CM(M^*CM)^\dagger M^*C] \\ &= i_\pm\begin{bmatrix} M^*CM & M^*C \\ CM & C \end{bmatrix} - i_\pm(M^*CM), \end{aligned} \tag{3.36}$$

$$\begin{aligned} r(A\widehat{X}B - C) = i_-(A\widehat{X}B - C) &= r[CM(M^*CM)^\dagger M^*C] \\ &= r\begin{bmatrix} M^*CM & M^*C \\ CM & 0 \end{bmatrix} - r(M^*CM). \end{aligned} \tag{3.37}$$

Applying elementary matrix operations, congruence matrix operations and (2.29), we obtain

$$\begin{aligned} r\begin{bmatrix} M^*CM & M^*CB^\dagger \\ A^\dagger CM & A^\dagger CB^\dagger \end{bmatrix} &= r\left(\begin{bmatrix} E_A \\ F_B \\ A^\dagger \end{bmatrix} C[E_A, F_B, B^\dagger]\right) \\ &= r\left(\begin{bmatrix} I_m \\ F_B \\ A^\dagger \end{bmatrix} C[E_A, I_m, B^\dagger]\right) = r(C), \end{aligned} \tag{3.38}$$

$$i_\pm\begin{bmatrix} M^*CM & M^*C \\ CM & C \end{bmatrix} = i_\pm\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} = i_\pm(C), \tag{3.39}$$

$$r\begin{bmatrix} M^*CM & M^*C \\ CM & 0 \end{bmatrix} = 2r(M^*C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - 2r(A) - 2r(B). \tag{3.40}$$

Substituting these formulas into (3.35)–(3.37) yields (3.27)–(3.29). Results (b) can be shown similarly. □

COROLLARY 3.4. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and M and N be as given in* (3.1)*. Also assume that* (3.2) *is feasible, and define*

$$\mathscr{S}_1 = \{X \in \mathbb{C}^{p\times q} \mid AXB \succcurlyeq C\}. \tag{3.41}$$

Then, the following hold.

(a) *The minimal matrices of AXB and $AXB - C$ subject to $X \in \mathscr{S}_1$ in the Löwner partial ordering are given by*

$$\min_{\succcurlyeq}\{AXB \mid X \in \mathscr{S}_1\} = C - CM(M^*CM)^\dagger M^*C, \tag{3.42}$$

$$\min_{\succcurlyeq}\{AXB - C \mid X \in \mathscr{S}_1\} = -CM(M^*CM)^\dagger M^*C. \tag{3.43}$$

(b) *The maximal and minimal ranks and partial inertias of AXB and $AXB-C$ subject to $X \in \mathscr{S}_1$ are given by*

$$\max_{X\in\mathscr{S}_1} r(AXB) = \max_{X\in\mathscr{S}_1} i_+(AXB) = r(A)+r(B)-r[A, B^*], \tag{3.44}$$

$$\min_{X\in\mathscr{S}_1} r(AXB) = \min_{X\in\mathscr{S}_1} i_+(AXB) = r(A)+r(B)+i_+(C)-i_+(N), \tag{3.45}$$

$$\max_{X\in\mathscr{S}_1} i_-(AXB) = r(A)+r(B)+i_-(C)-i_-(N), \tag{3.46}$$

$$\min_{X\in\mathscr{S}_1} i_-(AXB) = 0, \tag{3.47}$$

$$\max_{X\in\mathscr{S}_1} r(AXB-C) = r(N)-r(A)-r(B)-r[A, B^*], \tag{3.48}$$

$$\min_{X\in\mathscr{S}_1} r(AXB-C) = r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(A)-r(B). \tag{3.49}$$

In consequence, the following hold.

(c) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $AXB \succ 0$ and $AXB \succcurlyeq C$ if and only if $r[A, B^*] = r(A)+r(B)-m$.*

(d) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $0 \succ AXB \succcurlyeq C$ if and only if $C \prec 0$ and $r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = r(A)+r(B)$.*

(e) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $0 \succcurlyeq AXB \succcurlyeq C$ if and only if $C \preccurlyeq 0$.*

(f) *There always exists an $X \in \mathbb{C}^{p\times q}$ such that $AXB \succcurlyeq 0$ and $AXB \succcurlyeq C$.*

Proof. From (3.18), both AXB and $AXB-C$ subject to $X \in \mathscr{S}_1$ can be written as

$$\begin{aligned} AXB = (AXB)^* &= C - CM(M^*CM)^\dagger M^*C + E_M UU^* E_M \\ &= A\widehat{X}B + E_M UU^* E_M, \end{aligned} \tag{3.50}$$

$$\begin{aligned} AXB - C = (AXB-C)^* &= -CM(M^*CM)^\dagger M^*C + E_M UU^* E_M \\ &= A\widehat{X}B - C + E_M UU^* E_M. \end{aligned} \tag{3.51}$$

Hence,

$$AXB \succcurlyeq C - CM(M^*CM)^\dagger M^*C, \quad AXB - C \succcurlyeq -CM(M^*CM)^\dagger M^*C \tag{3.52}$$

hold for any $U \in \mathbb{C}^{m\times m}$, which implies (3.42) and (3.43).

Applying elementary matrix operations, congruence matrix operations, (2.29) and (3.16), we obtain

$$r(E_M) = m - r(M) = r(A)+r(B)-r[A, B^*], \tag{3.53}$$

$$\begin{aligned} r[E_M, A\widehat{X}B] &= r(E_M) + r(A\widehat{X}BM) = r(E_M) + r[(A\widehat{X}B)^* E_A, A\widehat{X}BF_B] \\ &= r(E_M) = r(A)+r(B)-r[A, B^*], \end{aligned} \tag{3.54}$$

$$i_\pm\begin{bmatrix} A\widehat{X}B & E_M \\ E_M & 0 \end{bmatrix} = i_\pm\begin{bmatrix} 0 & E_M \\ E_M & 0 \end{bmatrix} = r(E_M) = r(A)+r(B)-r[A, B^*]. \tag{3.55}$$

Applying (2.64)–(2.69) to (3.50) and (3.51) and simplifying by (3.53)–(3.55), we obtain

$$
\begin{aligned}
\max_{X\in\mathscr{S}_1} r(AXB) &= \max_{U\in\mathbb{C}^{m\times m}} r(A\widehat{X}B + E_M UU^* E_M) \\
&= r[E_M, A\widehat{X}B] = r(A) + r(B) - r[A, B^*], \\
\min_{X\in\mathscr{S}_1} r(AXB) &= \min_{U\in\mathbb{C}^{m\times m}} r(A\widehat{X}B + E_M UU^* E_M) \\
&= i_+(A\widehat{X}B) + r[E_M, A\widehat{X}B] - i_+\begin{bmatrix} A\widehat{X}B & E_M \\ E_M & 0 \end{bmatrix} \\
&= r(A) + r(B) + i_+(C) - i_+(N), \\
\max_{X\in\mathscr{S}_1} i_+(AXB) &= \max_{U\in\mathbb{C}^{m\times m}} i_+(A\widehat{X}B + E_M UU^* E_M) \\
&= i_+\begin{bmatrix} A\widehat{X}B & E_M \\ E_M & 0 \end{bmatrix} = r(A) + r(B) - r[A, B^*], \\
\min_{X\in\mathscr{S}_1} i_+(AXB) &= \min_{U\in\mathbb{C}^{m\times m}} i_+(A\widehat{X}B + E_M UU^* E_M) \\
&= i_+(A\widehat{X}B) = r(A) + r(B) + i_+(C) - i_+(N), \\
\max_{X\in\mathscr{S}_1} i_-(AXB) &= \max_{U\in\mathbb{C}^{m\times m}} i_-(A\widehat{X}B + E_M UU^* E_M) \\
&= i_-(A\widehat{X}B) = r(A) + r(B) + i_-(C) - i_-(N), \\
\min_{X\in\mathscr{S}_1} i_-(AXB) &= \min_{U\in\mathbb{C}^{m\times m}} i_-(A\widehat{X}B + E_M UU^* E_M) \\
&= r[E_M, A\widehat{X}B] - i_+\begin{bmatrix} A\widehat{X}B & E_M \\ E_M & 0 \end{bmatrix} = 0,
\end{aligned}
$$

establishing (3.44)–(3.49). Note from (3.18), that

$$
\begin{aligned}
r(AXB - C) &= r[-CM(M^*CM)^{\dagger}M^*C + E_M UU^* E_M] \\
&= r[-CM(M^*CM)^{\dagger}M^*C, E_M UU^* E_M] = r[CM, E_M U].
\end{aligned}
$$

Hence, we can find from (2.27), (2.28), (3.40) and (3.14) that

$$
\begin{aligned}
\max_{X\in\mathscr{S}_1} r(AXB - C) &= \max_{U\in\mathbb{C}^{m\times m}} r[CM, E_M U] \\
&= r[CM, E_M] = r(M^*CM) + r(E_M) \\
&= r(N) - r(A) - r(B) - r[A, B^*], \\
\min_{X\in\mathscr{S}_1} r(AXB - C) &= \min_{U\in\mathbb{C}^{m\times m}} r[CM, E_M U] \\
&= r(CM) = r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(A) - r(B),
\end{aligned}
$$

establishing (3.48) and (3.49). Result (b) can be shown similarly. □

COROLLARY 3.5. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m}_{\mathrm{H}}$ be given, and M and N be as given in (3.1). Also assume that (3.21) is feasible, and define*

$$\mathscr{S}_2 = \{ X \in \mathbb{C}^{p\times q} \mid AXB \preccurlyeq C \}. \tag{3.56}$$

Then, the following hold.

(a) *The maximal matrices of AXB and $AXB - C$ subject to $X \in \mathscr{S}_2$ in the Löwner partial ordering are given by*

$$\max_{\succcurlyeq}\{AXB \mid X \in \mathscr{S}_2\} = C - CM(M^*CM)^{\dagger}M^*C, \tag{3.57}$$

$$\max_{\succcurlyeq}\{AXB - C \mid X \in \mathscr{S}_2\} = -CM(M^*CM)^{\dagger}M^*C. \tag{3.58}$$

(b) *The maximal and minimal ranks and partial inertias of AXB and $AXB - C$ subject to $X \in \mathscr{S}_2$ are given by*

$$\max_{X\in\mathscr{S}_2} r(AXB) = \max_{X\in\mathscr{S}_2} i_-(AXB) = r(A) + r(B) - r[A, B^*], \tag{3.59}$$

$$\min_{X\in\mathscr{S}_2} r(AXB) = \min_{X\in\mathscr{S}_2} i_-(AXB) = r(A) + r(B) + i_-(C) - i_-(N), \tag{3.60}$$

$$\max_{X\in\mathscr{S}_2} i_+(AXB) = r(A) + r(B) + i_+(C) - i_+(N), \tag{3.61}$$

$$\min_{X\in\mathscr{S}_2} i_+(AXB) = 0, \tag{3.62}$$

$$\max_{X\in\mathscr{S}_2} r(AXB - C) = r(N) - r(A) - r(B) - r[A, B^*], \tag{3.63}$$

$$\min_{X\in\mathscr{S}_2} r(AXB - C) = r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(A) - r(B). \tag{3.64}$$

In consequence, the following hold.

(c) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $AXB \prec 0$ and $AXB \preccurlyeq C$ if and only if $r[A, B^*] = r(A) + r(B) - m$.*

(d) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $0 \prec AXB \preccurlyeq C$ if and only if $C \succ 0$ and $r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = r(A) + r(B)$.*

(e) *There exists an $X \in \mathbb{C}^{p\times q}$ such that $0 \preccurlyeq AXB \preccurlyeq C$ if and only if $C \succcurlyeq 0$.*

(f) *There always exists an $X \in \mathbb{C}^{p\times q}$ such that $AXB \preccurlyeq 0$ and $AXB \preccurlyeq C$.*

In what follows, we give some consequences of Theorem 3.1 for different choice of C in (1.2).

THEOREM 3.6. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m}_{\mathrm{H}}$ be given, and M be as given in (3.1), and assume that $AXB = C$ is consistent. Then, the following hold.*

(a) *The general solution of* $AXB \succcurlyeq C$ *and the corresponding* AXB *can be written as*

$$X = A^{\dagger}CB^{\dagger} + A^{\dagger}E_{M}UU^{*}E_{M}B^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.65}$$

$$AXB = C + E_{M}UU^{*}E_{M}, \tag{3.66}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $AXB \succ C$ *if and only if* $r(A) = r(B) = m$. *In this case, the general solution* X *of* $AXB \succ C$ *and the corresponding* AXB *can be written as*

$$X = A^{\dagger}CB^{\dagger} + A^{\dagger}UB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.67}$$

$$AXB = C + U, \tag{3.68}$$

where $0 \prec U$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(c) *The general solution* X *of* $AXB \preccurlyeq C$ *and the corresponding* AXB *can be written as*

$$X = A^{\dagger}CB^{\dagger} - A^{\dagger}E_{M}UU^{*}E_{M}B^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.69}$$

$$AXB = C - E_{M}UU^{*}E_{M}, \tag{3.70}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $AXB \prec C$ *if and only if* $r(A) = r(B) = m$. *In this case, the general solution* X *of* $AXB \prec C$ *and the corresponding* AXB *can be written in the following parametric forms*

$$X = A^{\dagger}CB^{\dagger} - A^{\dagger}UB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.71}$$

$$AXB = C - U, \tag{3.72}$$

where $0 \prec U$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

COROLLARY 3.7. *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ *and* $C \in \mathbb{C}^{m\times m}$ *be given, and let* M *be as given in* (3.1). *Then, the following hold.*

(a) *The inequality*

$$AXB \succcurlyeq -CC^{*} \tag{3.73}$$

is always feasible; the general solution X *of* (3.73) *and the corresponding* AXB *can be written as*

$$X = -A^{\dagger}CC^{*}B^{\dagger} + A^{\dagger}C(M^{*}C)^{\dagger}(M^{*}C)C^{*}B^{\dagger} + A^{\dagger}E_{M}UU^{*}E_{M}B^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.74}$$

$$AXB = -CC^{*} + C(M^{*}C)^{\dagger}(M^{*}C)C^{*} + E_{M}UU^{*}E_{M}, \tag{3.75}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \succ -CC^* \tag{3.76}$$

if and only if $r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = m + r[A, B^*]$. *In this case, the general solution* X *of* (3.76) *can be written as* (3.74), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix such that* $r[CC^*M, E_MU] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

(c) *The inequality*

$$AXB \preccurlyeq CC^* \tag{3.77}$$

is always feasible; *the general solution* X *of* (3.77) *and the corresponding* AXB *can be written in the following parametric forms*

$$X = A^{\dagger}CC^*B^{\dagger} - A^{\dagger}C(M^*C)^{\dagger}(M^*C)C^*B^{\dagger} - A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.78}$$

$$AXB = CC^* - C(M^*C)^{\dagger}(M^*C)C^* + E_MUU^*E_M, \tag{3.79}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \prec CC^* \tag{3.80}$$

if and only if $r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = m + r[A, B^*]$. *In this case, the general solution* X *of* (3.80) *can be written as* (3.78), *in which* U *is any matrix such that* $r[CC^*M, E_MU] = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

COROLLARY 3.8. *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ *and* $C \in \mathbb{C}^{m\times m}$ *be given, and let* M *be as given in* (3.1). *Then, the following hold.*

(a) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \succcurlyeq CC^* \tag{3.81}$$

if and only if

$$\mathscr{R}(C) \subseteq \mathscr{R}(A) \ \text{ and } \ \mathscr{R}(C) \subseteq \mathscr{R}(B^*). \tag{3.82}$$

In this case, the general solution X *of* (3.81) *and the corresponding* AXB *can be written as*

$$X = A^{\dagger}CC^*B^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.83}$$

$$AXB = CC^* + E_MUU^*E_M, \tag{3.84}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \succ CC^* \tag{3.85}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution* X *of* (3.85) *can be written as* (3.83), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(E_M U) = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

(c) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \preccurlyeq -CC^* \tag{3.86}$$

if and only if (3.82) *holds. In this case, the general solution* X *of* (3.86) *can be written as*

$$X = -A^{\dagger}CC^*B^{\dagger} - A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.87}$$

$$AXB = -CC^* - E_MUU^*E_M, \tag{3.88}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \prec -CC^* \tag{3.89}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution* X *of* (3.89) *can be written as* (3.87), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(E_M U) = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

COROLLARY 3.9. *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ *be given, and let* M *be as given in* (3.1). *Then, the following hold.*

(a) *The general solution* $X \in \mathbb{C}^{p\times q}$ *of*

$$AXB \succcurlyeq 0 \tag{3.90}$$

and the corresponding AXB *can be written as*

$$X = A^{\dagger}E_MUU^*E_MB^{\dagger} + W - A^{\dagger}AWBB^{\dagger}, \tag{3.91}$$

$$AXB = E_MUU^*E_M, \tag{3.92}$$

where $U \in \mathbb{C}^{m\times m}$ *and* $W \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB \succ 0 \tag{3.93}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution* X *of* (3.93) *can be written as* (3.91), *in which* $U \in \mathbb{C}^{m\times m}$ *is any matrix such that* $r(E_M U) = m$, *and* $W \in \mathbb{C}^{p\times q}$ *is arbitrary.*

We next establish a group of formulas for calculating the ranks and inertias of $AXB-D$ subject to (3.2), and use the results obtained to derive necessary and sufficient conditions for the following two-sided inequality

$$D \succcurlyeq AXB \succcurlyeq C \tag{3.94}$$

and their variations to hold.

COROLLARY 3.10. *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ *and* $C, D \in \mathbb{C}_{\mathrm{H}}^{m}$ *be given, and let* $\mathscr{S}_1$ *be of the forms in* (3.41)*, and define*

$$K_1 = \begin{bmatrix} C & C & C & A & 0 \\ C & C & C & 0 & B^* \\ C & C & C-D & 0 & 0 \\ A^* & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} D & D & A & 0 \\ D & D & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} A & 0 & D \\ 0 & B^* & D \end{bmatrix}. \tag{3.95}$$

Then, the maximal and minimal ranks and partial inertias of $AXB-D$ *subject to* $X \in \mathscr{S}_1$ *are given by*

$$\max_{X\in\mathscr{S}_1} r(AXB-D) = r(K_3) - r[A, B^*], \tag{3.96}$$

$$\min_{X\in\mathscr{S}_1} r(AXB-D) = i_+(K_1) + r(K_3) - r(K_2), \tag{3.97}$$

$$\max_{X\in\mathscr{S}_1} i_+(AXB-D) = i_-(K_2) - r[A, B^*], \tag{3.98}$$

$$\max_{X\in\mathscr{S}_1} i_-(AXB-D) = i_-(K_1) - i_-(K_2), \tag{3.99}$$

$$\min_{X\in\mathscr{S}_1} i_+(AXB-D) = i_+(K_1) - i_+(K_2), \tag{3.100}$$

$$\min_{X\in\mathscr{S}_1} i_-(AXB-D) = r(K_3) - i_-(K_2). \tag{3.101}$$

In consequence, the following hold.

(a) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $AXB \succ D$ *and* $AXB \succcurlyeq C$ *if and only if* $i_-(K_2) = r[A, B^*] + m$.

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $D \succ AXB \succcurlyeq C$ *if and only if* $D \succ C$ *and* $i_-(K_1) = i_-(K_2) + m$.

(c) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $D \succcurlyeq AXB \succcurlyeq C$ *if and only if* $D \succcurlyeq C$ *and* $i_+(K_1) = i_+(K_2)$.

(d) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* $AXB \succcurlyeq C$ *and* $AXB \succcurlyeq D$ *f and only if* $r(K_3) = i_-(K_2)$.

Proof. From (3.17), $AXB-D$ subject to $X \in \mathscr{S}_1$ can be written as

$$\begin{aligned} AXB-D &= C-D-CM(M^*CM)^{\dagger}M^*C + E_MUU^*E_M \\ &= A\widehat{X}B - D + E_MUU^*E_M. \end{aligned} \tag{3.102}$$

where $\widehat{X}$ and $A\widehat{X}B$ are defined in (3.26) and (3.33). Applying (2.29), (3.13), (3.14), (3.53)–(3.55) and elementary matrix operations, we obtain

$$r[E_M, A\widehat{X}B - D] = r[E_M, D] = r(E_M) + r(DM) = r(E_M) + r[DE_A, DF_B] = r\begin{bmatrix} A & 0 & D \\ 0 & B^* & D \end{bmatrix} - r[A, B^*] = r(K_3) - r[A, B^*], \tag{3.103}$$

$$i_\pm \begin{bmatrix} A\widehat{X}B - D & E_M \\ E_M & 0 \end{bmatrix} = i_\pm \begin{bmatrix} -D & E_M \\ E_M & 0 \end{bmatrix} = i_\mp(M^*DM) + r(E_M) = i_\mp(K_2) - r[A, B^*], \tag{3.104}$$

$$\begin{aligned} i_\pm(A\widehat{X}B - D) &= i_\pm[C - D - CM(M^*CM)^\dagger M^*C] \\ &= i_\pm \begin{bmatrix} M^*CM & M^*C \\ CM & C - D \end{bmatrix} - i_\pm(M^*CM) \\ &= i_\pm \begin{bmatrix} C & C & C & A & 0 \\ C & C & C & 0 & B^* \\ C & C & C-D & 0 & 0 \\ A^* & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \end{bmatrix} - i_\pm \begin{bmatrix} C & C & A & 0 \\ C & C & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix} \\ &= i_\pm(K_1) - i_\pm(K_2). \end{aligned} \tag{3.105}$$

Applying (2.64)–(2.69) to (3.102) and simplifying by (3.103)–(3.105), we obtain

$$\begin{aligned} \max_{X \in \mathscr{S}_1} r(AXB - D) &= \max_{U \in \mathbb{C}^{m \times m}} r(A\widehat{X}B - D + E_M UU^* E_M) = r[E_M, A\widehat{X}B - D] \\ &= r(K_3) - r[A, B^*], \\ \min_{X \in \mathscr{S}_1} r(AXB - D) &= \min_{U \in \mathbb{C}^{m \times m}} r(A\widehat{X}B - D + E_M UU^* E_M) \\ &= i_+(A\widehat{X}B - D) + r[E_M, A\widehat{X}B - D] - i_+ \begin{bmatrix} A\widehat{X}B - D & E_M \\ E_M & 0 \end{bmatrix} \\ &= i_+(K_1) + r(K_3) - r(K_2), \\ \max_{X \in \mathscr{S}_1} i_+(AXB - D) &= \max_{U \in \mathbb{C}^{m \times m}} i_+(A\widehat{X}B - D + E_M UU^* E_M) = i_+ \begin{bmatrix} A\widehat{X}B - D & E_M \\ E_M & 0 \end{bmatrix} \\ &= i_-(K_2) - r[A, B^*], \\ \max_{X \in \mathscr{S}_1} i_-(AXB - D) &= \max_{U \in \mathbb{C}^{m \times m}} i_-(A\widehat{X}B - D + E_M UU^* E_M) = i_-(A\widehat{X}B - D) \\ &= i_-(K_1) - i_-(K_2), \\ \min_{X \in \mathscr{S}_1} i_+(AXB - D) &= \min_{U \in \mathbb{C}^{m \times m}} i_+(A\widehat{X}B - D + E_M UU^* E_M) = i_+(A\widehat{X}B - D) \\ &= i_+(K_1) - i_+(K_2), \end{aligned}$$

$$\begin{aligned}\min_{X\in\mathscr{S}_1} i_-(AXB-D) &= \min_{U\in\mathbb{C}^{m\times m}} i_-(A\widehat{X}B-D+E_MUU^*E_M)\\ &= r[E_M, A\widehat{X}B-D]-i_+\begin{bmatrix}A\widehat{X}B-D & E_M\\ E_M & 0\end{bmatrix}\\ &= r(K_3)-i_-(K_2),\end{aligned}$$

as required for (3.96)–(3.101). □

4. General Hermitian solution of $AXA^* \succcurlyeq B\,(\succ B, \preccurlyeq B, \prec B)$ and its properties

The LMIs in (1.3) are the simplest case of all LMIs with symmetric pattern. Due to the importance of matrix inequalities in the Löwner partial ordering, any contribution on this type of LMIs is valuable from both theoretical and practical points of view. Some previous work on solvability and general solutions of (1.3) and their applications in system and control theory were given in [20] by using SVDs of matrices. In a recent paper [24], necessary and sufficient conditions for the LMIs in (1.3) to hold were obtained by using some expansion formulas for the inertia of the matrix function $B-AXA^*$, while general Hermitian solution of $AXA^* \preccurlyeq B$ was established in [28]. In this section, we reconsider (1.3) and give a group of complete conclusions on Hermitian solutions of the LMIs and their algebraic properties.

THEOREM 4.1. *Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}_{\mathrm{H}}^{m}$ be given, and let* $N=\begin{bmatrix}B & A\\ A^* & 0\end{bmatrix}$.

(a) *Then, the following statements are equivalent:*

(i) *There exists an $X\in\mathbb{C}_{\mathrm{H}}^{n}$ such that*

$$AXA^* \succcurlyeq B. \tag{4.1}$$

(ii) $E_ABE_A \preccurlyeq 0$ *and* $\mathscr{R}(E_ABE_A)=\mathscr{R}(E_AB)$.

(iii) $i_-(E_ABE_A)=r(E_AB)$.

(iv) $i_+(N)=r(A)$ *and* $i_-(N)=r[A, B]$.

In this case, the general Hermitian solution X of (4.1) and the corresponding AXA^* can be written in the following parametric forms

$$X=A^\dagger B(A^\dagger)^*-A^\dagger BE_A(E_ABE_A)^\dagger E_AB(A^\dagger)^*+UU^*+W-A^\dagger AWA^\dagger A, \tag{4.2}$$

$$AXA^*=B-BE_A(E_ABE_A)^\dagger E_AB+AUU^*A^*, \tag{4.3}$$

where $U\in\mathbb{C}^{n\times n}$ and $W\in\mathbb{C}_{\mathrm{H}}^{n}$ are arbitrary.

(b) *There exists an $X\in\mathbb{C}_{\mathrm{H}}^{n}$ such that*

$$AXA^* \succ B \tag{4.4}$$

if and only if

$$E_ABE_A \preccurlyeq 0 \quad and \quad r(E_ABE_A) = r(E_A). \tag{4.5}$$

In this case, the general Hermitian solution X of (4.1) *can be written as* (4.2)*, in which U is any matrix such that $r[BE_A,\, AU] = m$, say, $U = I_n$, and $W \in \mathbb{C}^n_{\mathrm{H}}$ is arbitrary.*

(c) [28] *The following statements are equivalent*:

(i) *There exists an $X \in \mathbb{C}^n_{\mathrm{H}}$ such that*

$$AXA^* \preccurlyeq B. \tag{4.6}$$

(ii) $E_ABE_A \succcurlyeq 0$ *and* $\mathscr{R}(E_ABE_A) = \mathscr{R}(E_AB)$.

(iii) $i_+(E_ABE_A) = r(E_AB)$.

(iv) $i_+(N) = r[A,\, B]$ *and* $i_-(N) = r(A)$.

In this case, the general Hermitian solution X of (4.6) *and the corresponding AXA^* can be written in the following parametric forms*

$$X = A^{\dagger}B(A^{\dagger})^* - A^{\dagger}BE_A(E_ABE_A)^{\dagger}E_AB(A^{\dagger})^* - UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.7}$$

$$AXA^* = B - BE_A(E_ABE_A)^{\dagger}E_AB - AUU^*A^*, \tag{4.8}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^n_{\mathrm{H}}$ such that*

$$AXA^* \prec B \tag{4.9}$$

if and only if $E_ABE_A \succcurlyeq 0$ and $r(E_ABE_A) = r(E_A)$. In this case, the general Hermitian solution X of (4.9) *can be written as* (4.7)*, in which U is any matrix such that $r[BE_A,\, AU] = m$, say, $U = I_n$, and $W \in \mathbb{C}^n_{\mathrm{H}}$ is arbitrary.*

Proof. Inequality (4.1) can be relaxed to the following quadratic matrix equation

$$AXA^* = B + YY^*. \tag{4.10}$$

By Lemma 2.3(a), (4.10) is solvable for X if and only if $E_A(B+YY^*) = 0$, that is,

$$E_AYY^* = -E_AB. \tag{4.11}$$

By Lemma 2.1(b), (4.11) is solvable for YY^* if and only if $E_ABE_A \preccurlyeq 0$ and $r(E_ABE_A) = r(E_AB)$, establishing the equivalence of (i) and (ii) in (a). The equivalence of (ii) and (iii) in (a) follows from (2.33) and $i_-(E_ABE_A) \leqslant r(E_ABE_A) \leqslant r(E_AB)$. The equivalence of (iii) and (iv) in (a) follows from (2.33). In this case, the general solution YY^* of (4.11) can be written as

$$YY^* = -BE_A(E_ABE_A)^{\dagger}E_AB + AA^{\dagger}VV^*AA^{\dagger},$$

where V is an arbitrary matrix. Substituting the YY^* into (4.10) gives

$$AXA^* = B - BE_A(E_ABE_A)^{\dagger}E_AB + AA^{\dagger}VV^*AA^{\dagger}. \tag{4.12}$$

By Lemma 2.3(a), the general Hermitian solution X of (4.12) can be written as

$$X = A^{\dagger}B(A^{\dagger})^* - A^{\dagger}BE_A(E_ABE_A)^{\dagger}E_AB(A^{\dagger})^* + A^{\dagger}VV^*(A^{\dagger})^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.13}$$

where $V \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ are arbitrary. Replacing $A^{\dagger}VV^*(A^{\dagger})^*$ with UU^* gives (4.2), which is also the general solution X of (4.1).

It can be seen from (4.3) that (4.4) holds if and only if

$$-BE_A(E_ABE_A)^{\dagger}E_AB + AUU^*A^* \succ 0 \tag{4.14}$$

for some U. Under (ii) in (a), we have

$$\begin{aligned} r[\,-BE_A(E_ABE_A)^{\dagger}E_AB + AUU^*A^*\,] &= r[\,-BE_A(E_ABE_A)^{\dagger}E_AB,\ AUU^*A^*\,] \\ &= r[\,BE_A,\ AU\,]. \end{aligned}$$

Hence,

$$\begin{aligned} \max_U r[\,-BE_A(E_ABE_A)^{\dagger}E_AB + AUU^*A^*\,] &= \max_U r[\,BE_A,\ AU\,] \\ &= r[\,BE_A, A\,] = r(E_ABE_A) + r(A), \end{aligned}$$

so that (4.4) holds if and only if $r(E_ABE_A) + r(A) = m$. Thus (b) follows. Results (c) and (d) can be shown similarly. □

Concerning the constant term in (4.2), we have the consequence.

COROLLARY 4.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^m_{\mathrm{H}}$ be given, and let*

$$\widehat{X} = A^{\dagger}B(A^{\dagger})^* - A^{\dagger}BE_A(E_ABE_A)^{\dagger}E_AB(A^{\dagger})^*. \tag{4.15}$$

Then, the following hold.

(a) *Under the condition that (4.1) is feasible, $\widehat{X}$ is a Hermitian solution of (4.1), and*

$$i_+(\widehat{X}) = i_+(A\widehat{X}A^*) = i_+(B), \tag{4.16}$$

$$i_-(\widehat{X}) = i_-(A\widehat{X}A^*) = r(A) + i_-(B) - r[A, B], \tag{4.17}$$

$$r(\widehat{X}) = r(A\widehat{X}A^*) = r(A) + r(B) - r[A, B], \tag{4.18}$$

$$i_-(\,B - A\widehat{X}A^*\,) = r(\,B - A\widehat{X}A^*\,) = r(B) - r(\,A\widehat{X}A^*\,) = r[A, B] - r(A). \tag{4.19}$$

(b) *Under the condition that (4.6) is feasible, $\widehat{X}$ is a Hermitian solution of (4.6), and*

$$i_+(\widehat{X}) = i_+(A\widehat{X}A^*) = r(A) + i_+(B) - r[A, B], \tag{4.20}$$

$$i_-(\widehat{X}) = i_-(A\widehat{X}A^*) = i_-(B), \tag{4.21}$$

$$r(\widehat{X}) = r(A\widehat{X}A^*) = r(A) + r(B) - r[A, B], \tag{4.22}$$

$$i_+(\,B - A\widehat{X}A^*\,) = r(\,B - A\widehat{X}A^*\,) = r(B) - r(\,A\widehat{X}A^*\,) = r[A, B] - r(A). \tag{4.23}$$

Proof. Under the condition that (4.1) has a solution, set $U = W = 0$ in (4.2), we see that $\widehat{X}$ in (4.15) is a Hermitian solution of $AXA^* \succcurlyeq B$. In this case, applying (2.37) to (4.15) and simplifying by congruence matrix operations, we obtain

$$\begin{aligned} i_{\pm}(\widehat{X}) &= i_{\pm}[A^{\dagger}B(A^{\dagger})^* - A^{\dagger}BE_A(E_ABE_A)^{\dagger}E_AB(A^{\dagger})^*] \\ &= i_{\pm}\begin{bmatrix} E_ABE_A & E_AB(A^{\dagger})^* \\ A^{\dagger}BE_A & A^{\dagger}B(A^{\dagger})^* \end{bmatrix} - i_{\pm}(E_ABE_A) \\ &= i_{\pm}\begin{bmatrix} B & B(A^{\dagger})^* \\ A^{\dagger}B & A^{\dagger}B(A^{\dagger})^* \end{bmatrix} - i_{\pm}(E_ABE_A) \\ &= i_{\pm}\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} - i_{\pm}(E_ABE_A) = i_{\pm}(B) - i_{\pm}(E_ABE_A), \end{aligned} \tag{4.24}$$

$$\begin{aligned} i_{\pm}(A\widehat{X}A^*) &= i_{\pm}[B - BE_A(E_ABE_A)^{\dagger}E_AB] \\ &= i_{\pm}\begin{bmatrix} E_ABE_A & E_AB \\ BE_A & B \end{bmatrix} - i_{\pm}(E_ABE_A) \\ &= i_{\pm}\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} - i_{\pm}(E_ABE_A) \\ &= i_{\pm}(B) - i_{\pm}(E_ABE_A). \end{aligned} \tag{4.25}$$

In consequence,

$$\begin{aligned} i_{+}(\widehat{X}) &= i_{+}(A\widehat{X}A^*) = i_{+}(B), \\ i_{-}(\widehat{X}) &= i_{-}(A\widehat{X}A^*) = i_{-}(B) - i_{-}(E_ABE_A) \\ &= i_{-}(B) - r(E_AB) = i_{-}(B) + r(A) - r[A, B], \end{aligned}$$

establishing (4.16), (4.17) and (4.18). Applying (2.37) and simplifying by congruence matrix operations, we obtain

$$\begin{aligned} i_{\pm}(B - A\widehat{X}A^*) &= i_{\pm}[BE_A(E_ABE_A)^{\dagger}E_AB] \\ &= i_{\pm}\begin{bmatrix} -E_ABE_A & E_AB \\ BE_A & 0 \end{bmatrix} - i_{\mp}(E_ABE_A) \\ &= i_{\pm}\begin{bmatrix} 0 & E_AB \\ BE_A & 0 \end{bmatrix} - i_{\mp}(E_ABE_A) \\ &= r(E_AB) - i_{\mp}(E_ABE_A). \end{aligned} \tag{4.26}$$

In consequence,

$$\begin{aligned} i_{+}(B - A\widehat{X}A^*) &= r(E_AB) - i_{-}(E_ABE_A) = r(E_AB) - r(E_ABE_A) = 0, \\ i_{-}(B - A\widehat{X}A^*) &= r(E_AB) - i_{+}(E_ABE_A) = r[A, B] - r(A), \end{aligned}$$

establishing (4.19). Result (b) can be shown similarly. □

The following two corollaries are derived from Corollaries 3.3 and 3.4.

COROLLARY 4.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_{\mathrm{H}}^{m}$ be given. Then, the following hold.*

(a) *Under the condition that (4.1) is feasible, define*

$$\mathscr{S}_1 = \{X \in \mathbb{C}_{\mathrm{H}}^{n} \mid AXA^* \succcurlyeq B\}. \tag{4.27}$$

Then, the minimal matrices of AXA^* and $AXA^* - B$ subject to $X \in \mathscr{S}_1$ in the Löwner partial ordering are given by

$$\min_{\succcurlyeq}\{AXA^* \mid X \in \mathscr{S}_1\} = B - BE_A(E_ABE_A)^{\dagger}E_AB, \tag{4.28}$$

$$\min_{\succcurlyeq}\{AXA^* - B \mid X \in \mathscr{S}_1\} = -BE_A(E_ABE_A)^{\dagger}E_AB, \tag{4.29}$$

while the maximal and minimal ranks and partial inertias of AXA^* and $AXA^* - B$ subject to $X \in \mathscr{S}_1$ are given by

$$\max_{X\in\mathscr{S}_1} r(AXA^*) = \max_{X\in\mathscr{S}_1} i_{+}(AXA^*) = r(A), \tag{4.30}$$

$$\min_{X\in\mathscr{S}_1} r(AXA^*) = \min_{X\in\mathscr{S}_1} i_{+}(AXA^*) = i_{+}(B), \tag{4.31}$$

$$\max_{X\in\mathscr{S}_1} i_{-}(AXA^*) = r(A) + i_{-}(B) - r[A,\, B], \tag{4.32}$$

$$\min_{X\in\mathscr{S}_1} i_{-}(AXA^*) = 0, \tag{4.33}$$

$$\max_{X\in\mathscr{S}_1} r(AXA^* - B) = r[A,\, B], \tag{4.34}$$

$$\min_{X\in\mathscr{S}_1} r(AXA^* - B) = r[A,\, B] - r(A). \tag{4.35}$$

(b) *Under the condition that (4.6) is feasible, and define*

$$\mathscr{S}_2 = \{X \in \mathbb{C}_{\mathrm{H}}^{n} \mid AXA^* \preccurlyeq B\}. \tag{4.36}$$

Then, the maximal matrices of AXA^* and $AXA^* - B$ subject to $X \in \mathscr{S}_2$ in the Löwner partial ordering are given by

$$\max_{\succcurlyeq}\{AXA^* \mid X \in \mathscr{S}_2\} = B - BE_A(E_ABE_A)^{\dagger}E_AB, \tag{4.37}$$

$$\max_{\succcurlyeq}\{AXA^* - B \mid X \in \mathscr{S}_2\} = -BE_A(E_ABE_A)^{\dagger}E_AB, \tag{4.38}$$

while the maximal and minimal ranks and partial inertias of AXA^* and $AXA^* - B$ subject to $X \in \mathscr{S}_2$ are given by

$$\max_{X\in\mathscr{S}_2} r(AXA^*) = \max_{X\in\mathscr{S}_2} i_{-}(AXA^*) = r(A), \tag{4.39}$$

$$\min_{X\in\mathscr{S}_2} r(AXA^*) = \min_{X\in\mathscr{S}_2} i_{-}(AXA^*) = i_{-}(B), \tag{4.40}$$

$$\max_{X\in\mathscr{S}_2} i_{+}(AXA^*) = r(A) + i_{+}(B) - r[A,\, B], \tag{4.41}$$

$$\min_{X\in\mathscr{S}_2} i_+(AXA^*) = 0, \tag{4.42}$$

$$\max_{X\in\mathscr{S}_2} r(AXA^* - B) = r[A, B], \tag{4.43}$$

$$\min_{X\in\mathscr{S}_2} r(AXA^* - B) = r[A, B] - r(A). \tag{4.44}$$

COROLLARY 4.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^m_{\mathrm{H}}$ be given, and assume that $AXA^* = B$ is consistent. Then, the following hold.*

(a) *The general Hermitian solution X of $AXA^* \succcurlyeq B$ and the corresponding AXA^* can be written as*

$$X = A^\dagger B(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.45}$$

$$AXA^* = B + AUU^*A^*, \tag{4.46}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^n_{\mathrm{H}}$ such that $AXA^* \succ B$ if and only if $r(A) = m$. In this case, the general Hermitian solution X of $AXA^* \succ B$ and the corresponding AXA^* can be written as*

$$X = A^\dagger B(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.47}$$

$$AXA^* = B + AUU^*A^*, \tag{4.48}$$

where $U \in \mathbb{C}^{n\times n}$ is any matrix such that $r(AU) = m$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ is arbitrary.

(c) *The general Hermitian solution X of $AXA^* \preccurlyeq B$ and the corresponding AXA^* can be written as*

$$X = A^\dagger B(A^\dagger)^* - UU^* + W - A^\dagger AWA^\dagger A, \tag{4.49}$$

$$AXA^* = B - AUU^*A^*, \tag{4.50}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^n_{\mathrm{H}}$ such that $AXA^* \prec B$ if and only if $r(A) = m$. In this case, the general Hermitian solution X of $AXA^* \prec B$ and the corresponding AXA^* can be written as*

$$X = A^\dagger B(A^\dagger)^* - UU^* + W - A^\dagger AWA^\dagger A, \tag{4.51}$$

$$AXA^* = B - AUU^*A^*, \tag{4.52}$$

where $U \in \mathbb{C}^{n\times n}$ is any matrix such that $r(AU) = m$ and and $W \in \mathbb{C}^n_{\mathrm{H}}$ is arbitrary.

THEOREM 4.5. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times m}$ be given. Then, the following hold.*

(a) *The inequality*

$$AXA^* \succcurlyeq -BB^* \tag{4.53}$$

is always feasible; the general Hermitian solution X of (4.53) and the corresponding AXA^* can be written in the following parametric forms

$$X = A^\dagger B(E_AB)^\dagger(E_AB)B^*(A^\dagger)^* - A^\dagger BB^*(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.54}$$

$$AXA^* = B(E_AB)^\dagger(E_AB)B^* - BB^* + AUU^*A^*, \tag{4.55}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}_{\mathrm{H}}^{n}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXA^* \succ -BB^* \tag{4.56}$$

if and only if $r[A, B] = m$. In this case, the general Hermitian solution X of (4.56) can be written as (4.54), in which U is any matrix such that $r(AU) = r(A)$, say, $U = I_n$, and $W \in \mathbb{C}_{\mathrm{H}}^{n}$ is arbitrary.

(c) *The inequality*

$$AXA^* \preccurlyeq BB^* \tag{4.57}$$

is always feasible; the general Hermitian solution X of (4.57) and the corresponding AXA^* can be written in the following parametric forms

$$X = A^\dagger BB^*(A^\dagger)^* - A^\dagger B(E_AB)^\dagger(E_AB)B^*(A^\dagger)^* - UU^* + W - A^\dagger AWA^\dagger A, \tag{4.58}$$

$$AXA^* = BB^* - B(E_AB)^\dagger(E_AB)B^* - AUU^*A^*, \tag{4.59}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}_{\mathrm{H}}^{n}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXA^* \prec BB^* \tag{4.60}$$

if and only if $r[A, B] = m$. In this case, the general Hermitian solution X of (4.60) can be written as (4.58), in which U is any matrix such that $r(AU) = r(A)$, say, $U = I_n$, and $W \in \mathbb{C}_{\mathrm{H}}^{n}$ is arbitrary.

COROLLARY 4.6. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times m}$ be given. Then, the following hold.*

(a) *There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXA^* \succcurlyeq BB^* \tag{4.61}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general Hermitian solution X of (4.61) and the corresponding AXA^* can be written in the following parametric forms

$$X = A^\dagger BB^*(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.62}$$

$$AXA^* = BB^* + AUU^*A^*, \tag{4.63}$$

where $U \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXA^* \succ BB^* \tag{4.64}$$

if and only if $r(A) = m$*. In this case, the general Hermitian solution* X *of* (4.64) *can be written as* (4.62)*, in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix such that* $r(AU) = m$*, and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *is arbitrary.*

(c) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that*

$$AXA^* \preccurlyeq -BB^* \tag{4.65}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$*. In this case, the general Hermitian solution* X *of* (4.65) *and the corresponding* AXA^* *can be written in the following parametric forms*

$$X = -A^{\dagger}BB^*(A^{\dagger})^* - UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.66}$$

$$AXA^* = -BB^* - AUU^*A^*, \tag{4.67}$$

where $U \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that*

$$AXA^* \prec -BB^* \tag{4.68}$$

if and only if $r(A) = m$*. In this case, the general Hermitian solution* X *of* (4.68) *can be written as* (4.66)*, in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix such that* $r(AU) = m$*, and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *is arbitrary.*

COROLLARY 4.7. ([24]) *Let* $A \in \mathbb{C}^{m\times n}$ *be given. Then, the following hold.*

(a) *The general solution* X *of* $AXA^* \succcurlyeq 0$ *and the corresponding* AXA^* *can be written in the following parametric forms*

$$X = UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.69}$$

$$AXA^* = AUU^*A^*, \tag{4.70}$$

where $U \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $AXA^* \succ 0$ *if and only if* $r(A) = m$*. In this case, the general Hermitian solution* X *of* $AXA^* \succ 0$ *can be written as* (4.69)*, in which* $U \in \mathbb{C}^{n\times n}$ *with* $r(AU) = m$ *and* $W \in \mathbb{C}_{\mathrm{H}}^{n}$ *are arbitrary.*

(c) *The general Hermitian solution X of $AXA^* \preccurlyeq 0$ and the corresponding AXA^* can be written in the following parametric forms*

$$X = -UU^* + W - A^\dagger AWA^\dagger A, \tag{4.71}$$

$$AXA^* = -AUU^*A^*, \tag{4.72}$$

where $U \in \mathbb{C}^{n\times n}$ and $W \in \mathbb{C}^n_{\mathrm{H}}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^n_{\mathrm{H}}$ such that $AXA^* \prec 0$ if and only if $r(A) = m$. In this case, the general Hermitian solution X of $AXA^* \prec 0$ can be written as (4.71), in which $U \in \mathbb{C}^{n\times n}$ is any matrix such that $r(AU) = m$, and $W \in \mathbb{C}^n_{\mathrm{H}}$ is arbitrary.*

THEOREM 4.8. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times m}$ be given. Then, the following hold.*

(a) *There exists an $X \in \mathbb{C}^{m\times m}$ such that*

$$AXX^*A^* \succcurlyeq BB^* \tag{4.73}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general solution XX^* of (4.73) and the corresponding AXX^*A^* can be written in the following parametric forms

$$XX^* = [A^\dagger(BB^* + AUU^*A^*)^{1/2} + F_AW][A^\dagger(BB^* + AUU^*A^*)^{1/2} + F_AW]^*, \tag{4.74}$$

$$AXX^*A^* = BB^* + AUU^*A^*, \tag{4.75}$$

where $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{m\times m}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{m\times m}$ such that*

$$AXX^*A^* \succ BB^* \tag{4.76}$$

if and only if $r(A) = m$. In this case, the general solution XX^* of (4.76) can be written as

$$XX^* = A^{-1}(BB^* + UU^*)A^{-1}, \tag{4.77}$$

$$AXX^*A^* = BB^* + UU^*, \tag{4.78}$$

where $U \in \mathbb{C}^{m\times m}$ is any matrix with $r(U) = m$.

Proof. The solvability condition of (4.73) follows from Corollary 4.6(a). In this case, (4.73) is equivalent to

$$AXX^*A^* = BB^* + AUU^*A^*$$

by (4.63), and its general solution is (4.74) by Lemma 2.3(c). □

An application to partitioned matrices is given below.

COROLLARY 4.9. *Let*

$$\phi(X) = \begin{bmatrix} AXA^* & B \\ B^* & CC^* \end{bmatrix}, \tag{4.79}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ *and* $C \in \mathbb{C}^{p\times p}$ *are given. Then, the following hold.*

(a) *There exists an* $X \in \mathbb{C}_{\rm H}^{n}$ *such that* $\phi(X) \succcurlyeq 0$ *if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \ and \ \ \mathscr{R}(B^*) \subseteq \mathscr{R}(C). \tag{4.80}$$

In this case, the general solution X *of* $\phi(X) \succcurlyeq 0$ *can be written in the following parametric form*

$$X = A^{\dagger}B(CC^*)^{\dagger}B^*(A^{\dagger})^* + UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.81}$$

where $U \in \mathbb{C}^{n\times n}$ *and* $W \in \mathbb{C}_{\rm H}^{n}$ *are arbitrary.*

(b) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that* $\phi(X) \succ 0$ *in* (4.79) *if and only if*

$$r(A) = m \ \ and \ \ r(C) = p. \tag{4.82}$$

In this case, the general solution X *of* $\phi(X) \succ 0$ *can be written in the following parametric form*

$$X = A^{\dagger}B(CC^*)^{-1}B^*(A^{\dagger})^* + UU^* + W - A^{\dagger}AWA^{\dagger}A, \tag{4.83}$$

where $U \in \mathbb{C}^{n\times n}$ *is any matrix such* $r(AU) = m$ *and* $W \in \mathbb{C}_{\rm H}^{n}$ *is arbitrary.*

Proof. It is easily seen from Lemma 2.9(e) and (f) that

$$\phi(X) \succcurlyeq 0 \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A), \ \ \mathscr{R}(B^*) \subseteq \mathscr{R}(C) \ \text{ and } \ AXA^* \succcurlyeq B(CC^*)^{+}B^*, \tag{4.84}$$

$$\phi(X) \succ 0 \Leftrightarrow r(A) = m, \ \ r(C) = p \ \text{ and } \ AXA^* \succ B(CC^*)^{+}B^*. \tag{4.85}$$

Solving the two inequalities in (4.84) and (4.85) by Theorem 4.1 leads to (a) and (b). □

We next solve $AXX^*A^* \preccurlyeq BB^*$. It is obvious that the inequality has a trivial solution $X = 0$. However, the inequality may only have zero solution in some cases. For example, the inequality

$$\begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} \preccurlyeq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

only has a solution $x = 0$.

THEOREM 4.10. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times m}$ be given. Then, the following hold.*

(a) *There exists an $X \in \mathbb{C}^{n\times n}$ such that both $AX \neq 0$ and*

$$AXX^*A^* \preccurlyeq BB^* \tag{4.86}$$

if and only if

$$\mathscr{R}(A) \cap \mathscr{R}(B) \neq \{0\}. \tag{4.87}$$

In this case, a solution XX^* of (4.86) and the corresponding AXX^*A^* can be written in the following parametric forms

$$XX^* = [A^\dagger(BF_{B_1}VF_{B_1}B^*)^{1/2} + F_AW][A^\dagger(BF_{B_1}VF_{B_1}B^*)^{1/2} + F_AW]^*, \tag{4.88}$$
$$AXX^*A^* = BF_{B_1}VF_{B_1}B^*, \tag{4.89}$$

where $B_1 = E_AB$, V is any matrix satisfying $0 \prec V \preccurlyeq I_m$, and $W \in \mathbb{C}^{n\times m}$ is arbitrary. The rank of (4.89) is

$$\max_{AXX^*A^* \preccurlyeq BB^*} r(AXX^*A^*) = r(A) + r(B) - r[A,\, B]. \tag{4.90}$$

(b) *There exists an $X \in \mathbb{C}^{n\times n}$ such that $AX \neq 0$ and*

$$AXX^*A^* \prec BB^* \tag{4.91}$$

if and only if

$$A \neq 0 \ \text{ and } \ r(B) = m. \tag{4.92}$$

In this case, a solution XX^* of (4.91) can be written as (4.88), in which V is any matrix satisfying $0 \prec V \prec I_m$, and $W \in \mathbb{C}^{n\times m}$ is arbitrary.

(c) *Under the condition $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, there always exists an $X \in \mathbb{C}^{n\times n}$ such that both $AX \neq 0$ and*

$$AXX^*A^* \preccurlyeq BB^*, \tag{4.93}$$

and a solution XX^* of (4.93) and the corresponding AXX^*A^* can be written in the following parametric forms

$$XX^* = [A^\dagger(BVB^*)^{1/2} + F_AW][A^\dagger(BVB^*)^{1/2} + F_AW]^*, \tag{4.94}$$
$$AXX^*A^* = BVB^*, \tag{4.95}$$

where V is any matrix satisfying $0 \prec V \preccurlyeq I_m$, and $W \in \mathbb{C}^{n\times m}$ is arbitrary.

(d) *Under the condition $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, there exists an $X \in \mathbb{C}^{n\times n}$ such that $AX \neq 0$ and*

$$AXX^*A^* \prec BB^* \tag{4.96}$$

if and only if $r(B) = m$. In this case, a solution XX^* of (4.96) can be written as (4.94), in which V is any matrix satisfying $0 \prec V \prec I_m$, and $W \in \mathbb{C}^{n\times m}$ is arbitrary.

Proof. It can be seen from Lemma 2.9(g) that if there exists an X such that $AX \neq 0$ and (4.86) hold, then $\mathscr{R}(AX) \subseteq \mathscr{R}(B)$, which obviously implies that (4.87) holds. On the other hand, it can be derived from $E_ABF_{E_AB} = 0$ that

$$AA^{\dagger}BF_{E_AB} = BF_{E_AB}, \tag{4.97}$$

and from (2.27) and (2.28) that

$$r(BF_{E_AB}) = r\begin{bmatrix} B \\ E_AB \end{bmatrix} - r(E_AB) = r(A) + r(B) - r[A,\, B] = \dim[\mathscr{R}(A) \cap \mathscr{R}(B)]. \tag{4.98}$$

Hence if (4.87) holds, then $BF_{E_AB} \neq 0$ and $\mathscr{R}(BF_{E_AB}) = \mathscr{R}(A) \cap \mathscr{R}(B)$ by (4.97) and (4.98). In this case,

$$AA^{\dagger}BF_{E_AB}VF_{E_AB}B^*(A^{\dagger})^*A = BF_{E_AB}VF_{E_AB}B^*.$$

Thus we can derive from (4.88) and Lemma 2.10(c) that

$$BB^* - AXX^*A^* = BB^* - BF_{E_AB}VF_{E_AB}B^* = B(I_m - F_{E_AB}VF_{E_AB})B^* \succcurlyeq 0,$$

that is, (4.88) is a solution of (4.86). The two conditions in (4.92) are obvious under the condition that both $AX \neq 0$ and (4.91) hold. Conversely, if (4.92) holds, we can derive from (4.91) and Lemma 2.10(d) that $I_m - F_{E_AB}VF_{E_AB} \succ 0$ and

$$BB^* - AXX^*A^* = BB^* - BF_{E_AB}VF_{E_AB}B^* = B(I_m - F_{E_AB}VF_{E_AB})B^* \succ 0.$$

Results (c) and (d) are direct consequences of (a) and (b). □

A direct consequence of Corollary 3.10 is given below.

COROLLARY 4.11. *Let $A \in \mathbb{C}^{m\times n}$ and $B,\, C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and let $\mathscr{S}_1$ be of the form in (4.27), and define*

$$K_1 = \begin{bmatrix} B & B & A \\ B & B-C & 0 \\ A^* & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}. \tag{4.99}$$

Then, the maximal and minimal ranks and partial inertias of $AXA^* - C$ subject to $X \in \mathscr{S}_1$ are given by

$$\max_{X\in\mathscr{S}_1} r(AXA^* - C) = r[A,\, C], \tag{4.100}$$

$$\min_{X\in\mathscr{S}_1} r(AXA^* - C) = i_+(K_1) - r(K_2) + r[A,\, C], \tag{4.101}$$

$$\max_{X\in\mathscr{S}_1} i_+(AXA^* - C) = i_-(K_2), \tag{4.102}$$

$$\max_{X\in\mathscr{S}_1} i_-(AXA^* - C) = i_-(K_1) - i_-(K_2), \tag{4.103}$$

$$\min_{X\in\mathscr{S}_1} i_+(AXA^* - C) = i_+(K_1) - i_+(K_2), \tag{4.104}$$

$$\min_{X\in\mathscr{S}_1} i_-(AXA^* - C) = r[A,\, C] - i_-(K_2). \tag{4.105}$$

In consequence, the following hold.

(a) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $AXA^* \succcurlyeq B$ *and* $AXA^* \succ C$ *if and only if* $i_-(K_2) = m$.

(b) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $C \succ AXA^* \succcurlyeq B$ *if and only if* $C \succ B$ *and* $i_-(K_1) = i_-(K_2) + m$.

(c) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $C \succcurlyeq AXA^* \succcurlyeq B$ *if and only if* $C \succcurlyeq B$ *and* $i_+(K_1) = i_+(K_2)$.

(d) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{n}$ *such that* $AXA^* \succcurlyeq B$ *and* $AXA^* \succcurlyeq C$ *f and only if* $i_-(K_2) = r[A, C]$.

5. General solution of $AX + (AX)^* \succcurlyeq B\,(\succ B, \preccurlyeq B, \prec B)$ and its properties

The inequality in (1.4) was approached in [25] by using a relaxation method and their general solutions were given analytically. In this section, we reconsider this inequality and give some new conclusions on algebraic properties of its solutions.

THEOREM 5.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_{\mathrm{H}}^{m}$ *be given, and let* $M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}$. *Then, the following hold.*

(a) [25] *The following statements are equivalent*:

(i) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* \succcurlyeq B. \tag{5.1}$$

(ii) $E_ABE_A \preccurlyeq 0$.

(iii) $i_+(M) = r(A)$.

In this case, the general solution X *of* (5.1) *and the corresponding* $AX + (AX)^*$ *can be written in the following parametric forms*

$$X = \frac{1}{2}A^{\dagger}B\widehat{A} + \frac{1}{2}A^{\dagger}(AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*\widehat{A} + VA^* + F_AW, \tag{5.2}$$

$$AX + (AX)^* = B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*, \tag{5.3}$$

where $J = -E_ABE_A$, $\widehat{A} = 2I_m - AA^{\dagger}$, *and* U, $W \in \mathbb{C}^{n\times m}$ *and* $V \in \mathbb{C}_{\mathrm{SH}}^{n}$ *are arbitrary.*

(b) [25] *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* \succ B \tag{5.4}$$

if and only if

$$E_ABE_A \preccurlyeq 0 \ \text{and} \ \mathscr{R}(E_ABE_A) = \mathscr{R}(E_A), \tag{5.5}$$

or equivalently, $i_-(M) = m$. *In this case, the general solution* X *of* (5.4) *can be written as* (5.2), *in which* U *is any matrix such that* $r(AU + J^{\frac{1}{2}}) = m$, *say,* $U = A^*$, $V \in \mathbb{C}_{\mathrm{SH}}^{n}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *Under* (a), *let*

$$\mathscr{S}_1 = \{ X \in \mathbb{C}^{n\times m} \mid AX + (AX)^* \succcurlyeq B \}. \tag{5.6}$$

Then, the maximal and minimal ranks and partial inertias of $AX + (AX)^*$ *and* $AX + (AX)^* - B$ *subject to* $X \in \mathscr{S}_1$ *are given by*

$$\max_{X\in\mathscr{S}_1} r[AX + (AX)^*] = \min\{2r(A),\ \ r[A, B]\}, \tag{5.7}$$

$$\begin{aligned}\min_{X\in\mathscr{S}_1} r[AX + (AX)^*] = \max\{ & 2r(A) + 2r[A, B] - 2r(M),\ \ r(B) - m, \\ & i_+(B) + r(A) + r[A, B] - r(M), \\ & i_-(B) + r(A) + r[A, B] - r(M) - m\},\end{aligned} \tag{5.8}$$

$$\max_{X\in\mathscr{S}_1} i_+[AX + (AX)^*] = r(A), \tag{5.9}$$

$$\max_{X\in\mathscr{S}_1} i_-[AX + (AX)^*] = \min\{r(A),\ \ i_-(B)\}, \tag{5.10}$$

$$\min_{X\in\mathscr{S}_1} i_+[AX + (AX)^*] = \max\{r[A, B] + r(A) - r(M),\ \ i_+(B)\}, \tag{5.11}$$

$$\min_{X\in\mathscr{S}_1} i_-[AX + (AX)^*] = r[A, B] + r(A) - r(M), \tag{5.12}$$

$$\max_{X\in\mathscr{S}_1} r[AX + (AX)^* - B] = r(M) - r(A), \tag{5.13}$$

$$\min_{X\in\mathscr{S}_1} r[AX + (AX)^* - B] = r(M) - 2r(A). \tag{5.14}$$

In consequence, the following hold.

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $AX + (AX)^* \succ 0$ *and* $AX + (AX)^* \succcurlyeq B$ *if and only if* $r(A) = m$.

(e) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $0 \succ AX + (AX)^* \succcurlyeq B$ *if and only if* $r(A) = m$ *and* $B \prec 0$.

(f) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $AX + (AX)^* \succcurlyeq 0$ *and* $AX + (AX)^* \succcurlyeq B$ *if and only if* $r(N) = r[A, B] + r(A)$.

(g) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $0 \succcurlyeq AX + (AX)^* \succcurlyeq B$ *if and only if* $B \preccurlyeq 0$.

Proof. Inequality (5.1) can be relaxed to the following quadratic matrix equation

$$AX + (AX)^* = B + YY^*, \tag{5.15}$$

where $Y \in \mathbb{C}^{m\times m}$. From Lemma 2.4(a), there exists an X that satisfies (5.15) if and only if YY^* satisfies $E_A(B + YY^*)E_A = 0$, that is,

$$E_AYY^*E_A = -E_ABE_A = J. \tag{5.16}$$

Further by Lemma 2.3(c), there exists a YY^* that satisfies (5.16) if and only if (ii) of (a) holds, in which case, the general solution of (5.16) can be written as

$$YY^* = (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*, \tag{5.17}$$

where $U \in \mathbb{C}^{n\times m}$ is arbitrary. Substituting this YY^* into (5.15) gives

$$AX + (AX)^* = B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*. \tag{5.18}$$

Applying Lemma 2.4(a) to this equation, we obtain (5.2).

Setting (5.13) equal to m gives $r(M) - r(A) = m$, i.e., $r(E_ABE_A) = r(E_A)$ by (2.33), which is further equivalent to (5.5). The equivalence of $i_-(M) = m$ and (5.5) follows from (2.33) and $i_-(E_ABE_A) \leqslant r(E_ABE_A) \leqslant r(E_A)$.

Applying Lemma 2.13(a) to (5.18), we obtain

$$\max_{U\in\mathbb{C}^{n\times m}} r[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] = \min\left\{ r[A, B, J^{\frac{1}{2}}],\ r\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix},\ r(B) + m \right\}, \tag{5.19}$$

$$\min_{U\in\mathbb{C}^{n\times m}} r[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] = 2r[A, B, J^{\frac{1}{2}}] + \max\{h_1,\ h_2,\ h_3,\ h_4\}, \tag{5.20}$$

$$\max_{U\in\mathbb{C}^{n\times m}} i_+[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] = \min\left\{ i_+\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix},\ i_+(B) + m \right\}, \tag{5.21}$$

$$\max_{U\in\mathbb{C}^{n\times m}} i_-[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] = \min\left\{ i_-\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix},\ i_-(B) \right\}, \tag{5.22}$$

$$\begin{aligned} &\min_{U\in\mathbb{C}^{n\times m}} i_+[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] \\ &= r[A, B, J^{\frac{1}{2}}] + \max\left\{ i_+\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix},\ i_+(B) - r[A, B] \right\}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} &\min_{U\in\mathbb{C}^{n\times m}} i_-[B + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*] \\ &= r[A, B, J^{\frac{1}{2}}] + \max\left\{ i_-\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix},\ i_-(B) - r[A, B] - m \right\}, \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} h_1 &= r\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} - 2r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix}, \\ h_2 &= r(B) - 2r[A, B] - m, \\ h_3 &= i_-\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} + i_+(B) - r[A, B], \\ h_4 &= i_+\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} + i_-(B) - r[A, B] - m. \end{aligned}$$

Simplifying the ranks and partial inertias of the block matrices in (5.19)–(5.24) gives

$$r[A, B, J^{\frac{1}{2}}] = r[A, B, J] = r[A, B, E_ABE_A] = r[A, B], \tag{5.25}$$

$$i_\pm\begin{bmatrix} B+J & A \\ A^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} B - E_ABE_A & A \\ A^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = r(A), \tag{5.26}$$

$$r\begin{bmatrix} B & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B & A & J \\ A^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B & A & E_ABE_A \\ A^* & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} B & A & BE_A \\ A^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B & A & B \\ A^* & 0 & 0 \\ 0 & 0 & A^* \end{bmatrix} - r(A)$$
$$= r\begin{bmatrix} B & A & 0 \\ A^* & 0 & 0 \\ 0 & 0 & A^* \end{bmatrix} - r(A) = r\begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}. \tag{5.27}$$

Substituting (5.25)–(5.27) into (5.19)–(5.24) gives (5.7)–(5.12). It can be seen from (5.18) that

$$r[AX + (AX)^* - B] = r(AU + J^{\frac{1}{2}}). \tag{5.28}$$

Hence, we derive from (2.43) that

$$\begin{aligned} \max r[AX + (AX)^* - B] &= \max_U r(AU + J^{\frac{1}{2}}) = r[A, J^{\frac{1}{2}}] \\ &= r[A, E_ABE_A] = r(A) + r(E_ABE_A) \ \text{(by (2.27))} \\ \min r[AX + (AX)^* - B] &= \min_U r(AU + J^{\frac{1}{2}}) = r[A, J^{\frac{1}{2}}] - r(A) \\ &= r[A, E_ABE_A] - r(A) = r(E_ABE_A) \ \text{(by (2.27))}, \end{aligned}$$

establishing (5.13) and (5.14). □

The following results can be shown similarly.

THEOREM 5.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B,\ C \in \mathbb{C}^m_{\mathrm{H}}$ be given, and let $M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}$. Then, the following hold.*

(a) [25] *The following statements are equivalent*:

(i) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \preccurlyeq B. \tag{5.29}$$

(ii) $E_ABE_A \succcurlyeq 0$.

(iii) $i_-(M) = r(A)$.

In this case, the general solution X of (5.29) and the corresponding $AX + (AX)^*$ can be written in the following parametric forms

$$X = \frac{1}{2}A^\dagger B\widehat{A} - \frac{1}{2}A^\dagger(AU + K^{\frac{1}{2}})(AU + K^{\frac{1}{2}})^*\widehat{A} + VA^* + F_AW, \tag{5.30}$$
$$AX + (AX)^* = B - (AU + K^{\frac{1}{2}})(AU + K^{\frac{1}{2}})^*, \tag{5.31}$$

where $K = E_ABE_A$, $\widehat{A} = 2I_m - AA^\dagger$, $U, W \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}^n_{\mathrm{SH}}$ are arbitrary.

(b) [25] *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* \prec B \tag{5.32}$$

if and only if

$$E_A B E_A \succcurlyeq 0 \ \ and \ \ \mathscr{R}(E_A B E_A) = \mathscr{R}(E_A), \tag{5.33}$$

or equivalently, $i_+(M) = m$. *In this case, the general solution* X *of* (5.32) *can be written as* (5.30)*, in which* U *is any matrix such that* $r(AU + K^{\frac{1}{2}}) = m$, *say,* $U = A^*$, $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *Under* (a)*, let*

$$\mathscr{S}_2 = \{ X \in \mathbb{C}^{n\times m} \mid AX + (AX)^* \preccurlyeq B \}. \tag{5.34}$$

Then,

$$\max_{X\in\mathscr{S}_2} r[AX + (AX)^*] = \min\{2r(A), \ \ r[A, B]\}, \tag{5.35}$$

$$\min_{X\in\mathscr{S}_2} r[AX + (AX)^*] = \max\{2r(A) + 2r[A, B] - 2r(N), \ \ r(B) - m,$$
$$i_+(B) + r(A) + r[A, B] - r(N) - m,$$
$$i_-(B) + r(A) + r[A, B] - r(N)\}, \tag{5.36}$$

$$\max_{X\in\mathscr{S}_2} i_+[AX + (AX)^*] = \min\{r(A), \ \ i_+(B)\}, \tag{5.37}$$

$$\max_{X\in\mathscr{S}_2} i_-[AX + (AX)^*] = r(A), \tag{5.38}$$

$$\min_{X\in\mathscr{S}_2} i_+[AX + (AX)^*] = r[A, B] + r(A) - r(N), \tag{5.39}$$

$$\min_{X\in\mathscr{S}_2} i_-[AX + (AX)^*] = \max\{r[A, B] + r(A) - r(N), \ \ i_-(B)\}, \tag{5.40}$$

$$\max_{X\in\mathscr{S}_2} r[AX + (AX)^* - B] = r(N) - r(A), \tag{5.41}$$

$$\min_{X\in\mathscr{S}_2} r[AX + (AX)^* - B] = r(N) - 2r(A). \tag{5.42}$$

In consequence, the following hold.

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $0 \prec AX + (AX)^* \preccurlyeq B$ *if and only if* $r(A) = m$ *and* $B \succ 0$.

(e) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $AX + (AX)^* \prec 0$ *and* $AX + (AX)^* \preccurlyeq B$ *if and only if* $r(A) = m$.

(f) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $AX + (AX)^* \preccurlyeq 0$ *and* $AX + (AX)^* \preccurlyeq B$ *if and only if* $r(N) = r[A, B] + r(A)$.

(g) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $0 \preccurlyeq AX + (AX)^* \preccurlyeq B$ *if and only if* $B \succcurlyeq 0$.

Theorem 5.1 established identifying conditions for the LMI in (1.4) to be solvable, and gave general expression of the matrix X satisfying (1.4). In particular, the general solutions in (5.2) and (5.30) are represented in closed-form by using generalized inverses of the given matrices and arbitrary matrices. Hence, they can be directly used to deal with various problems on the inequality in (1.4) and its properties. In what follows, we present some consequences of Theorem 5.1 when A and B satisfy some more conditions.

COROLLARY 5.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_{\rm H}^{m}$ be given, and assume that there exists an $X \in \mathbb{C}^{n\times m}$ such that $AX + (AX)^* = B$. Then, the following hold.*

(a) *The general solution $X \in \mathbb{C}^{n\times m}$ of*

$$AX + (AX)^* \succcurlyeq B \tag{5.43}$$

and the corresponding $AX + (AX)^*$ can be written in the following parametric forms

$$X = \frac{1}{2}A^{\dagger}B(\,2I_m - AA^{\dagger}\,) + UU^*A^* + VA^* + F_AW, \tag{5.44}$$

$$AX + (AX)^* = B + 2AUU^*A^*, \tag{5.45}$$

where $U \in \mathbb{C}^{n\times n}$, $W \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}_{\rm SH}^{n}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \succ B \tag{5.46}$$

if and only if $r(A) = m$. In this case, the general solution X of (5.46) can be written as (5.44), in which U is any matrix such that $r(AU) = m$, and $W \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}_{\rm SH}^{n}$ are arbitrary.

(c) *The general solution $X \in \mathbb{C}^{n\times m}$ of*

$$AX + (AX)^* \preccurlyeq B \tag{5.47}$$

and the corresponding $AX + (AX)^*$ can be written in the following parametric forms

$$X = \frac{1}{2}A^{\dagger}B(\,2I_m - AA^{\dagger}\,) - UU^*A^* + VA^* + F_AW, \tag{5.48}$$

$$AX + (AX)^* = B - 2AUU^*A^*, \tag{5.49}$$

where $U \in \mathbb{C}^{n\times n}$, $W \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}_{\rm SH}^{n}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \prec B \tag{5.50}$$

if and only if $r(A) = m$. In this case, the general solution X of (5.50) can be written as (5.48), in which U is any matrix such that $r(AU) = m$, and $W \in \mathbb{C}^{n\times m}$ and $V \in \mathbb{C}_{\rm SH}^{n}$ are arbitrary.

COROLLARY 5.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then, the following hold.*

(a) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \succcurlyeq BB^* \tag{5.51}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general solution X of (5.51) and the corresponding $AX + (AX)^*$ can be written as

$$X = \frac{1}{2}A^{\dagger}BB^* + UU^*A^* + VA^* + F_AW, \tag{5.52}$$

$$AX + (AX)^* = BB^* + 2AUU^*A^*, \tag{5.53}$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^n_{\mathrm{SH}}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \succ BB^* \tag{5.54}$$

if and only if both $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $r(A) = m$. In this case, the general solution X of (5.54) can be written as (5.52), in which U is any matrix with $r(AU) = m$, and $V \in \mathbb{C}^n_{\mathrm{SH}}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(c) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \preccurlyeq -BB^* \tag{5.55}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$. In this case, the general solution X of (5.55) and the corresponding $AX + (AX)^*$ can be written as

$$X = -\frac{1}{2}A^{\dagger}BB^* - UU^*A^* + VA^* + F_AW, \tag{5.56}$$

$$AX + (AX)^* = -BB^* - 2AUU^*A^*, \tag{5.57}$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^n_{\mathrm{SH}}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(d) *There exists an $X \in \mathbb{C}^{n\times m}$ such that*

$$AX + (AX)^* \prec -BB^* \tag{5.58}$$

if and only if both $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $r(A) = m$. In this case, the general solution X of (5.58) can be written as (5.56), in which U is any matrix with $r(AU) = m$, and $V \in \mathbb{C}^n_{\mathrm{SH}}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

COROLLARY 5.5. *Let $B \in \mathbb{C}^{n\times n}$ be given. Then, the following hold.*

(a) *The general solution $X \in \mathbb{C}^{n\times n}$ of*

$$X + X^* \succcurlyeq BB^* \tag{5.59}$$

and the corresponding $X + X^*$ can be written as

$$X = \frac{1}{2}BB^* + UU^* + V - V^*, \tag{5.60}$$

$$X + X^* = BB^* + 2UU^*, \tag{5.61}$$

where $U,\ V \in \mathbb{C}^{n\times n}$ are arbitrary.

(b) *The general solution $X \in \mathbb{C}^{n\times n}$ of*

$$X + X^* \succ BB^* \tag{5.62}$$

can be written as (5.60), in which $U,\ V \in \mathbb{C}^{n\times n}$ are arbitrary with $r(U) = n$.

(c) *The general solution $X \in \mathbb{C}^{n\times n}$ of*

$$X + X^* \preccurlyeq BB^* \tag{5.63}$$

and the corresponding $X + X^*$ can be written as

$$X = \frac{1}{2}BB^* - UU^* + V - V^*, \tag{5.64}$$

$$X + X^* = BB^* - 2UU^*, \tag{5.65}$$

where $U,\ V \in \mathbb{C}^{n\times n}$ are arbitrary.

(d) *The general solution $X \in \mathbb{C}^{n\times n}$ of the inequality*

$$X + X^* \prec BB^* \tag{5.66}$$

can be written as (5.64), in which $U,\ V \in \mathbb{C}^{n\times n}$ are arbitrary with $r(U) = n$.

COROLLARY 5.6. *Let $A \in \mathbb{C}^{m\times n}$ be given. Then, the following hold.*

(a) *The general solution $X \in \mathbb{C}^{n\times m}$ of*

$$AX + (AX)^* \succcurlyeq 0 \tag{5.67}$$

and the corresponding $AX + (AX)^*$ can be written as

$$X = UU^*A^* + VA^* + F_AW, \tag{5.68}$$

$$AX + (AX)^* = 2AUU^*A^*, \tag{5.69}$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{n}_{\mathrm{SH}}$ and $W \in \mathbb{C}^{n\times m}$ are arbitrary.

(b) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* \succ 0 \tag{5.70}$$

if and only if $r(A) = m$. *In this case, the general solution* X *can be written as in* (5.68), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(c) *The general solution* $X \in \mathbb{C}^{n\times m}$ *of*

$$AX + (AX)^* \preccurlyeq 0 \tag{5.71}$$

and the corresponding $AX + (AX)^*$ *can be written as*

$$X = -UU^*A^* + VA^* + F_A W, \tag{5.72}$$

$$AX + (AX)^* = -2AUU^*A^*, \tag{5.73}$$

where $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that*

$$AX + (AX)^* \prec 0 \tag{5.74}$$

if and only if $r(A) = m$. *In this case, the general solution* X *can be written as* (5.72), *in which* $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, *and* $V \in \mathbb{C}^n_{\mathrm{SH}}$ *and* $W \in \mathbb{C}^{n\times m}$ *are arbitrary. In particular, if* A *is square and nonsingular, then the general solution* X *of* (5.74) *can be written as*

$$X = -UU^*A^* + VA^*, \tag{5.75}$$

where $U \in \mathbb{C}^{n\times n}$ *is any matrix with* $r(AU) = m$, *and* $V \in \mathbb{C}^n_{\mathrm{SH}}$ *is arbitrary.*

As an application of Theorems 5.1 and 5.2, we next give solutions of the inequality $(A+B)X + X^*(A+B)^* \succcurlyeq AB + BA$, which was considered for $A \succcurlyeq 0$ and $B \succcurlyeq 0$ in Chan and Kwong [6].

COROLLARY 5.7. *Let* $A, B \in \mathbb{C}^{m\times n}$ *be given. Then, there always exists an* $X \in \mathbb{C}^{n\times m}$ *that satisfies*

$$(A+B)X + X^*(A+B)^* \succcurlyeq AB^* + BA^*. \tag{5.76}$$

The general solution X *of* (5.76) *and the corresponding* $(A+B)X + X^*(A+B)^*$ *can be written as*

$$X = \frac{1}{2}(A+B)^* + \frac{1}{2}(\,UU^* + V - V^*)(A+B)^* + F_{(A+B)}W, \tag{5.77}$$

$$(A+B)X + X^*(A+B)^* = (A+B)(A+B)^* + (A+B)UU^*(A+B)^*, \tag{5.78}$$

where $U, V, W \in \mathbb{C}^{n\times n}$ *are arbitrary. In particular, there exists an* $X \in \mathbb{C}^{m\times m}$ *such that*

$$(A+B)X + X^*(A+B)^* \succ AB + BA \tag{5.79}$$

if and only if $r(A+B) = m$.

We next establish a group of formulas for calculating the ranks and inertias of $AX+(AX)^*-C$ subject to (5.1), and use the results obtained to derive necessary and sufficient conditions for the following two-sided LMI

$$C \succcurlyeq AX+(AX)^* \succcurlyeq B \tag{5.80}$$

and their variations to hold.

THEOREM 5.8. *Let $A \in \mathbb{C}^{m\times n}$ and $B,\ C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and assume that (5.1) is solvable for X. Also let $\mathscr{S}_1$ be as given in (5.6), and let*

$$N = \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} B & C & A \\ A^* & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} B & C & A \\ A^* & A^* & 0 \end{bmatrix}. \tag{5.81}$$

Then, the maximal and minimal ranks and partial inertias of $AX+(AX)^*-C$ subject to $X \in \mathscr{S}_1$ are given by

$$\max_{X\in\mathscr{S}_1} r[AX+(AX)^*-C] = \min\{\, r(K_2)-r(A),\ \ r(N)\,\}, \tag{5.82}$$

$$\min_{X\in\mathscr{S}_1} r[AX+(AX)^*-C] = \max\{t_1,\ t_2,\ t_4,\ t_4\}, \tag{5.83}$$

$$\max_{X\in\mathscr{S}_1} i_+[AX+(AX)^*-C] = i_-(N), \tag{5.84}$$

$$\max_{X\in\mathscr{S}_1} i_-[AX+(AX)^*-C] = \min\{\, i_-(B-C),\ \ i_+(N)\,\}, \tag{5.85}$$

$$\min_{X\in\mathscr{S}_1} i_+[AX+(AX)^*-C] = \max\{r(K_2)+i_-(N)-r(K_1),\ r(K_2)+i_+(B-C)-r[A, B-C]-r(A)\,\}, \tag{5.86}$$

$$\min_{X\in\mathscr{S}_1} i_-[AX+(AX)^*-C] = \max\{r(K_2)+i_+(N)-r(K_1),\ r(K_2)+i_-(B-C)-r[A, B-C]-r(A)-m\,\}, \tag{5.87}$$

where

$$t_1 = 2r(K_2)+r(N)-2r(K_1),$$
$$t_2 = 2r(K_2)+r(B-C)-2r[A, B-C]-2r(A)-m,$$
$$t_3 = 2r(K_2)+i_+(N)+i_+(B-C)-r(A)-r[A, B-C]-r(K_1),$$
$$t_4 = 2r(K_2)+i_-(N)+i_-(B-C)-r(A)-r[A, B-C]-r(K_1)-m.$$

In consequence, the following hold.

(a) *There exists an $X \in \mathbb{C}^{n\times m}$ such that $C \succ AX+(AX)^* \succcurlyeq B$ if and only if $i_+(N) \succcurlyeq m$ and $C \succ B$.*

(b) *There exists an $X \in \mathbb{C}^{n\times m}$ such that $AX+(AX)^* \succcurlyeq B$ and $AX+(AX)^* \succ C$ if and only if $i_-(N) \succcurlyeq m$.*

(c) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $AX + (AX)^* \not\succcurlyeq B$ *and* $AX + (AX)^* \not\succcurlyeq C$ *if and only if*

$$r(K_2) + i_+(N) - r(K_1) = 0 \ \ \text{and} \ \ r(K_2) + i_-(B-C) = r[A,\, B-C] + r(A) + m.$$

(d) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $C \succcurlyeq AX + (AX)^* \succcurlyeq B$ *if and only if*

$$C \succcurlyeq B,\ i_-(N) = r(A) \ \ \text{and} \ \ r(K_2) = r[A,\, B-C] + r(A).$$

Proof. Note from (5.3) that

$$AX + (AX)^* - C = B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*. \tag{5.88}$$

Applying Lemma 2.13(a) to (5.88), we obtain

$$\max_{U\in\mathbb{C}^{n\times m}} r[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= \min\left\{ r[A,\, B-C,\, J^{\frac{1}{2}}],\ r\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix},\ r(B-C)+m \right\}, \tag{5.89}$$

$$\min_{U\in\mathbb{C}^{n\times m}} r[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= 2r[A,\, B-C,\, J^{\frac{1}{2}}] + \max\{h_1,\ h_2,\ h_3,\ h_4\}, \tag{5.90}$$

$$\max_{U\in\mathbb{C}^{n\times m}} i_+[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= \min\left\{ i_+\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix},\ i_+(B-C)+m \right\}, \tag{5.91}$$

$$\max_{U\in\mathbb{C}^{n\times m}} i_-[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= \min\left\{ i_-\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix},\ i_-(B-C) \right\}, \tag{5.92}$$

$$\min_{U\in\mathbb{C}^{n\times m}} i_+[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= r[A,\, B-C,\, J^{\frac{1}{2}}]$$
$$+ \max\left\{ i_+\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix},\ i_+(B-C) - r[A,\, B-C] \right\}, \tag{5.93}$$

$$\min_{U\in\mathbb{C}^{n\times m}} i_-[B - C + (AU + J^{\frac{1}{2}})(AU + J^{\frac{1}{2}})^*]$$
$$= r[A,\, B-C,\, J^{\frac{1}{2}}]$$
$$+ \max\left\{ i_-\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} - r\begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix},\ i_-(B-C) - r[A,\, B-C] - m \right\}, \tag{5.94}$$

where

$$h_1 = r\begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} - 2r\begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix},$$

$$h_2 = r(B-C) - 2r[A, B-C] - m,$$
$$h_3 = i_- \begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} - r \begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} + i_+(B-C) - r[A, B-C],$$
$$h_4 = i_+ \begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} - r \begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} + i_-(B-C) - r[A, B-C] - m.$$

Simplifying the ranks and partial inertias of the block matrices in (5.89)–(5.94) gives

$$r[A, B-C, J^{\frac{1}{2}}] = r[A, B-C, J] = r[A, B-C, E_ABE_A] = r[A, B-C, BE_A]$$
$$= r \begin{bmatrix} A & B-C & B \\ 0 & 0 & A^* \end{bmatrix} - r(A) = r \begin{bmatrix} B & C & A \\ A^* & A^* & 0 \end{bmatrix} - r(A), \tag{5.95}$$
$$i_\pm \begin{bmatrix} B-C+J & A \\ A^* & 0 \end{bmatrix} = i_\pm \begin{bmatrix} B-C-E_ABE_A & A \\ A^* & 0 \end{bmatrix} = i_\mp \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}, \tag{5.96}$$
$$r \begin{bmatrix} B-C & A & J^{\frac{1}{2}} \\ A^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} B-C & A & J \\ A^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} B-C & A & E_ABE_A \\ A^* & 0 & 0 \end{bmatrix}$$
$$= r \begin{bmatrix} B-C & A & BE_A \\ A^* & 0 & 0 \end{bmatrix} = r \begin{bmatrix} B-C & A & B \\ A^* & 0 & 0 \\ 0 & 0 & A^* \end{bmatrix} - r(A)$$
$$= r(K_1) - r(A). \tag{5.97}$$

Substituting (5.95)–(5.97) into (5.89)–(5.94) gives (5.82)–(5.87). □

The following result can be shown similarly.

THEOREM 5.9. *Let $A \in \mathbb{C}^{m\times n}$ and $B,\ C \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, and assume that (5.29) is solvable for X. Also let $\mathscr{S}_2$ be as given in (5.34), and let*

$$N = \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}, \ K_1 = \begin{bmatrix} B & C & A \\ A^* & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix}, \ K_2 = \begin{bmatrix} B & C & A \\ A^* & A^* & 0 \end{bmatrix}.$$

Then, the maximal and minimal ranks and partial inertias of $AX + (AX)^* - C$ subject to $X \in \mathscr{S}_2$ are given by

$$\max_{X\in\mathscr{S}_2} r[AX + (AX)^* - C] = \min\{ r(K_2) - r(A), \ r(N)\},$$
$$\min_{X\in\mathscr{S}_2} r[AX + (AX)^* - C] = \max\{ t_1, \ t_2, \ t_4, \ t_4\},$$
$$\max_{X\in\mathscr{S}_2} i_+[AX + (AX)^* - C] = \min\{ i_+(B-C), \ i_-(N)\},$$
$$\max_{X\in\mathscr{S}_2} i_-[AX + (AX)^* - C] = i_+(N),$$
$$\min_{X\in\mathscr{S}_2} i_+[AX + (AX)^* - C] = \max\{ r(K_2) + i_-(N) - r(K_1),$$
$$r(K_2) + i_+(B-C) - r[A, B-C] - r(A) - m\},$$

$$\min_{X\in\mathscr{S}_2} i_-[AX+(AX)^*-C] = \max\{r(K_2)+i_+(N)-r(K_1),\\ r(K_2)+i_-(B-C)-r[A,B-C]-r(A)\},$$

where

$$t_1 = 2r(K_2)+r(N)-2r(K_1),$$
$$t_2 = 2r(K_2)+r(B-C)-2r[A,B-C]-2r(A)-m,$$
$$t_3 = 2r(K_2)+i_-(N)+i_-(B-C)-r(A)-r[A,B-C]-r(K_1),$$
$$t_4 = 2r(K_2)+i_+(N)+i_+(B-C)-r(A)-r[A,B-C]-r(K_1)-m.$$

In consequence, the following hold.

(a) *There exists an* $X\in\mathbb{C}^{n\times m}$ *such that* $C\prec AX+(AX)^*\preccurlyeq B$ *if and only if* $i_-(N)\geqslant m$ *and* $B\succ C$.

(b) *There exists an* $X\in\mathbb{C}^{n\times m}$ *such that* $AX+(AX)^*\preccurlyeq B$ *and* $AX+(AX)^*\prec C$ *if and only if* $i_+(N)\geqslant m$.

(c) *There exists an* $X\in\mathbb{C}^{n\times m}$ *such that* $AX+(AX)^*\preccurlyeq B$ *and* $AX+(AX)^*\preccurlyeq C$ *if and only if*

$$r(K_2)+i_-(N)-r(K_1)=0 \ \text{and}\ r(K_2)+i_+(B-C)=r[A,B-C]+r(A)+m.$$

(d) *There exists an* $X\in\mathbb{C}^{n\times m}$ *such that* $C\preccurlyeq AX+(AX)^*\preccurlyeq B$ *if and only if*

$$C\preccurlyeq B,\ i_+(N)=r(A)\ \text{and}\ r(K_2)=r[A,B-C]+r(A).$$

6. Concluding remarks

In the previous sections, we started a fundamental work on finding analytical solutions of three types of simple LMIs in (1.2)–(1.4). We first converted the LMIs to some equivalent quadratic matrix equations and then derive necessary and sufficient conditions for these LMIs to be feasible and obtained general solutions of these LMIs by using the theory of generalized inverses of matrices. Since the results obtained in the previous sections are represented in closed form by using the ranks, inertias and ordinary operations of the given matrices and their generalized inverses, they can be easily used to approach various problems related to these basic LMIs in matrix theory and applications. In particular, they can be used to solve mathematical programming and optimization problems subject to LMIs in (1.2)–(1.4).

Based on the results in the previous sections, it is not hard to establish analytical solutions of the following constrained LMIs:

(a) $AXB\succcurlyeq C\,(\succ C,\preccurlyeq C,\prec C)$ subject to $PX=Q$ and/or $XR=S$;

(b) $AXA^*\succcurlyeq B\,(\succ B,\preccurlyeq B,\prec B)$ subject to $PX=Q$ and $X=X^*$, or $PXP^*=Q$ and $X=X^*$;

(c) $AX + (AX)^* \succcurlyeq C\,(\succ C, \preccurlyeq C, \prec C)$ subject to $PX = Q$.

The results obtained altogether will greatly enrich the fundamental theory of LMIs. In addition, the work in this paper will also motivate finding possible analytical solutions of some general LMIs, such as,

(d) $AX + YB \succcurlyeq C\,(\succ C, \preccurlyeq C, \prec C)$;

(e) $AXA^* + BYB^* \succcurlyeq C\,(\succ C, \preccurlyeq C, \prec C)$;

(f) $AXA^* \succcurlyeq B\,(\succ B, \preccurlyeq B, \prec B)$ and $CXC^* \succcurlyeq D\,(\succ D, \preccurlyeq D, \prec D)$;

(g) $AXB + (AXB)^* \succcurlyeq C\,(\succ C, \preccurlyeq C, \prec C)$,

which are equivalent to the following linear-quadratic matrix equations:

(d1) $AX + YB = C \pm UU^*$;

(e1) $AXA^* + BYB^* = C \pm UU^*$;

(f1) $AXA^* = B \pm UU^*$ and $CXC^* = D \pm VV^*$;

(g1) $AXB + (AXB)^* = C \pm UU^*$.

A special case of (g) for $C \succcurlyeq 0$ was solved in [32].

In system and control theory, minimizing or maximizing the rank of matrix with variable entries (partially-specified matrix) over a set defined by matrix inequalities in the Löwner partial ordering is referred to as a rank minimization or maximization problem, and is denoted collectively by RMPs. RMPs now are known to be NP-hard in general case, and a satisfactory characterization of solution set of a general RMP is not available. Notice, however, from the results in this paper that for some types of matrix inequality in the Löwner partial ordering, their general solutions can be written in closed form by using the given matrices and their generalized inverses in the inequalities. Hence, it is expected that the results in this paper can be used to solve certain general RMPs. These further developments are beyond the scope of the present paper and will be the subjects of separate studies.

After a half century's development of the theory of generalized inverses of matrices, people now are widely using generalized inverses of matrices to solve a huge amount of problems in matrix theory and applications. In particular, one can utilize them to represent solutions of matrix equations and inequalities. Since linear algebra is a successful theory with essential applications in most scientific fields, the methods and results in matrix theory are prototypes of many concepts and content in other advanced branches of mathematics. In particular, matrix equations and matrix inequalities in the Löwner partial ordering, as well as generalized inverses of matrices were sufficiently extended to their counterparts for operators in a Hilbert space, or elements in a ring with involution, and their algebraic properties were extensively studied in the literature. In most cases, the conclusions on the complex matrices and their counterparts in general algebraic settings are analogous. Also, note that the results in this paper are derived from ordinary algebraic operations of the given matrices and their generalized inverses.

Hence, it is no doubt that most of the conclusions except those on ranks and inertias of matrices in this paper can trivially be extended to the corresponding equations and inequalities for linear operators on a Hilbert space or elements in a ring with involution.

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