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A CERTAIN FUNCTIONAL INEQUALITY DERIVED FROM AN OPERATOR INEQUALITY

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Abstract. We will show a certain functional inequality involving fractional powers, making use of the grand Furuta inequality and Tanahashi's argument.

1. Introduction

Each capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive semidefinite (denoted by $0 \leqslant T$) if $0 \leqslant (Tx,x)$ for all vectors x. We write $0 < T$ if T is positive semidefinite and invertible.

The celebrated Löwner-Heinz Theorem includes

THEOREM 1.1. [7], [4] *Let $0 \leqslant p \leqslant 1$. If $0 \leqslant B \leqslant A$, then $B^p \leqslant A^p$.*

For $1 < p$, $0 \leqslant B \leqslant A$ does not always ensure $B^p \leqslant A^p$. Furuta obtained an epochmaking extension of the Löwner-Heinz inequality by using itself.

THEOREM 1.2. [2] *Let $0 \leqslant p$, $1 \leqslant q$ and $0 \leqslant r$ with $p+r \leqslant (1+r)q$. If $0 \leqslant B \leqslant A$ holds, then*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}}.$$

The following result by Tanahashi is a full description of best possibility of the range

$$p+r \leqslant (1+r)q \quad \text{and} \quad 1 \leqslant q$$

as far as all parameters are positive.

THEOREM 1.3. [8] *Let p,q,r be positive real numbers. If $(1+r)q < p+r$ or $0 < q < 1$, then there exist 2×2 matrices A,B with $0 < B \leqslant A$ that do not satisfy the inequality*

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}}.$$

Furuta gave a unifying extension of both Theorem 1.2 and the Ando-Hiai inequality [1], which is often called the grand Furuta inequality.

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THEOREM 1.4. [3] *Let* $1 \leqslant p,\ 1 \leqslant s,\ 0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *If* $0 \leqslant B \leqslant A$ *with* $0 < A$, *then the following inequality holds:*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A^{1-t+r}. \tag{1}$$

Tanahashi showed that the outside powers in this theorem are the best possible.

THEOREM 1.5. [9] *Let* $1 \leqslant p,\ 1 \leqslant s,\ 0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *If* $1 < \alpha$, *then there exist* 2×2 *matrices* A, B *with* $0 < B \leqslant A$ *that do not satisfy the inequality*

$$\left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \leqslant A^{(1-t+r)\alpha}.$$

The purpose of this article is to show a functional inequality making use of the grand Furuta inequality and Tanahashi's argument.

2. Results

We set

$$\alpha(2n) = 1 - t_1 + t_2 - \cdots - t_{2n-1} + t_{2n}$$

$$\psi(0) = 1$$

$$\psi(2n-1) = \{\cdots(((p_1 - t_1)p_2 + t_2)\,p_3 - t_3)\,p_4 + \cdots + t_{2n-2}\}\,p_{2n-1} - t_{2n-1}$$

$$\psi(2n) = \{\cdots(((p_1 - t_1)p_2 + t_2)\,p_3 - t_3)\,p_4 + \cdots - t_{2n-1}\}\,p_{2n} + t_{2n}.$$

THEOREM 2.1. *Let* n *be a natural number. Let* $1 \leqslant p_j\ \ (j = 1, \cdots, 2n),\ 0 \leqslant t_{2k-1} \leqslant 1$ *and* $t_{2k-1} \leqslant t_{2k}\ \ (k = 1, \cdots, n)$. *Then, for arbitrary* $1 < x$,

$$(x^{\alpha(2n)} - 1)\prod_{j=1}^{2n}(x^{\psi(j-1)\,p_j} - 1) \leqslant \frac{\alpha(2n)}{\psi(2n)}(x-1)\prod_{j=1}^{2n}p_j(x^{\psi(j)} - 1). \tag{2}$$

The next Theorem is just the case $n = 1$ of Theorem 2.1.

THEOREM 2.2. *Let* $1 \leqslant p,\ 1 \leqslant s,\ 0 \leqslant t \leqslant 1$ *and* $t \leqslant r$. *Then, for arbitrary* $1 < x$,

$$(x^{p} - 1)(x^{(p-t)s} - 1)(x^{1-t+r} - 1) \leqslant \frac{(1-t+r)ps}{(p-t)s+r}(x-1)(x^{p-t} - 1)(x^{(p-t)s+r} - 1). \tag{3}$$

Although the mathematical meaning of this inequality is not sufficiently clarified at this stage, and the efficiency, possible applications to other branches of mathematics are still to be examined, this inequality represents the relation between some products of $x^{j} - 1$, where the powers j are combinations of the parameters in the grand Furuta inequality, namely, p, $(p-t)s$, $1-t+r$, 1, $p-t$, $(p-t)s+r$. The validity of the functional inequality (3) for arbitrary $1 < x$ prescribes the parameters p, s, t, r which make the grand Furuta inequality valid.

One notices the coincidence between the assumption on parameters in Theorem 1.4 and Theorem 2.2. As a matter of fact, the inequality (3) is a particular conclusion of the grand Furuta inequality. We should point out that Tanahashi's argument in [9] is almost sufficient to deduce the former from the latter. Our proof of Theorem 2.2 has a major part which is parallel to [9]. However, there is an essential difference between [9]'s and ours. Theorem 1.5 includes $1 < \alpha$ in the power, ours is α-free.

Our matrix A is a little different from that in [9], we use a variable y instead of ε and δ. The benefits of this modification of matrix A is that it considerably simplifies the proof. Tanahashi's proof in [9] needs the coefficients $c_1, \cdots, c_{11}$. On the other hand, $k_1, \cdots, k_6$ are sufficient in our proof. Moreover, Tanahashi's proof in [9] has finished with obtaining a contradiction in a refutation. In contrast, we will prove a functional inequality by applying l'Hopital's rule.

Throughout this paper, we assume that $1 < a < b$ and $0 < y$.

Proof of Theorem 2.2. We put

$$\alpha = 1 - t + r, \qquad \psi = (p-t)s + r.$$

We will consider matrices

$$A = \begin{pmatrix} a & \sqrt{(a-1)y} \\ \sqrt{(a-1)y} & b+y \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

Then we have $0 < B \leqslant A$. The eigenvalues of A are $\dfrac{a+b+y \pm \sqrt{d}}{2}$, where $d = a^2 + b^2 + y^2 - 2ab + 2(a+b-2)y$.

Let

$$c = \frac{-2\sqrt{(a-1)y}}{a-b-y-\sqrt{d}} \quad (>0)$$

and

$$U = \frac{1}{\sqrt{c^2+1}} \begin{pmatrix} c & 1 \\ 1 & -c \end{pmatrix}.$$

Then U is unitary and

$$U^*AU = \frac{1}{2} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where

$$d_1 = a+b+y+\sqrt{d}, \qquad d_2 = a+b+y-\sqrt{d}.$$

By the assumption and Theorem 1.4, A and B satisfy the inequality (1). Then

$$\left\{ U^*A^{\frac{r}{2}}U \left(U^*A^{-\frac{t}{2}}UU^*B^pUU^*A^{-\frac{t}{2}}U \right)^s U^*A^{\frac{r}{2}}U \right\}^{\frac{\alpha}{\psi}} \leqslant U^*A^{\alpha}U,$$

hence we have

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\left[\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix}\right]^s \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}}$$
$$\leqslant 2^{-\frac{\alpha p s}{\psi}} \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}. \tag{4}$$

Denote

$$\begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} d_1^{-\frac{t}{2}} & 0 \\ 0 & d_2^{-\frac{t}{2}} \end{pmatrix} = \frac{1}{c^2+1}\begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix},$$

where

$$A_1 = d_1^{-t}(c^2+b^p)$$
$$A_2 = d_2^{-t}(1+b^p c^2)$$
$$A_3 = d_1^{-\frac{t}{2}} d_2^{-\frac{t}{2}} c(1-b^p) \quad (<0).$$

Since $1<a<b$, $A_1 \to 2^{-t}b^{p-t}$ and $A_2 \to 2^{-t}a^{-t}$ as $y \to +0$. It follows from $t \leqslant p$ that $A_2 < A_1$ for sufficiently small $0<y$.

Let

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}}\begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & -\sqrt{\varepsilon_1} \\ -\sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix}$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1-A_2)^2 + 4A_3^2}.$$

Then it is easy to see that $A_3 = -\sqrt{(A_1 - A_2 + \varepsilon_1)\varepsilon_1}$, V is unitary and

$$V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix}.$$

Then (4) implies

$$\left\{\begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}(c^2+1)^{-s} V \begin{pmatrix} (A_1+\varepsilon_1)^s & 0 \\ 0 & (A_2-\varepsilon_1)^s \end{pmatrix} V^* \begin{pmatrix} d_1^{\frac{r}{2}} & 0 \\ 0 & d_2^{\frac{r}{2}} \end{pmatrix}\right\}^{\frac{\alpha}{\psi}}$$
$$\leqslant 2^{-\frac{\alpha p s}{\psi}} \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}. \tag{5}$$

Write the left-hand matrix as

$$(c^2+1)^{-s\frac{\alpha}{\psi}} (A_1 - A_2 + 2\varepsilon_1)^{-\frac{\alpha}{\psi}} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}^{\frac{\alpha}{\psi}},$$

where

$$B_1 = d_1^r \{(A_1 - A_2 + \varepsilon_1)(A_1 + \varepsilon_1)^s + \varepsilon_1 (A_2 - \varepsilon_1)^s\}$$
$$B_2 = d_2^r \{\varepsilon_1 (A_1 + \varepsilon_1)^s + (A_1 - A_2 + \varepsilon_1)(A_2 - \varepsilon_1)^s\}$$
$$B_3 = -d_1^{\frac{r}{2}} d_2^{\frac{r}{2}} \sqrt{A_1 - A_2 + \varepsilon_1}\sqrt{\varepsilon_1}\,\{(A_1 + \varepsilon_1)^s - (A_2 - \varepsilon_1)^s\}.$$

It follows that

$$B_1 \to 2^{r-t-ts} b^{(p-t)s+r}\left(b^{p-t} - a^{-t}\right)$$
$$B_2 \to 2^{r-t-ts} a^{r-st}\left(b^{p-t} - a^{-t}\right)$$

as $y \to +0$. Hence we have $B_2 < B_1$ for sufficiently small $0 < y$.

Let

$$W = \frac{1}{\sqrt{B_1 - B_2 + 2\varepsilon_2}} \begin{pmatrix} \sqrt{B_1 - B_2 + \varepsilon_2} & -\sqrt{\varepsilon_2} \\ -\sqrt{\varepsilon_2} & -\sqrt{B_1 - B_2 + \varepsilon_2} \end{pmatrix}$$

where

$$2\varepsilon_2 = -B_1 + B_2 + \sqrt{(B_1 - B_2)^2 + 4B_3^2}.$$

Then it is easy to see that $B_3 = -\sqrt{(B_1 - B_2 + \varepsilon_2)\,\varepsilon_2}$, W is unitary and

$$W^* \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} W = \begin{pmatrix} B_1 + \varepsilon_2 & 0 \\ 0 & B_2 - \varepsilon_2 \end{pmatrix}.$$

The following lemma is one of the most important points in Tanahashi's argument. Although the substance is presented in [8] and [9], we should restate and prove it in our context for the readers' convenience.

LEMMA.

$$\varepsilon_2 \left\{ \gamma_1 d_1^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \right\} \left\{ (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma_1 d_2^{\alpha} \right\}$$
$$\leqslant (B_1 - B_2 + \varepsilon_2) \left\{ \gamma_1 d_1^{\alpha} - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} \right\} \left\{ \gamma_1 d_2^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \right\}, \tag{6}$$

where

$$\gamma_1 = \left\{ \left(\frac{c^2 + 1}{2^p} \right)^s (A_1 - A_2 + 2\varepsilon_1) \right\}^{\frac{\alpha}{\psi}}. \tag{7}$$

Proof. The formula (5) implies

$$W \begin{pmatrix} (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} & 0 \\ 0 & (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \end{pmatrix} W^* \leqslant \gamma_1 \begin{pmatrix} d_1^{\alpha} & 0 \\ 0 & d_2^{\alpha} \end{pmatrix}. \tag{8}$$

Write the left-hand matrix as

$$(B_1 - B_2 + 2\varepsilon_2)^{-1} \begin{pmatrix} C_1 & C_3 \\ C_3 & C_2 \end{pmatrix},$$

where

$$\begin{aligned} C_1 &= (B_1 - B_2 + \varepsilon_2)(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} + \varepsilon_2 (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \\ C_2 &= \varepsilon_2 (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} + (B_1 - B_2 + \varepsilon_2)(B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} \\ C_3 &= -\sqrt{B_1 - B_2 + \varepsilon_2}\sqrt{\varepsilon_2}\left\{(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}. \end{aligned}$$

Then, by the inequality (8), we have

$$0 \leqslant \begin{pmatrix} \gamma_1 (B_1 - B_2 + 2\varepsilon_2) d_1^{\alpha} - C_1 & -C_3 \\ -C_3 & \gamma_1 (B_1 - B_2 + 2\varepsilon_2) d_2^{\alpha} - C_2 \end{pmatrix}.$$

So its determinant is also non-negative. We expand it to obtain

$$\begin{aligned} 0 \leqslant\ & \gamma_1^2 (B_1 - B_2 + 2\varepsilon_2)^2 d_1^{\alpha} d_2^{\alpha} - \gamma_1 (B_1 - B_2 + 2\varepsilon_2) d_1^{\alpha} C_2 \\ & - \gamma_1 (B_1 - B_2 + 2\varepsilon_2) d_2^{\alpha} C_1 + C_1 C_2 - C_3^2. \end{aligned} \tag{9}$$

Now

$$C_1 C_2 - C_3^2 = (B_1 - B_2 + 2\varepsilon_2)^2 (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}.$$

Hence, the formula (9) implies

$$\begin{aligned} 0 \leqslant\ & (B_1 - B_2 + 2\varepsilon_2)\left\{\gamma_1^2 (B_1 - B_2 + 2\varepsilon_2) d_1^{\alpha} d_2^{\alpha} - \gamma_1 d_1^{\alpha} C_2 - \gamma_1 d_2^{\alpha} C_1\right\} \\ & + (B_1 - B_2 + 2\varepsilon_2)^2 (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}. \end{aligned}$$

Cancel the common positive factor $B_1 - B_2 + 2\varepsilon_2$ and substitute the definitions for C_1 and C_2. Then a simple calculation shows that

$$\begin{aligned} & -\varepsilon_2 \left\{\gamma_1^2 d_1^{\alpha} d_2^{\alpha} - \gamma_1 d_1^{\alpha} (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma_1 d_2^{\alpha} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} + (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\} \\ \leqslant\ & (B_1 - B_2 + \varepsilon_2)\left\{\gamma_1^2 d_1^{\alpha} d_2^{\alpha} - \gamma_1 d_1^{\alpha} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} - \gamma_1 d_2^{\alpha} (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} + (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}. \end{aligned}$$

By factorizing, we have

$$\begin{aligned} & -\varepsilon_2 \left\{\gamma_1 d_1^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{\gamma_1 d_2^{\alpha} - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}}\right\} \\ \leqslant\ & (B_1 - B_2 + \varepsilon_2)\left\{\gamma_1 d_1^{\alpha} - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}\left\{\gamma_1 d_2^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}}\right\}. \end{aligned}$$

This completes the proof of Lemma. □

Now we estimate each term of the inequality (6) with respect to $y \to +0$. A key point in making use of the inequality (6) is that both estimations of the factor ε_2 in the left-hand side and the factor $\gamma_1 d_1^{\alpha} - (B_1+\varepsilon_2)^{\frac{\alpha}{\psi}}$ in the right-hand side contain a common subfactor y. After the cancellation of this y, we will derive the desired functional inequality by letting $y \to +0$, $a \to 1+0$ and applying l'Hopital's rule. Terms in other factors can be roughly estimated.

As usual notation, $f(y) = o(y^{\beta})$ means that $\dfrac{f(y)}{y^{\beta}} \to 0 \ (y \to +0)$.

One can establish the following formulae:

$$\sqrt{d} = (b-a)\left\{1+\frac{a+b-2}{(b-a)^2}y+o(y)\right\},$$

$$d_1^{-t} = (2b)^{-t}\left\{1-\frac{t(b-1)}{b(b-a)}y+o(y)\right\},$$

$$d_2^{-t} = (2a)^{-t}\left\{1+\frac{t(a-1)}{a(b-a)}y+o(y)\right\},$$

$$c = \frac{-2\sqrt{(a-1)y}}{a-b-y-\left(b-a+\dfrac{a+b-2}{b-a}y+o(y)\right)} = \sqrt{y}\cdot\frac{\sqrt{a-1}}{b-a}(1+o(1)),$$

$$c^2+1 = 1+\frac{a-1}{(b-a)^2}y+o(y),$$

$$(c^2+1)^s d_1^{\alpha} = (2b)^{\alpha}\left\{1+\frac{1}{b(b-a)^2}\Big(s(a-1)b+\alpha(b-1)(b-a)\Big)y+o(y)\right\},$$

$$(c^2+1)^s d_2^{\alpha} = (2a)^{\alpha}(1+o(1)).$$

Next

$$A_1 = 2^{-t}b^{p-t}\left\{1+\frac{1}{b(b-a)^2}\Big(-t(b-1)(b-a)+b^{1-p}(a-1)\Big)y+o(y)\right\},$$

$$A_2 = (2a)^{-t}\left\{1+\frac{a-1}{a(b-a)^2}\Big(t(b-a)+ab^p\Big)y+o(y)\right\},$$

$$A_3^2 = 4^{-t}a^{-t}b^{-t}\frac{a-1}{(b-a)^2}(1-b^p)^2\,y\,(1+o(1)),$$

so we have

$$A_1-A_2 = 2^{-t}(b^{p-t}-a^{-t})(1+o(1)), \qquad (A_1-A_2)\left(\frac{4A_3^2}{(A_1-A_2)^2}\right)^2 = o(y),$$

$$\begin{aligned}\varepsilon_1 &= \frac{1}{2}(A_1-A_2)\left(-1+\sqrt{1+\frac{4A_3^2}{(A_1-A_2)^2}}\right)\\ &= \frac{1}{2}(A_1-A_2)\left\{\binom{\frac{1}{2}}{1}\frac{4A_3^2}{(A_1-A_2)^2}+\binom{\frac{1}{2}}{2}\left(\frac{4A_3^2}{(A_1-A_2)^2}\right)^2+\cdots\right\}\\ &= \frac{A_3^2}{A_1-A_2}+o(y) = \frac{4^{-t}a^{-t}b^{-t}(a-1)(b-a)^{-2}(1-b^p)^2y(1+o(1))}{2^{-t}(b^{p-t}-a^{-t})(1+o(1))}+o(y)\\ &= \frac{2^{-t}a^{-t}b^{-t}(a-1)(1-b^p)^2}{(b-a)^2(b^{p-t}-a^{-t})}y(1+o(1)),\end{aligned}$$

hence

$$A_1+\varepsilon_1 = 2^{-t}b^{p-t}\left\{1+\frac{k_1}{b(b-a)^2}y+o(y)\right\},$$

where

$$k_1 = -t(b-1)(b-a)+b^{1-p}(a-1)+\frac{a^{-t}b^{1-p}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}}.$$

Further

$$(A_1+\varepsilon_1)^s = 2^{-ts}b^{(p-t)s}\left\{1+\frac{sk_1}{b(b-a)^2}y+o(y)\right\},$$

$$(A_2-\varepsilon_1)^s = 2^{-ts}a^{-ts}(1+o(1)),$$

$$A_1-A_2+\varepsilon_1 = 2^{-t}(b^{p-t}-a^{-t})\left\{1+\frac{k_2}{b(b-a)^2(b^{p-t}-a^{-t})}y+o(y)\right\},$$

where

$$k_2 = b^{p-t}k_1-a^{-1-t}(a-1)\Big(tb(b-a)+ab^{1+p}\Big),$$

and so

$$\begin{aligned}&A_1-A_2+2\varepsilon_1\\ &=2^{-t}(b^{p-t}-a^{-t})\left\{1+\frac{1}{b(b-a)^2(b^{p-t}-a^{-t})}\left(k_2+\frac{a^{-t}b^{1-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}}\right)y+o(y)\right\}.\end{aligned}$$

Now

$$\begin{aligned}&(A_1-A_2+\varepsilon_1)(A_1+\varepsilon_1)^s+\varepsilon_1(A_2-\varepsilon_1)^s\\ &=2^{-t-ts}b^{(p-t)s}(b^{p-t}-a^{-t})\left\{1+\frac{k_3}{b(b-a)^2(b^{p-t}-a^{-t})}y+o(y)\right\},\end{aligned}$$

where

$$k_3 = k_2+sk_1(b^{p-t}-a^{-t})+\frac{a^{-t-ts}b^{1-(p-t)s-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}}.$$

Further

$$
\begin{aligned}
B_1 &= 2^{r-t-ts} b^{(p-t)s+r} (b^{p-t} - a^{-t}) \\
&\quad \cdot \left\{ 1 + \frac{1}{b(b-a)^2(b^{p-t}-a^{-t})} \Big(r(b-1)(b-a)(b^{p-t}-a^{-t}) + k_3 \Big) y + o(y) \right\}, \\
B_2 &= 2^{r-t-ts} a^{r-ts} (b^{p-t} - a^{-t})(1+o(1)), \\
B_3^2 &= (2b)^r (1+o(1))(2a)^r (1+o(1)) 2^{-t} (b^{p-t}-a^{-t})(1+o(1)) \\
&\quad \cdot \frac{2^{-t} a^{-t} b^{-t} (a-1)(1-b^p)^2}{(b-a)^2 (b^{p-t}-a^{-t})} y(1+o(1)) \left\{ 2^{-ts} b^{(p-t)s} (1+o(1)) - 2^{-ts} a^{-ts} (1+o(1)) \right\}^2 \\
&= 2^{2r-2t-2ts} a^{r-t} b^{r-t} (a-1)(1-b^p)^2 (b-a)^{-2} (b^{(p-t)s} - a^{-ts})^2 y(1+o(1)),
\end{aligned}
$$

$$
\begin{aligned}
B_1 - B_2 &= 2^{r-t-ts} (b^{p-t} - a^{-t})(b^{(p-t)s+r} - a^{r-ts})(1+o(1)), \\
(B_1 - B_2) &\left(\frac{4B_3^2}{(B_1-B_2)^2} \right)^2 = o(y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\varepsilon_2 &= \frac{1}{2} \left(-B_1 + B_2 + \sqrt{(B_1-B_2)^2 + 4B_3^2} \right) \\
&= \frac{1}{2}(B_1 - B_2) \left\{ \binom{\frac{1}{2}}{1} \frac{4B_3^2}{(B_1-B_2)^2} + \binom{\frac{1}{2}}{2} \left(\frac{4B_3^2}{(B_1-B_2)^2} \right)^2 + \cdots \right\} \\
&= \frac{B_3^2}{B_1 - B_2} + o(y) \\
&= \frac{2^{r-t-ts} a^{r-t} b^{r-t} (a-1)(1-b^p)^2 (b^{(p-t)s} - a^{-ts})^2}{(b-a)^2 (b^{p-t} - a^{-t})(b^{(p-t)s+r} - a^{r-ts})} y(1+o(1)).
\end{aligned}
$$

Hence

$$
B_1 + \varepsilon_2 = 2^{r-t-ts} b^{(p-t)s+r} (b^{p-t} - a^{-t}) \left\{ 1 + \frac{k_4}{b(b-a)^2(b^{p-t}-a^{-t})} y + o(y) \right\},
$$

where

$$
k_4 = r(b-1)(b-a)(b^{p-t}-a^{-t}) + k_3 + \frac{a^{r-t} b^{1-(p-t)s-t} (a-1)(1-b^p)^2 (b^{(p-t)s} - a^{-ts})^2}{(b^{p-t}-a^{-t})(b^{(p-t)s+r} - a^{r-ts})},
$$

and so

$$
(B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} = 2^{(r-t-ts)\frac{\alpha}{\psi}} b^{\alpha} (b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}} \left\{ 1 + \frac{\alpha}{\psi} \cdot \frac{k_4}{b(b-a)^2(b^{p-t}-a^{-t})} y + o(y) \right\}.
$$

$$
(B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} = 2^{(r-t-ts)\frac{\alpha}{\psi}} a^{(r-ts)\frac{\alpha}{\psi}} (b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}} (1+o(1)).
$$

Furthermore, by (7),

$$
\begin{aligned}
\gamma_1 &= \left\{\left(\frac{c^2+1}{2^p}\right)^s (A_1 - A_2 + 2\varepsilon_1)\right\}^{\frac{\alpha}{\psi}} = 2^{-ps\frac{\alpha}{\psi}}(c^2+1)^{s\frac{\alpha}{\psi}}(A_1 - A_2 + 2\varepsilon_1)^{\frac{\alpha}{\psi}} \\
&= 2^{-ps\frac{\alpha}{\psi}}\left(1 + \frac{\alpha}{\psi}\cdot\frac{s(a-1)}{(b-a)^2}y + o(y)\right)2^{-t\frac{\alpha}{\psi}}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}} \\
&\quad \cdot\left\{1 + \frac{\alpha}{\psi}\cdot\frac{1}{b(b-a)^2(b^{p-t}-a^{-t})}\left(k_2 + \frac{a^{-t}b^{1-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}}\right)y + o(y)\right\} \\
&= 2^{(-ps-t)\frac{\alpha}{\psi}}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}\left\{1 + \frac{\alpha}{\psi}\cdot\frac{k_5}{b(b-a)^2(b^{p-t}-a^{-t})}y + o(y)\right\},
\end{aligned}
$$

where

$$
k_5 = s(a-1)b(b^{p-t} - a^{-t}) + k_2 + \frac{a^{-t}b^{1-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}},
$$

and so

$$
\gamma_1 d_1^{\alpha} = 2^{(r-t-ts)\frac{\alpha}{\psi}} b^{\alpha}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}\left\{1 + \frac{\alpha}{\psi}\cdot\frac{k_6}{b(b-a)^2(b^{p-t}-a^{-t})}y + o(y)\right\},
$$

where

$$
k_6 = k_5 + \psi(b-1)(b-a)(b^{p-t} - a^{-t}).
$$

The following 4 factors in the formula (6) can be roughly estimated.

$$
\begin{aligned}
\gamma_1 d_1^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{(r-t-ts)\frac{\alpha}{\psi}}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}(b^{\alpha} - a^{(r-ts)\frac{\alpha}{\psi}})(1 + o(1)) \\
\gamma_1 d_2^{\alpha} - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{(r-t-ts)\frac{\alpha}{\psi}}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}(a^{\alpha} - b^{\alpha})(1 + o(1)) \\
B_1 - B_2 + \varepsilon_2 &= 2^{r-t-ts}(b^{p-t} - a^{-t})(b^{(p-t)s+r} - a^{r-ts})(1 + o(1)) \\
\gamma_1 d_2^{\alpha} - (B_2 - \varepsilon_2)^{\frac{\alpha}{\psi}} &= 2^{(r-t-ts)\frac{\alpha}{\psi}}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}(a^{\alpha} - a^{(r-ts)\frac{\alpha}{\psi}})(1 + o(1))
\end{aligned}
$$

Now we have the estimation of the most delicate factor in the formula (6), whose main term is canceled by subtraction.

$$
\begin{aligned}
&\gamma_1 d_1^{\alpha} - (B_1 + \varepsilon_2)^{\frac{\alpha}{\psi}} \\
&= 2^{(r-t-ts)\frac{\alpha}{\psi}} b^{\alpha}(b^{p-t} - a^{-t})^{\frac{\alpha}{\psi}}\cdot\frac{\alpha}{\psi}\cdot\frac{k_6 - k_4}{b(b-a)^2(b^{p-t}-a^{-t})}y(1 + o(1)).
\end{aligned}
$$

We apply these estimations to the inequality (6). Cancelling y and letting $y \to +0$, we have

$$
\begin{aligned}
&a^{r-t}(1-b^p)^2(b^{(p-t)s} - a^{-ts})^2\cdot(b^{\alpha} - a^{(r-ts)\frac{\alpha}{\psi}})\cdot(b^{\alpha} - a^{\alpha}) \\
&\leqslant (b^{p-t} - a^{-t})(b^{(p-t)s+r} - a^{r-ts})^2\cdot\frac{\alpha}{\psi}\cdot(k_6 - k_4)\cdot\frac{a^{\alpha} - a^{(r-ts)\frac{\alpha}{\psi}}}{a-1}. \qquad (10)
\end{aligned}
$$

Let us write down the coefficient $k_6 - k_4$ explicitly.

$$\begin{aligned}
& k_6 - k_4 \\
=& s(a-1)b(b^{p-t}-a^{-t}) + k_2 + \frac{a^{-t}b^{1-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}} + \psi(b-1)(b-a)(b^{p-t}-a^{-t}) \\
& - r(b-1)(b-a)(b^{p-t}-a^{-t}) \\
& - k_2 - sk_1(b^{p-t}-a^{-t}) - \frac{a^{-t-ts}b^{1-(p-t)s-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}} \\
& - \frac{a^{r-t}b^{1-(p-t)s-t}(a-1)(1-b^p)^2(b^{(p-t)s}-a^{-ts})^2}{(b^{p-t}-a^{-t})(b^{(p-t)s+r}-a^{r-ts})} \\
=& s(a-1)b(b^{p-t}-a^{-t}) + \frac{a^{-t}b^{1-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}} + ps(b-1)(b-a)(b^{p-t}-a^{-t}) \\
& - sb^{1-p}(a-1)(b^{p-t}-a^{-t}) - sa^{-t}b^{1-p}(a-1)(1-b^p)^2 \\
& - \frac{a^{-t-ts}b^{1-(p-t)s-t}(a-1)(1-b^p)^2}{b^{p-t}-a^{-t}} - \frac{a^{r-t}b^{1-(p-t)s-t}(a-1)(1-b^p)^2(b^{(p-t)s}-a^{-ts})^2}{(b^{p-t}-a^{-t})(b^{(p-t)s+r}-a^{r-ts})}.
\end{aligned}$$

Letting $a \to 1+0$ in the inequality (10), we have

$$k_6 - k_4 \to ps(b-1)^2(b^{p-t}-1) \qquad \text{and} \qquad \frac{a^{\alpha} - a^{(r-ts)\frac{\alpha}{\psi}}}{a-1} \to \frac{\alpha ps}{\psi},$$

and we can obtain

$$(b^p-1)^2(b^{(p-t)s}-1)^2(b^{\alpha}-1)^2 \leqslant \left(\frac{\alpha ps}{\psi}\right)^2 (b-1)^2(b^{p-t}-1)^2(b^{(p-t)s+r}-1)^2.$$

This completes the proof of Theorem 2.2. $\square$

Our proof of Theorem 2.1 uses the previous Theorem and an argument which is similar to that for the proof of [5, Theorem 7] and [6, Proposition 7].

Proof of Theorem 2.1. We just proved the case $n=1$. Suppose that the inequality (2) holds. Let $1 \leqslant p_{2n+1}$, p_{2n+2}, $0 \leqslant t_{2n+1} \leqslant 1$, $t_{2n+1} \leqslant t_{2n+2}$. Put

$$p = \frac{\psi(2n)}{\alpha(2n)} p_{2n+1}, \quad t = \frac{t_{2n+1}}{\alpha(2n)}, \quad r = \frac{t_{2n+2}}{\alpha(2n)}, \quad s = p_{2n+2}.$$

Then it is easy to check that $1 \leqslant p$, $1 \leqslant s$, $0 \leqslant t \leqslant 1$ and $t \leqslant r$. Applying Theorem 2.2, we have

$$(y^{1-t+r}-1)(y^p-1)(y^{(p-t)s}-1) \leqslant \frac{(1-t+r)ps}{(p-t)s+r}(y-1)(y^{p-t}-1)(y^{(p-t)s+r}-1) \tag{11}$$

for arbitrary $1 < y$. Substitute $y = x^{\alpha(2n)}$ in (11). Then it is elementary to calculate that

$$\begin{aligned} y^{1-t+r} &= x^{\alpha(2n)-t_{2n+1}+t_{2n+2}} = x^{\alpha(2n+2)} \\ y^{p} &= x^{\psi(2n)p_{2n+1}} \\ y^{p-t} &= x^{\psi(2n)p_{2n+1}-t_{2n+1}} = x^{\psi(2n+1)} \\ y^{(p-t)s} &= x^{(\psi(2n)p_{2n+1}-t_{2n+1})p_{2n+2}} = x^{\psi(2n+1)p_{2n+2}} \\ y^{(p-t)s+r} &= x^{(\psi(2n)p_{2n+1}-t_{2n+1})p_{2n+2}+t_{2n+2}} = x^{\psi(2n+2)} \end{aligned}$$

and

$$\frac{(1-t+r)ps}{(p-t)s+r} = \frac{\alpha(2n+2)\psi(2n)p_{2n+1}p_{2n+2}}{\psi(2n+2)\alpha(2n)}.$$

So we have

$$\begin{aligned} &(x^{\alpha(2n+2)}-1)(x^{\psi(2n)p_{2n+1}}-1)(x^{\psi(2n+1)p_{2n+2}}-1) \\ &\leqslant \frac{\alpha(2n+2)\psi(2n)p_{2n+1}p_{2n+2}}{\psi(2n+2)\alpha(2n)}(x^{\alpha(2n)}-1)(x^{\psi(2n+1)}-1)(x^{\psi(2n+2)}-1). \qquad (12) \end{aligned}$$

Take the product of each side of the inequalities (2) and (12), and cancel the factor $x^{\alpha(2n)}-1$. We conclude that

$$\begin{aligned} &(x^{\alpha(2n+2)}-1)(x^{p_1}-1)(x^{(p_1-t_1)p_2}-1)\cdots(x^{\psi(2n)p_{2n+1}}-1)(x^{\psi(2n+1)p_{2n+2}}-1) \\ &\leqslant \frac{\alpha(2n+2)}{\psi(2n+2)}p_1\cdots p_{2n+2}(x-1)(x^{p_1-t_1}-1)(x^{(p_1-t_1)p_2+t_2}-1)\cdots(x^{\psi(2n+1)}-1)(x^{\psi(2n+2)}-1). \end{aligned}$$

This completes the proof. $\square$

Here is an application. At least to the author, it seems not easy to give an elementary proof of the following inequality.

COROLLARY 2.3. *For arbitrary* $1 < x$,

$$(x^{\sqrt{2}}-1)(x^{\sqrt{3}}-1)(x^{\sqrt{5}}-1) \leqslant \frac{\sqrt{30}(\sqrt{2}+1)}{\sqrt{3}+\sqrt{5}}(x-1)(x^{\sqrt{2}-1}-1)(x^{\sqrt{3}+\sqrt{5}}-1).$$

REMARK 2.4. Theorem 2.2 is an extension of the case $p+r=(1+r)q$ of [10, Theorem 1.1]. Indeed, by putting $s=1$ and $t=0$ in Theorem 2.2, if $1 \leqslant p$, $0 \leqslant r$, then

$$(x^p-1)(x^{1+r}-1) \leqslant \frac{(1+r)p}{p+r}(x-1)(x^{p+r}-1)$$

for arbitrary $1 < x$.

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REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl. 197/198 (1994), 113–131.

[2] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc. 101 (1987), no. 1, 85–88.

[3] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. 219 (1995), 139–155.

[4] E. HEINZ, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. 123 (1951), 415–438.

[5] T. KOIZUMI AND K. WATANABE, *A remark on extension of order preserving operator inequality*, J. Math. Inequal. **6** (2012), 119–124.

[6] T. KOIZUMI AND K. WATANABE, *On monotonicity of some operator functions related to order preserving operator inequalities*, J. Math. Inequal. **6** (2012), 579–587.

[7] K. LÖWNER, *Über monotone Matrixfunktionen*, Math. Z. 38 (1934), 177–216.

[8] K. TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. 124 (1996), 141–146.

[9] K. TANAHASHI, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc. 128 (2000), 511–519.

[10] K. WATANABE, *An application of matrix inequalities to certain functional inequalities involving fractional powers*, J. Inequal. Appl. 2012, **2012**:221.

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