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# JORDAN TYPE INEQUALITIES USING MONOTONY OF FUNCTIONS

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*Abstract.* In this paper we will consider Jordan type inequalities involving hyperbolic trigonometric functions.

## 1. Introduction

In this section we give a brief overview of known results which pertain to the main results of this paper.

The following inequalities

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1, \quad (0 < |x| < \frac{\pi}{2}),$$

are due to Jordan ([7], p. 33). Its have attracted the attention of several mathematicians (See, e.g., [1–6], [8–14]).

Lazarević [5] gives us the following inequality:

$$\left(\frac{\sinh x}{x}\right)^q < \cosh x, \quad (x \neq 0,\ q \geqslant 3). \tag{1}$$

The following inequalities

$$\sinh x < x + \frac{x^3}{5}, \quad (0 < x < 1), \tag{2}$$

$$\frac{\sinh kx}{kx} \leqslant \frac{\sinh x}{x}, \quad (x > 0), \tag{3}$$

$$\frac{1}{\cosh x} < 1 - \frac{x^2}{3}, \quad (0 < x < 1), \tag{4}$$

and

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \quad (0 < x < \frac{\pi}{2}), \tag{5}$$

have established by R. Klén, M. Visuri, and M. Vuorinen [4].

The inequality

$$\left(\frac{x}{\sinh x}\right)^\alpha < (1-\eta) + \eta\left(\frac{1}{\cosh x}\right)^\alpha, \tag{6}$$

where $x > 0$, $\alpha > 0$, and $\eta \leqslant 1/3$ was studied recently by Zhu in [13].

In Section 2 we give upper and lower bounds for $(\sin x)/x$. Some Jordan's type inequalities with hyperbolic trigonometric functions are presented in Section 3.

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## 2. Jordan's inequality

In this section we will find upper and lower bounds for $(\sin x)/x$ by using hyperbolic functions.

THEOREM 1. *For* $x \in (0,\infty)$,

$$\frac{\sin x}{x} < \sqrt{\cosh x}.$$

*Proof.* The inequality holds true if the function $f(x) = x\sqrt{\cosh x} - \sin x$ is positive on $(0,\infty)$. Since

$$f''(x) = \frac{1}{\sqrt{\cosh x}}\left[\sin\left(x\sqrt{\cosh x}\right)(2\cosh x + x\sinh x) + 3\sqrt{\cosh x}\sinh x + x\cosh^{\frac{3}{2}} x\right],$$

we have $f''(x) > 0$ for $x \in (0,\infty)$ and $f'(x)$ is increasing on $(0,\infty)$. Therefore

$$f'(x) = \frac{1}{2\sqrt{\cosh x}}\left(2\cosh x + x\sinh x - 2\cos x\sqrt{\cosh x}\right) > f'(0) = 0,$$

and the function $f(x)$ is increasing on $(0,\infty)$. Now $f(x) > f(0) = 0$ for $x \in (0,\infty)$. □

COROLLARY 2. *For* $x \in \mathbb{R}\backslash\{0\}$,

$$\frac{\sin x}{x} < \cosh x.$$

*Proof.* Because $f(x) = \frac{\sinh x}{x} - \cosh x$ is even function we proof the inequality on $(0,\infty)$. Using Theorem 1 and the fact that $\sqrt{\cosh x} < \cosh x$ the assertion follows. In Figure 1 we present the graphics of functions $\frac{\sin x}{x}$ and $\cosh x$. □

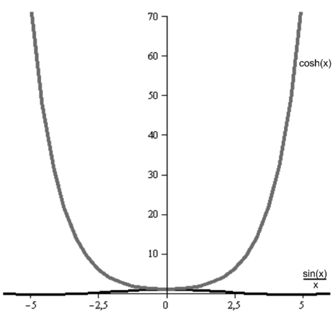


Figure 1.

In the above picture, with red is plotted the graph of the function $\frac{\sin x}{x}$ and with black is plotted the graph of the function $\cosh x$.

REMARK 3. Klén, Visuri, and Vuorinen have shown in [4] that the inequalities

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x} \tag{7}$$

are true for $x \in (0, \pi/2)$. By Corollary 2 and (7) we have

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \cosh x,$$

for $x \in (0, \pi/2)$.

THEOREM 4. *For* $x, k \in (0, \infty)$,

$$\frac{\sin x}{x} < \frac{\sinh kx}{kx}.$$

*Proof.* The inequality holds true if the function $f(x) = \sinh kx - k \sin x$ is positive on $(0, \infty)$. Since

$$f'(x) = k(\cosh kx - \cos x),$$

we have $f'(x) > 0$ for $x \in (0, \infty)$. Therefore that the function $f(x)$ is increasing on $(0, \infty)$. Now $f(x) > f(0) = 0$ for $x \in (0, \infty)$. $\square$

COROLLARY 5. *For* $x \in (0, \infty)$ *and* $k \in [1, \infty)$ *the following inequality*

$$\frac{\sin x}{x} < \frac{\sinh kx}{x}$$

*holds.*

REMARK 6. Similar inequalities to Theorem 4 have been considered by R. Klén, M. Visuri, and M. Vuorinen in [4, Theorem 4.3].

## 3. Hyperbolic Jordan's inequality

In this section we gave some hyperbolic Jordan's type inequalities with hyperbolic trigonometric functions.

THEOREM 7. *Let* $x \in (0,1)$, $k \in (0, \infty)$ *and* $q \geqslant 3$, $q$ *is natural number. Then the double inequality*

$$\sqrt[q]{\frac{3}{3 - k^2x^2}} < \frac{\sinh x}{x} < 1 + \frac{x^2}{5}$$

*holds.*

*Proof.* From (1) result that

$$\sqrt[q]{\cosh x} < \frac{\sinh x}{x}. \tag{8}$$

By (3) and (8) we obtain

$$\sqrt[q]{\cosh kx} < \frac{\sinh kx}{kx} \leqslant \frac{\sinh x}{x} \tag{9}$$

The inequality (4) is equivalent with

$$\frac{3}{3-x^2} < \cosh x,$$

therefore

$$\sqrt[q]{\frac{3}{3-k^2x^2}} < \sqrt[q]{\cosh kx} \tag{10}$$

By (2), (9) and (10) we obtain the conclusion. □

THEOREM 8. *Let $x \in (0,1)$, $\alpha > 0$ and $\eta \leqslant 1/3$, then the following inequality*

$$\left(\frac{x}{\sinh x}\right)^{\alpha} < (1-\eta) + \eta\left(1 - \frac{x^2}{3}\right)^{\alpha}$$

*holds.*

*Proof.* By (4) and (6) the assertion follows. □

When letting $\alpha = 1$ in Theorem 8, one can obtain the following result.

COROLLARY 9. *Let $x \in (0,1)$, and $\eta \leqslant 1/3$, then the following inequality*

$$\frac{x}{\sinh x} < 1 - \eta\frac{x^2}{3}$$

*holds.*

THEOREM 10. *For $x \in (0,1)$ the following inequalities*

$$1 - \frac{x^2}{2} \leqslant \frac{1}{\cosh x} \leqslant 1 - \frac{x^2}{3}$$

*hold.*

*Proof.* By the series expansion of $\frac{1}{\cosh x}$ we have

$$1 - \frac{x^2}{2} \leqslant \frac{1}{\cosh x} \leqslant 1 - \frac{x^2}{2} + 5 \cdot \frac{x^4}{24}. \tag{11}$$

By (4) and (11) we obtain the conclusion. □

THEOREM 11. *Let $x > 0$. Then the function*

$$f(t) = \frac{1}{\cosh^t \frac{x}{t}}$$

*is decreasing on $(0, \infty)$.*

*Proof.* Let us consider instead of $f(x)$ the function

$$f_1(y) = -\frac{x}{y} \log\cosh y$$

for $y \in (0, \infty)$. Note that $f(t) = \exp(f_1(\frac{x}{t}))$ and therefore the claim is equivalent to the function $f_1(y)$ being decreasing on $(0, \infty)$. We have

$$f_1'(y) = \frac{x}{y^2}(\log\cosh y - y\tanh y),$$

and

$$\frac{d(\log\cosh y - y\tanh y)}{dy} = -\frac{y}{\cosh^2 y} < 0.$$

and $\log\cosh y - y\tanh y \leqslant 0$. Therefore the function $f_1(y)$ is decreasing on $(0, \infty)$, and $f(t)$ is increasing on $(0, \infty)$. $\square$

COROLLARY 12. *Let $x \in (0, \infty)$. Then the following inequality*

$$\cosh\frac{x}{3} < \frac{2}{3} + \frac{1}{3}\cosh x$$

*holds.*

*Proof.* By Theorem 11 we obtain that

$$\frac{1}{\cosh^2 \frac{x}{2}} > \frac{1}{\cosh^3 \frac{x}{3}}. \tag{12}$$

By using the identities

$$\cosh^2\frac{x}{2} = \frac{1 + \cosh x}{2},$$

and

$$\cosh^3\frac{x}{3} = \frac{3}{4}\cosh\frac{x}{3} + \frac{1}{4}\cosh x$$

in relation (12) we obtain the conclusion. $\square$

THEOREM 13. *Let $x \in \mathbb{R}\backslash\{0\}$. Then the following inequality*

$$\frac{\sinh x}{x} > \frac{1}{\cosh\frac{x}{3}}$$

*holds.*

*Proof.* Because

$$f(x) = \frac{\sinh x}{x} - \frac{1}{\cosh \frac{x}{3}}$$

is even function we proof the inequality on $(0,\infty)$. The inequality holds true if the function

$$g(x) = \sinh x \cosh \frac{x}{3} - x$$

is positive on $(0,\infty)$. Since

$$g'(x) = \frac{1}{3}\cosh \frac{2}{3}x + \frac{2}{3}\cosh \frac{4}{3}x - 1,$$

we have $g'(x) > 0$ for $x \in (0,\infty)$. Therefore that the function $g(x)$ is increasing on $(0,\infty)$. Now $g(x) > g(0) = 0$ for $x \in (0,\infty)$.

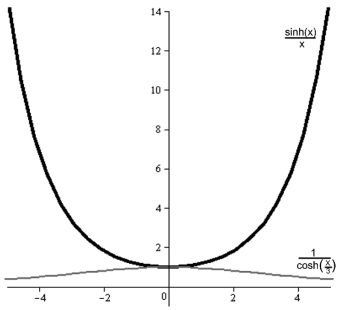


Figure 2.

In Figure 2 we present the graphics of functions $\frac{\sinh x}{x}$ and $\frac{1}{\cosh \frac{x}{3}}$. □

In the above picture, with red is plotted the graph of the function $\frac{\sinh x}{x}$ and with black is plotted the graph of the function $\frac{1}{\cosh \frac{x}{3}}$.

COROLLARY 14. *Let $x \in (0,\infty)$. Then the following inequality*

$$\frac{x}{\sinh x} < \frac{2}{3} + \frac{1}{3}\cosh x$$

*holds.*

*Proof.* By Corollary 12 and Theorem 13 we obtain the conclusion. □

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