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A GEOMETRIC INEQUALITY WITH ONE PARAMETER FOR A POINT IN THE PLANE OF A TRIANGLE

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Abstract. With the help of mathematical software Maple for calculations, we establish a new geometric inequality with one parameter on a given interval involving an arbitrary point in the plane of a triangle. Two related interesting conjectures checked by the computer are put forward.

1. Introduction and main result

Let P be an arbitrary point in the plane of triangle ABC. Denote by R_1, R_2, R_3 the distance of P from the vertices A,B,C, and r_1, r_2, r_3 the distances of P from the sidelines BC, CA, AB respectively. If P lies inside triangle ABC, then we have the following famous Erdös-Mordell inequality (see [1–5], [8–10], [12], [17–20]) and references therein):

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3), \tag{1.1}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

There are many methods to prove inequality (1.1) (see e.g. [1–3], [8–10], [16], [18]). In a recent paper [12], the author gave a new proof, in which the following inequality (1.2) holding for any point P in the plane was used:

$$R_2+R_3 \geqslant \sqrt{a^2+4r_1^2}, \tag{1.2}$$

where $a=BC$. Equality in (1.2) holds only when $R_2=R_3$.

Starting from inequality (1.2), the author recently find a new geometric inequality with one parameter. Let us introduce our original idea as follows:

Noting that $R_2+R_3 \leqslant \sqrt{2(R_2^2+R_3^2)}$, it follows from (1.2) that

$$R_2^2+R_3^2-2r_1^2 \geqslant \frac{1}{2}a^2. \tag{1.3}$$

Using (1.3) and its two analogues and adding them, we obtain the symmetric inequality:

$$\frac{R_2^2+R_3^2-2r_1^2}{a^2}+\frac{R_3^2+R_1^2-2r_2^2}{b^2}+\frac{R_1^2+R_2^2-2r_3^2}{c^2} \geqslant \frac{3}{2}, \tag{1.4}$$

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where a,b,c are the lengths of the sides BC, CA, AB respectively. Equality in (1.4) holds if and only if $R_1 = R_2 = R_3$, i.e. P is the circumcenter of ABC.

The inequality (1.4) first inspires the author to find the similar inequality:

$$\frac{R_2^2+R_3^2+2r_1^2}{a^2}+\frac{R_3^2+R_1^2+2r_2^2}{b^2}+\frac{R_1^2+R_2^2+2r_3^2}{c^2}\geqslant\frac{5}{2}. \tag{1.5}$$

It is interesting that the equality of this inequality also holds if and only if P is a special point of triangle ABC. More exactly, the equality in (1.5) holds if and only if the barycentric coordinates of P with respect to $\triangle ABC$ is (a^2,b^2,c^2), namely P coincide with the Lhuilier-Lemoine point K of ABC (see [17, p. 278]).

We also find that

$$\frac{R_2^2+R_3^2-r_1^2}{a^2}+\frac{R_3^2+R_1^2-r_2^2}{b^2}+\frac{R_1^2+R_2^2-r_3^2}{c^2}\geqslant\frac{7}{4} \tag{1.6}$$

and

$$\frac{R_2^2+R_3^2+r_1^2}{a^2}+\frac{R_3^2+R_1^2+r_2^2}{b^2}+\frac{R_1^2+R_2^2+r_3^2}{c^2}\geqslant\frac{9}{4} \tag{1.7}$$

hold and both of equality conditions are the same as in (1.1).

Considering the unified generalization of these inequalities, we find the following conclusion:

THEOREM 1. *Let ABC be a triangle with sides a,b,c. If $-2\leqslant\lambda\leqslant 2$ be a real number, then the inequality:*

$$\frac{R_2^2+R_3^2+\lambda r_1^2}{a^2}+\frac{R_3^2+R_1^2+\lambda r_2^2}{b^2}+\frac{R_1^2+R_2^2+\lambda r_3^2}{c^2}\geqslant\frac{\lambda+8}{4} \tag{1.8}$$

holds for any point P in the plane. When $\lambda=-2$, the equality in (1.8) holds if and only if P is the circumcenter of ABC. When $\lambda=2$, the equality in (1.8) holds if and only if P is the Lhuilier-Lemoine point of ABC. When $-2<\lambda<2$, the equality in (1.8) holds if and only if $\triangle ABC$ is equilateral and P is its center.

In particular, when $\lambda=0$, (1.8) becomes

$$\frac{R_2^2+R_3^2}{a^2}+\frac{R_3^2+R_1^2}{b^2}+\frac{R_1^2+R_2^2}{c^2}\geqslant 2, \tag{1.9}$$

which does not include the distances r_1, r_2 and r_3. This inequality can also be deduced by adding up (1.4) and (1.5) or (1.6) and (1.7). In [13], the author obtained the following weighted generalization of (1.9):

$$\frac{R_2^2+R_3^2}{a^2}x^2+\frac{R_3^2+R_1^2}{b^2}y^2+\frac{R_1^2+R_2^2}{c^2}z^2\geqslant\frac{2}{3}(yz+zx+xy), \tag{1.10}$$

where x,y,z are arbitrary real numbers.

The main purpose of this paper is to prove Theorem 1. In addition, we also propose two related conjectures in the last section.

2. Several lemmas

In order to prove our theorem, we first give several lemmas.

LEMMA 1. *For any real numbers x,y,z and triangle ABC withe sides a,b,c, we have*

$$a^2x^2+y^2b^2+z^2c^2 \geqslant yz(b^2+c^2-a^2)+zx(c^2+a^2-b^2)+xy(a^2+b^2-c^2), \quad (2.1)$$

with equality holding if and only if $x=y=z$.

The inequality (2.1) was first given by J. Wolstenholme in [21] and it has several equivalent forms. For example, the equivalent trigonometric form:

$$x^2+y^2+z^2 \geqslant 2(yz\cos A+zx\cos B+xy\cos C) \quad (2.2)$$

(A,B,C are the angles of triangle ABC) which can be used to establish the weighted Erdös-Mordell inequality (see [17]):

$$x^2R_1+y^2R_2+z^2R_3 \geqslant 2(yzr_1+zxr_2+xyr_3), \quad (2.3)$$

with equality holding if and only if $x:y:z=\sin A:\sin B:\sin C$ and P is the circumcenter of triangle ABC.

LEMMA 2. *For any triangle ABC with sides a,b,c, we have*

$$\begin{aligned}Q_0 \equiv{}& (b^4+c^4)a^8-2(b^2+c^2)(2b^4-b^2c^2+2c^4)a^6+(6b^8-2b^6c^2+6b^4c^4\\&-2b^2c^6+6c^8)a^4-2(b^2+c^2)(2b^4-b^2c^2+2c^4)(b^2-bc-c^2)(b^2+bc-c^2)a^2\\&+b^{12}-6b^{10}c^2+13b^8c^4-14b^6c^6+13b^4c^8-6b^2c^{10}+c^{12} \geqslant 0, \quad (2.4)\end{aligned}$$

with equality holding if and only if $a:b:c=\sqrt{(3+\sqrt{13})/2}:1:1$.

Proof. Let $b+c-a=2u$, $c+a-b=2v$, $a+b-c=2w$, then $a=v+w$, $b=w+u$, $c=u+v$. Substituting them into the expression of Q_0 and using important mathematical software Maple (we used Maple 15), one obtains

$$\begin{aligned}Q_0 ={}& 2u^{12}+12(v+w)u^{11}+(34v^2+112vw+34w^2)u^{10}+20(v+w)(3v^2+17vw\\&+3w^2)u^9+(72v^4+804v^3w+1622v^2w^2+804vw^3+72w^4)u^8\\&+4(v+w)(15v^4+246v^3w+620v^2w^2+246vw^3+15w^4)u^7\\&+(34v^6+888v^5w+5290v^4w^2+5008v^3w^3+5290v^2w^4+888vw^5+34w^6)u^6\\&+4(v+w)(3v^6+111v^5w+1397v^4w^2-320v^3w^3+1397v^2w^4+111vw^5\\&+3w^6)u^5+(2v^8+116v^7w+4370v^6w^2+3780v^5w^3-794v^4w^4+3780v^3w^5\\&+4370v^2w^6+116vw^7+2w^8)u^4+4(v+w)(2v^6+420v^5w+148v^4w^2\\&-461v^3w^3+148v^2w^4+420vw^5+2w^6)u^3vw+2(134v^6+260v^5w-125v^4w^2\\&-480v^3w^3-125v^2w^4+260vw^5+134w^6)u^2v^2w^2+4(v+w)(2v^4-11v^3w\\&-15v^2w^2-11vw^3+2w^4)uv^3w^3+2(v^2+vw+w^2)^2v^4w^4, \quad (2.5)\end{aligned}$$

where $u > 0$, $v > 0$, $w > 0$. Since v, w are symmetric with respect to Q_0, we may assume that $v \geqslant w$ and let

$$v = w + m, \tag{2.6}$$

where $m \geqslant 0$. Substituting (2.6) into (2.5) and making use of Maple, we get the following identity:

$$Q_0 = 2(Q_1 + wQ_2)m^3 + 2Q_3(u+w)^2, \tag{2.7}$$

where

$$\begin{aligned}
Q_1 = {} & (u^4 + 4u^3w + 134u^2w^2 + 4uw^3 + w^4)m^5 + (6u^5 + 66u^4w + 876u^3w^2 + 1332u^2w^3 \\
& + 14uw^4 + 10w^5)m^4 + (17u^4 + 2619u^2w^2 + 7156uw^3 + 5447w^4)m^3u^2 \\
& + 2(15u^4 + 273u^3w + 2255u^2w^2 + 8137uw^3 + 12069w^4)m^2u^3 \\
& + 4u^3(168u^4w + 9u^5 + 1280u^3w^2 + 10982uw^4 + 10776w^5 + 5216u^2w^3)m \\
& + 2(u+w)(15u^5 + 258u^4w + 1802u^3w^2 + 7130u^2w^3 + 15720uw^4 + 16774w^5)u^3, \\
Q_2 = {} & 3(90u^5 - 22uw^4 + 15w^5)m^3 + 2(5867u^2 - 259uw + 59w^2)m^2w^4 \\
& + 4(3520u^2 - 352uw + 49w^2)mw^5 + 2(u+w)(5654u^2 - 1122uw + 105w^2)w^5, \\
Q_3 = {} & (17u^8 + 256u^7w + 1704u^6w^2 + 6696u^5w^3 + 15626u^4w^4 + 18160u^3w^5 \\
& + 6448u^2w^6 - 1944uw^7 + 141w^8)m^2 + 2(u^4 + 4u^3w + 18u^2w^2 + 28uw^3 \\
& - 3w^4)(3u^4 + 24u^3w + 78u^2w^2 + 64uw^3 - 9w^4)(u+w)m + (u^4 + 4u^3w \\
& + 18u^2w^2 + 28uw^3 - 3w^4)^2(u+w)^2.
\end{aligned}$$

Obviously, inequality $Q_1 > 0$ holds strictly for $u, v, w > 0$ and $m \geqslant 0$. By using the monotonicity property of the function, it is easy to prove that $90x^5 - 22x + 15 > 0$ holds for $x > 0$. Taking $x = u/w$, it follows that $90u^5 - 22uw^4 + 15w^5 > 0$. Again, note that $5867u^2 - 259uw + 59w^2 > 0$, $3520u^2 - 352uw + 49w^2 > 0$ and $5654u^2 - 1122uw + 105w^2 > 0$, hence we see that $Q_2 > 0$ also holds strictly.

Next, we prove $Q_3 \geqslant 0$. Q_3 is a quadratic function for m and its constant term is nonnegative, to show its quadratic term is positive it suffices to show that

$$18160u^3w^5 + 6448u^2w^6 - 1944uw^7 + 141w^8 > 0, \tag{2.8}$$

which can be proved by the method to prove $90u^5 - 22uw^4 + 15w^5 > 0$ as above (we omit the details here). So, it remains to prove that the discriminant F_m of Q_3 is less than or equal to zero. Through the calculations using Maple software, we easily get

$$\begin{aligned}
F_m = {} & -16(2u^8 + 28u^7w + 165u^6w^2 + 642u^5w^3 + 1631u^4w^4 + 2152u^3w^5 + 939u^2w^6 \\
& - 198uw^7 + 15w^8)(u^4 + 4u^3w + 18u^2w^2 + 28uw^3 - 3w^4)^2(u+w)^2.
\end{aligned} \tag{2.9}$$

Since $939u^2w^6 - 198uw^7 + 15w^8 = 3w^6(313u^2 - 66uw + 5w^2) > 0$, $F_m \leqslant 0$ follows and inequality $Q_3 \geqslant 0$ is proved.

By $Q_1 > 0$, $Q_2 > 0$, $Q_3 \geqslant 0$ and the identity (2.7), we know that $Q_0 \geqslant 0$ holds true. Also, the equality in $Q_0 \geqslant 0$ occurs only when $m = 0$ and $Q_3 = 0$. Since also the equality in $Q_3 \geqslant 0$ holds if and only if $F_m = 0$ and

$$2(17u^8+256u^7w+1704u^6w^2+6696u^5w^3+15626u^4w^4+18160u^3w^5+6448u^2w^6 \\ -1944uw^7+141w^8)m+2(u^4+4u^3w+18u^2w^2+28uw^3-3w^4)(3u^4+24u^3w \\ +78u^2w^2+64uw^3-9w^4)(u+w)=0. \tag{2.10}$$

So, the equality in $Q_0 \geqslant 0$ holds if and only if $m=0$, $F_m=0$ and (2.10) holds. From $F_m=0$, we conclude that

$$u^4+4u^3w+18u^2w^2+28uw^3-3w^4=0, \tag{2.11}$$

which is equivalent to

$$(b+c-a)^4+4(b+c-a)^3(a+b-c)+18(b+c-a)^2(a+b-c)^2 \\ +28(b+c-a)(a+b-c)^3-3(a+b-c)^4=0. \tag{2.12}$$

Note that $m=0$ and (2.11) yields (2.10), while $m=0$ means that $v=w$ and then $b=c$, hence the equality in $Q_0 \geqslant 0$ holds if and only if $b=c$ and (2.12) is valid. Using $b=c$ in (2.12) and simplifying, we obtain

$$a^4-3a^2b^2-b^4=0,$$

and then $a^2=\frac{3+\sqrt{13}}{2}b^2$. Therefore, equality in (2.4) holds if and only if $a:b:c=\sqrt{(3+\sqrt{13})/2}:1:1$. This completes the proof of Lemma 2. □

LEMMA 3. *Let $0<t\leqslant 4$ be a real number, then for any triangle ABC with the sides a,b,c and the area S, we have*

$$f_0 \equiv a_0t^2+b_0t+c_0 \geqslant 0, \tag{2.13}$$

where

$$\begin{aligned} a_0 &= 16(2b^2c^2+2c^2a^2+2a^2b^2-a^4-2b^4-2c^4)S^2, \\ b_0 &= -4a^8+20(b^2+c^2)a^6-12(3b^4+2b^2c^2+3c^4)a^4+4(b^2+c^2)(7b^4-12b^2c^2 \\ &\qquad +7c^4)a^2-8(b^2+bc-c^2)(b^2-bc-c^2)(b^4-b^2c^2+c^4), \\ c_0 &= 64(4b^2c^2+4c^2a^2+4a^2b^2-a^4-b^4-c^4)S^2. \end{aligned}$$

If $a_0=0$, then the equality in $f_0\geqslant 0$ holds if and only if $t=4$, $A=\pi/2$ and $b^2+c^2-\sqrt{6}bc=0$. If $a_0\neq 0$, then the equality in $f_0\geqslant 0$ holds if and only if $t=4$ and $A=\pi/2$.

Proof. We consider three cases: $a_0=0$, $a_0>0$ and $a_0<0$. Let

$$k_0=2b^2c^2+2c^2a^2+2a^2b^2-a^4-2b^4-2c^4, \tag{2.14}$$

then we obviously have $k_0 = 0$, $k_0 > 0$ and $k_0 < 0$ under the above three cases, respectively.

Case 1. $a_0 = 0$.

In this case, it follows from the hypothesis that

$$a^4 - 2(b^2 + c^2)a^2 + 2(b^4 - b^2c^2 + c^4) = 0, \tag{2.15}$$

and we have to prove

$$b_0 t + c_0 \geqslant 0. \tag{2.16}$$

From the known equivalent form of Heron's formula:

$$16S^2 = 2b^2c^2 + 2a^2b^2 + 2c^2a^2 - a^4 - b^4 - c^4, \tag{2.17}$$

where S is the area of triangle ABC, we see that $c_0 > 0$. Hence, if $b_0 > 0$ then (2.16) holds for all positive numbers t. If $b_0 \leqslant 0$, then we obviously need to prove the case $t = 4$, i.e.

$$4b_0 + c_0 \geqslant 0. \tag{2.18}$$

Using the expressions of b_0 and c_0, it is easy to verify the following identity:

$$4b_0 + c_0 = 4(m_1 k_0 + n_1), \tag{2.19}$$

where

$$m_1 = 3a^4 - 8(b^2 + c^2)a^2 + 4(b^2 - bc - c^2)(b^2 + bc - c^2),$$
$$n_1 = (4b^2c^2 - b^4 - c^4)(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - b^4 - c^4).$$

By the hypothesis $a_0 = 0$ we have $k_0 = 0$. So, it follows from (2.19) that $4b_0 + c_0 = 4n_1$. Thus, we need to prove $n_1 \geqslant 0$. Since

$$2b^2c^2 + 2c^2a^2 + 2a^2b^2 - b^4 - c^4 > 0,$$

which is clear by (2.17). Hence, it remains to show that

$$4b^2c^2 - b^4 - c^4 \geqslant 0. \tag{2.20}$$

Noting that the quadratic equation (2.15) has real roots for a^2, the discriminant of it is greater than or equal to zero, i.e.

$$4(b^2 + c^2)^2 - 8(b^4 - b^2c^2 + c^4) \geqslant 0,$$

which is equivalent with the required (2.20). Thus, inequality (2.16) is proved and its equality occurs only when $t = 4$ and $n_1 = 0$. Again, $n_1 = 0$ yields

$$4b^2c^2 - b^4 - c^4 = 0, \tag{2.21}$$

which is equivalent to

$$b^2 + c^2 - \sqrt{6}bc = 0. \tag{2.22}$$

Solving the quadratic equation (2.15) in a^2 and using (2.21), we obtain $a^2 = b^2 + c^2$ which means $A = \pi/2$. Therefore, the equality in (2.16) holds if and only if $t = 4$, $A = \pi/2$ and (2.22) is valid. This completes the proof under the first case.

Case 2. $a_0 > 0$.

In this case, the parabola $f_0 = a_0 t^2 + b_0 t + c_0$ opens up. The discriminant of f_0 is

$$F_0 = b_0^2 - 4a_0 c_0. \tag{2.23}$$

We will prove that

$$F_0 \geqslant 0. \tag{2.24}$$

Using the expressions each of a_0, b_0, c_0 and making use of Maple, it is easy to obtain the following identity:

$$F_0 = 32 m_0 Q_0, \tag{2.25}$$

where

$$m_0 = a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2,$$

and Q_0 is the same as in Lemma 2. By (2.25), the inequality $Q_0 \geqslant 0$ of Lemma 2 and the simple inequality $m_0 \geqslant 0$, we conclude that (2.24) is true.

If $F_0 = 0$, then $f_0 \geqslant 0$ holds for any real number t under the hypothesis $a_0 > 0$. Also, it is not difficult to know that the equality in $f_0 \geqslant 0$ holds only when two cases occur, i.e. $a : b : c = 1 : 1 : 1$ and $t = 6$ or $a : b : c = \sqrt{(3+\sqrt{13})/2} : 1 : 1$ and $t = 8 + 2\sqrt{3}$. Therefore, if $a_0 > 0$, $F_0 = 0$ and $0 < t \leqslant 4$, then there is no equality in $f_0 \geqslant 0$, i.e. $f_0 > 0$ holds strictly on the interval $(0, 4]$.

If $F_0 > 0$, then there exist two crossover points $T_1(t_1, 0)$ and $T_2(t_2, 0)$ between the parabola f_0 and t axis (see Figure 1). Assuming that $t_2 > t_1$, then we have

$$t_1 = \frac{-b_0 - \sqrt{F_0}}{2a_0}, \tag{2.26}$$

$$t_2 = \frac{-b_0 + \sqrt{F_0}}{2a_0}. \tag{2.27}$$

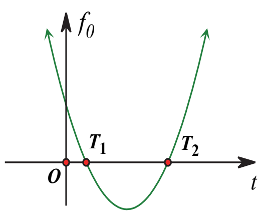

Figure 1

We will prove that two points T_1 and T_2 lie on the positive t axis (see Figure 1). For this, we have to prove $t_1 > 0$, namely,

$$-b_0 > \sqrt{F_0}. \tag{2.28}$$

We first show that

$$-b_0 > 0. \tag{2.29}$$

In fact, $-b_0$ can be written as follow:

$$-b_0 = 4(m_2k_0 + n_2), \tag{2.30}$$

where

$$m_2 = 3(b^2+c^2)a^2 + 4b^2c^2 - a^4 - b^4 - c^4,$$
$$n_2 = (b^2+c^2)(b-c)^2(b+c)^2a^2 + 2b^2c^2(b^4 - b^2c^2 + c^4).$$

By (2.17), we see that $m_2 > 0$. Since also $n_2 > 0$ and $k_0 > 0$ from the hypothesis, hence $-b_0 > 0$ follows from (2.30) at once. We now prove inequality (2.28). By (2.29), we need to prove $b_0^2 - F_0 > 0$. It is easy to verify that

$$b_0^2 - F_0 = 4096k_0n_0S^4, \tag{2.31}$$

where

$$n_0 = 4b^2c^2 + 4c^2a^2 + 4a^2b^2 - a^4 - b^4 - c^4.$$

Hence $b_0^2 - F_0 > 0$ holds since we have $k_0 > 0$ and $n_0 > 0$ by (2.17). The inequality (2.28) is proved.

The double inequality $t_2 > t_1 > 0$ shows that f_0 is greater than or equal to zero when t lies in the interval $(0, t_1]$ (see Figure 1), namely $f_0 \geqslant 0$ holds for $0 < t \leqslant t_1$, and the equality holds if and only if $t = t_1$. Thus, to prove $f_0 \geqslant 0$ for $0 < t \leqslant 4$ we have to prove that

$$t_1 \geqslant 4. \tag{2.32}$$

By (2.26) and the hypothesis, it is equivalent to

$$-(8a_0 + b_0) \geqslant \sqrt{F_0}. \tag{2.33}$$

But, we have the following identity:

$$-(8a_0 + b_0) = 4(k_0m_3 + n_3), \tag{2.34}$$

where

$$m_3 = a^4 + b^4 + c^4 - (b^2+c^2)a^2,$$
$$n_3 = a^2(b^2+c^2)(b+c)^2(b-c)^2 + 2b^2c^2(b^4 - b^2c^2 + c^4).$$

Clearly, $m_3 > 0$ and $n_3 > 0$ are valid, hence $-(8a_0 + b_0) > 0$ holds and the proof of (2.33) turns to

$$(8a_0 + b_0)^2 - F_0 \geqslant 0. \tag{2.35}$$

Again, using Maple software we easily obtain that

$$(8a_0 + b_0)^2 - F_0 = 256k_0(a^4 + b^4 + c^4)(b^2 + c^2 - a^2)^2S^2, \tag{2.36}$$

which shows that (2.35) holds if $k_0 > 0$. So, inequality (2.33) is proved, and its equality holds only when $b^2+c^2-a^2=0$, i.e. $A=\pi/2$. Further, it is easily seen that if $a_0>0$ and $F_0>0$ then the equality in $f_0 \geqslant 0$ holds if and only if $t=4$ and $A=\pi/2$.

Combining the above arguments under the two cases, $F_0=0$ and $F_0>0$. We know that if $a_0>0$ then $f_0 \geqslant 0$ holds for $0<t\leqslant 4$ and the equality in $f_0 \geqslant 0$ holds if and only if $t=4$ and $A=\pi/2$.

Case 3. $a_0<0$.

In this case, the parabola $f_0=a_0t^2+b_0t+c_0$ opens down. Also, it is easy to know that inequality (2.24) holds strictly (we can show that the case $F_0=0$ is incompatible with $a_0<0$). Therefore, there exist two crossover points $T_1(t_1,0)$ and $T_2(t_2,0)$ between the parabola f_0 and t axis. We will show that the crossover points $T_1(t_1,0)$ lies on the positive t axis and $T_2(t_2,0)$ lies on the non-negative t axis (see Figure 2). It is sufficient to prove the double inequality:

$$t_1>0\geqslant t_2. \tag{2.37}$$

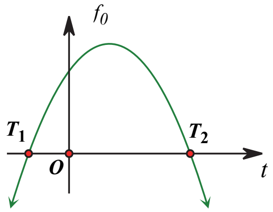

Figure 2

By (2.26) and the hypothesis, inequality $t_1>0$ is equivalent with

$$\sqrt{F_0}>-b_0. \tag{2.38}$$

If $b_0>0$, then (2.38) holds tritely. If $b_0\leqslant 0$, then we need to prove $F_0-b_0^2>0$. Under the hypothesis we have $k_0<0$, hence $F_0-b_0^2>0$ follows from (2.31) and $t_1>0$ is proved. In the same way, we can prove the second inequality $t_2\leqslant 0$ in (2.37).

The double inequality (2.37) shows that $f_0\geqslant 0$ holds if t is on the interval $(0,t_2]$. Thus, to prove $f_0\geqslant 0$ for $0<t\leqslant 4$, we need to prove $t_2\geqslant 4$. By (2.27) and the hypothesis, inequality $t_2\geqslant 4$ is equivalent to

$$\sqrt{F_0}\geqslant 8a_0+b_0. \tag{2.39}$$

Obviously, is suffices to prove for $8a_0+b_0>0$. So, the inequality to be proved is equivalent with $F_0-(8a_0+b_0)^2\geqslant 0$, which is obtained by identity (2.36) and $k_0<0$ from the hypothesis $a_0<0$. Hence (2.39) is proved and it is easily known that the equality in $f_0\geqslant 0$ holds if and only if $t=4$ and $A=\pi/2$ under the third case.

Combining the arguments of the three cases above, Lemma 3 is proved. □

LEMMA 4. *Let p_1, p_2, p_3, q_1, q_2, q_3 be real numbers such that $p_1>0$, $p_2>0$, $p_3>0$, $4p_2p_3-q_1^2>0$, $4p_3p_1-q_2^2>0$, $4p_1p_2-q_3^2>0$ and*

$$D_0\equiv 4p_1p_2p_3-(q_1q_2q_3+p_1q_1^2+p_2q_2^2+p_3q_3^2)\geqslant 0. \tag{2.40}$$

Then the inequality

$$p_1x^2+p_2y^2+p_3z^2 \geqslant yzq_1+zxq_2+xyq_3 \tag{2.41}$$

holds for all real numbers x,y,z. *If* $x,y,z \neq 0$, *then the equality in (2.41) holds if and only if* $D_0=0$ *and*

$$(2p_1q_1+q_2q_3)x=(2p_2q_2+q_3q_1)y=(2p_3q_3+q_1q_2)z. \tag{2.42}$$

Proof. Putting

$$E_0=p_1x^2+p_2y^2+p_3z^2-(yzq_1+zxq_2+xyq_3),$$

it is easy to verify the following identity:

$$E_0=\frac{(2p_1x-q_2y-q_3z)^2}{4p_1}+\frac{4p_1p_2-q_3^2}{4p_1}\left(y-\frac{2p_1q_1+q_2q_3}{4p_1p_2-q_3^2}z\right)^2+\frac{D_0x^2}{4p_2p_3-q_1^2}. \tag{2.43}$$

Using (2.43) and its two analogues and adding them, we get

$$3E_0=E_1+E_2+E_3, \tag{2.44}$$

where

$$\begin{aligned}
E_1&=\frac{(2p_1x-q_2z-q_3y)^2}{4p_1}+\frac{(2p_2y-q_3x-q_1z)^2}{4p_2}+\frac{(2p_3z-q_1y-q_2x)^2}{4p_3}\\
E_2&=\frac{4p_1p_2-q_3^2}{4p_1}\left(y-\frac{2p_1q_1+q_2q_3}{4p_1p_2-q_3^2}z\right)^2+\frac{4p_2p_3-q_1^2}{4p_2}\left(z-\frac{2p_2q_2+q_3q_1}{4p_2p_3-q_1^2}x\right)^2\\
&\quad+\frac{4p_3p_1-q_2^2}{4p_3}\left(x-\frac{2p_3q_3+q_1q_2}{4p_3p_1-q_2^2}y\right)^2,\\
E_3&=D_0\left(\frac{x^2}{4p_2p_3-q_1^2}+\frac{y^2}{4p_3p_1-q_2^2}+\frac{z^2}{4p_1p_2-q_3^2}\right).
\end{aligned}$$

By identity (2.44) and the given conditions, inequality $E_0 \geqslant 0$ is proved.

We now discuss the equality conditions of (2.41). If $x,y,z \neq 0$, from identity (2.44) we see that the equality in (2.41) holds if and only if $E_1=E_2=E_3=0$. By $E_3=0$ we conclude that $D_0=0$. By $E_2=0$ we have

$$\begin{cases}
y-\dfrac{2p_1q_1+q_2q_3}{4p_1p_2-q_3^2}z=0,\\
z-\dfrac{2p_2q_2+q_3q_1}{4p_2p_3-q_1^2}x=0,\\
x-\dfrac{2p_3q_3+q_1q_2}{4p_3p_1-q_2^2}y=0.
\end{cases} \tag{2.45}$$

Using (2.45) and $D_0 = 0$, we get

$$\begin{aligned}
&2p_1x - q_2z - q_3y \\
&\quad = 2p_1 \cdot \frac{(2p_3q_3 + q_1q_2)y}{4p_3p_1 - q_2^2} - q_2 \cdot \frac{(4p_1p_2 - q_3^2)y}{2p_1q_1 + q_2q_3} - q_3y \\
&\quad = \frac{-4yp_1q_2(4p_1p_2p_3 - q_1q_2q_3 - p_1q_1^2 - p_2q_2^2 - p_3q_3^2)}{(4p_3p_1 - q_2^22)(2p_1q_1 + q_2q_3)} \\
&\quad = 0.
\end{aligned}$$

Similarly, we obtain $2p_2y - q_3x - q_1z = 0$ and $2p_3z - q_1y - q_2x = 0$. Thus, we have $E_1 = 0$ by (2.45) and $D_0 = 0$. On the other hand, by $D_0 = 0$ and (2.45) one has

$$\begin{aligned}
&(2p_1q_1 + q_2q_3)x - (2p_2q_2 + q_3q_1)y \\
&\quad = (2p_1q_1 + q_2q_3) \cdot \frac{(2p_3q_3 + q_1q_2)y}{4p_3p_1 - q_2^2} - (2p_2q_2 + q_3q_1)y \\
&\quad = \frac{-2yq_2(4p_1p_2p_3 - q_1q_2q_3 - p_1q_1^2 - p_2q_2^2 - p_3q_3^2)}{(4p_3p_1 - q_2^2)} \\
&\quad = 0.
\end{aligned}$$

whence $(2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y$. Similarly, we get $(2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z$ and then the continuous identity (2.42) is obtained.

Combing the above arguments, we know that if $x,y,z \neq 0$ then the equality in (2.41) holds if and only if $D_0 = 0$ and (2.42) is valid. This completes the proof of Lemma 4.

REMARK 1. Lemma 4 gives a sufficient condition of the quadratic inequality (2.41) holding for all real numbers x,y,z. In fact, using other methods we can prove that the necessary and sufficient condition of inequality (2.41) holding for all real numbers x,y,z is $p_1 \geqslant 0$, $p_2 \geqslant 0$, $p_3 \geqslant 0$, $4p_2p_3 - q_1^2 \geqslant 0$, $4p_3p_1 - q_2^2 \geqslant 0$, $4p_1p_2 - q_3^2 \geqslant 0$ and $D_0 \geqslant 0$.

REMARK 2. As an application of Lemma 4, the previous Wolstenholme inequality (2.1) can be derived readily. Some other applications, see [14], [15], [22].

3. Proof of Theorem 1

For simplicity, we denote by $\sum$ cyclic sums in the next proof.

Proof. Clearly, the inequality (1.8) is equivalent to

$$\sum R_1^2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \lambda \sum \frac{r_1^2}{a^2} \geqslant \frac{(\lambda + 8)}{4}. \tag{3.1}$$

For the distances R_1 and r_1, we have the following known formulae (see, e.g. [6], [7], [17]):

$$(x + y + z)^2R_1^2 = (x + y + z)(yc^2 + zb^2) - (yza^2 + zxb^2 + xyc^2) \tag{3.2}$$

and

$$r_1 = \left| \frac{2xS}{(x+y+z)a} \right|, \tag{3.3}$$

where x,y,z are real numbers and related real triple (x,y,z) denotes the barycentric coordinates of the point P with respect to $\triangle ABC$. For R_2, R_3 and r_1, r_2, we also have similar formulae.

By (3.2) and (3.3), we know inequality (3.1) is equivalent to

$$\sum \left[\frac{yc^2+zb^2}{x+y+z} - \frac{yza^2+zxb^2+xyc^2}{(x+y+z)^2} \right] \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{4\lambda S^2}{(x+y+z)^2} \sum \frac{x^2}{a^4} \geqslant \frac{(\lambda+8)}{4},$$

namely,

$$\frac{1}{\sum x} \sum (yc^2+zb^2) \left(\frac{1}{b^2} + \frac{1}{c^2} \right) - \frac{2\sum yza^2}{(\sum x)^2} \sum \frac{1}{a^2} + \frac{4\lambda S^2}{(\sum x)^2} \sum \frac{x^2}{a^4} - \frac{(\lambda+8)}{4} \geqslant 0. \tag{3.4}$$

If we make the substitutions: $x \to xa^2$, $y \to yb^2$, $z \to zc^2$, then (3.4) becomes

$$\frac{\sum (y+z)(b^2+c^2)}{\sum xa^2} - \frac{2\sum yz \sum b^2c^2}{(\sum xa^2)^2} + \frac{4\lambda S^2 \sum x^2}{(\sum xa^2)^2} - \frac{\lambda+8}{4} \geqslant 0. \tag{3.5}$$

Multiplying both sides by $4(\sum xa^2)^2$ and using area formula (2.17), that is

$$\begin{aligned} &4\sum xa^2 \sum (y+z)\left(b^2+c^2\right) - 8\sum yz \sum b^2c^2 + \lambda\left(2\sum b^2c^2 - \sum a^4\right)\sum x^2 \\ &-(\lambda+8)\left(\sum xa^2\right)^2 \geqslant 0. \end{aligned} \tag{3.6}$$

Expanding out, we obtain the following equivalent inequality required to prove:

$$e_1x^2 + e_2y^2 + e_3z^2 - f_1yz - f_2zx - f_3xy \geqslant 0, \tag{3.7}$$

where

$$\begin{aligned} e_1 &= 4a^2(b^2+c^2) + \lambda(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - b^4 - c^4 - 2a^4), \\ e_2 &= 4b^2(c^2+a^2) + \lambda(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - c^4 - a^4 - 2b^4), \\ e_3 &= 4c^2(a^2+b^2) + \lambda(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - 2c^4), \\ f_1 &= 4a^2(b^2+c^2) + 2\lambda b^2c^2 - 4(b+c)^2(b-c)^2, \\ f_2 &= 4b^2(c^2+a^2) + 2\lambda c^2a^2 - 4(c+a)^2(c-a)^2, \\ f_3 &= 4c^2(a^2+b^2) + 2\lambda a^2b^2 - 4(a+b)^2(a-b)^2. \end{aligned}$$

When $\lambda = -2$, inequality (1.8) becomes (1.4), which has been proved by a straightforward way in the first section. So, we now consider two cases $\lambda = 2$ and $-2 < \lambda < 2$ to finish the proof of inequality (3.7) and Theorem 1.

Case 1. $\lambda = 2$.

We will use Lemma 1 to prove this case.

If we let $\lambda = 2$ in (3.7) and divide both sides by 2, then the inequality becomes

$$g_1x^2 + g_2y^2 + g_3z^2 - h_1yz - h_2zx - h_3xy \geqslant 0, \tag{3.8}$$

where

$$\begin{aligned}
g_1 &= -2a^4 + 4(b^2+c^2)a^2 - (b+c)^2(b-c)^2,\\
g_2 &= -2b^4 + 4(c^2+a^2)b^2 - (c+a)^2(c-a)^2,\\
g_3 &= -2c^4 + 4(a^2+b^2)c^2 - (a+b)^2(a-b)^2,\\
h_1 &= 2(b^2c^2+c^2a^2+a^2b^2) - 2(b+c)^2(b-c)^2,\\
h_2 &= 2(b^2c^2+c^2a^2+a^2b^2) - 2(c+a)^2(c-a)^2,\\
h_3 &= 2(b^2c^2+c^2a^2+a^2b^2) - 2(a+b)^2(a-b)^2.
\end{aligned}$$

Next, we are going to prove that $g_1 > 0$, $g_2 > 0$, $g_3 > 0$ and $\sqrt{g_1}$, $\sqrt{g_2}$, $\sqrt{g_3}$ form a triangle. First observe that

$$g_1 = 4(v^2+6vw+w^2)u^2 + 4(v+w)(v^2+6vw+w^2)u + (v-w)^2(v+w)^2, \tag{3.9}$$

where $u = s-a > 0$, $v = s-b > 0$, $w = s-c > 0$. Thus, $g_1 > 0$ holds true. Similarly, we have $g_2 > 0$ and $g_3 > 0$. Let $l_1 = \sqrt{g_1}$, $l_2 = \sqrt{g_2}$, $l_3 = \sqrt{g_3}$, then by Heron's formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)} \tag{3.10}$$

and its equivalent form (2.17), we know that l_1, l_2, l_3 form a triangle $A'B'C'$ if and only if

$$G_0 \equiv 2g_2g_3 + 2g_3g_1 + 2g_1g_2 - g_1^2 - g_2^2 - g_3^2 > 0. \tag{3.11}$$

Using the expressions of g_1, g_2, g_3 and formula (2.17), it is easy to verify that

$$G_0 = 64n_0S^2, \tag{3.12}$$

where n_0 is the same as in (2.31) and $n_0 > 0$. Thus, $G_0 > 0$ holds and the claimed conclusion is proved.

Now, if we apply the Wolstenholme inequality (2.1) to triangle $A'B'C'$ and note that

$$l_2^2 + l_3^2 - l_1^2 = g_2 + g_3 - g_1 = h_1, \tag{3.13}$$
$$l_3^2 + l_1^2 - l_2^2 = g_3 + g_1 - g_2 = h_2, \tag{3.14}$$
$$l_1^2 + l_2^2 - l_3^2 = g_1 + g_2 - g_3 = h_3, \tag{3.15}$$

(which can be verified easily)then we obtain inequality (3.8) immediately, and by Lemma 1 we know its equality holds if and only if $x = y = z$. Hence, we complete the proof of the case $\lambda = 2$ in (3.1), namely, inequality (1.5) is proved. Moreover, it is easily seen that if $\lambda = 2$ then the quality in (3.1) holds if and only if $x : y : z = a^2 : b^2 : c^2$. Therefore, the equality in (1.5) holds if and only if the barycentric coordinates of the point P is (a^2, b^2, c^2), which means that P is the Lhuilier-Lemoine point of $\triangle ABC$. This completes the proof of Theorem 1 under the case $\lambda = 2$.

Case 2. $-2<\lambda<2$.

By Lemma 4, to prove (3.7) for this case, we first prove that

$$4e_2e_3-f_1^2>0. \tag{3.16}$$

After calculating with the help of Maple, one obtains

$$4e_2e_3-f_1^2=4(a_1\lambda^2+b_1\lambda+c_1), \tag{3.17}$$

where

$$\begin{aligned} a_1 &= 16(2b^2c^2+2c^2a^2+2a^2b^2-a^4-2b^4-2c^4)S^2,\\ b_1 &= -4(b^2+c^2)a^6+8(b^2+bc+c^2)(b^2-bc+c^2)a^4\\ &\quad -4(b^2+c^2)(b^4-4b^2c^2+c^4)a^2-8b^2c^2(b^4-b^2c^2+c^4),\\ c_1 &= -4(b+c)^2(b-c)^2a^4+8(b^2+c^2)(b^4+c^4)a^2-4(b^4+c^4)(b^4-4b^2c^2+c^4). \end{aligned}$$

Thus, we need to prove

$$a_1\lambda^2+b_1\lambda+c_1>0. \tag{3.18}$$

Letting $\lambda=2-t$, then by $-2<\lambda<2$ we have $0<t<4$ and (3.18) becomes

$$a_1(2-t)^2+b_1(2-t)+c_1>0,$$

which is equivalent to

$$a_0t^2+b_0t+c_0>0, \tag{3.19}$$

where a_0, b_0, c_0 are the same as in Lemma 3. According to Lemma 3, we know that inequality (3.19) strictly holds for $0<t<4$. Hence, the strict inequalities (3.18) and (3.16) are proved. Clearly, inequalities $4e_3e_1-f_2^2>0$ and $4e_1e_2-f_3^2>0$ similar to (3.16) are also valid.

Now, by Lemma 4, it remains to prove the inequality:

$$K_0\equiv 4e_1e_2e_3-(f_1f_2f_3+e_1f_1^2+e_2f_2^2+e_3f_3^2)\geqslant 0.$$

Applying Maple software and the formula (2.17), it is not difficult to verify the following identity:

$$K_0=512(2-\lambda)(2+\lambda)\left(\sum a^4-\sum b^2c^2\right)\left(\sum b^2c^2+4\lambda S^2\right)S^2. \tag{3.20}$$

By (2.17) and the fact $\sum a^4-\sum b^2c^2\geqslant 0$ with equality if and only if $a=b=c$, we see that $K_0\geqslant 0$ holds for $-2<\lambda<2$ and the equality in $K_0\geqslant 0$ occurs if and only if $a=b=c$, namely $\triangle ABC$ is equilateral. Further, by lemma 4 we know that the equality in (3.7) holds if and only if $a=b=c$ and $x=y=z$. Thus, when $-2<\lambda<2$ the equality in (3.1) holds if and only if $\triangle ABC$ is equilateral and P is its center. This completes the proof under the second case.

Finally, combing the above arguments, the proof of Theorem 1 is completed.

4. Two related conjectures

The inequality (1.4) inspires the author to conjecture the following similar inequality:

$$\frac{R_2+R_3-2r_1}{a}+\frac{R_3+R_1-2r_2}{b}+\frac{R_1+R_2-2r_3}{c}\geqslant\sqrt{3} \tag{4.1}$$

holds for any interior point P of triangle ABC. With the help of the computer for checking, we propose general conjecture:

CONJECTURE 1. Let λ be a real number such that $\frac{3}{5}\leqslant\lambda\leqslant 2$, then for any interior point P of triangle ABC we have

$$\frac{R_2+R_3-\lambda r_1}{a}+\frac{R_3+R_1-\lambda r_2}{b}+\frac{R_1+R_2-\lambda r_3}{c}\geqslant\frac{(4-\lambda)\sqrt{3}}{2}. \tag{4.2}$$

In addition, considering the generalization of Theorem 1, we pose the following conjecture:

CONJECTURE 2. Let k and λ be real numbers such that $k>2$ and $k\geqslant\lambda\geqslant-2$, then for any interior point P of triangle ABC we have

$$\frac{R_2^k+R_3^k+\lambda r_1^k}{a^k}+\frac{R_3^k+R_1^k+\lambda r_2^k}{b^k}+\frac{R_1^k+R_2^k+\lambda r_3^k}{c^k}\geqslant\frac{3\left(2^{k+1}+\lambda\right)}{(2\sqrt{3})^k}. \tag{4.3}$$

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