Journal of Mathematical Inequalities
Volume 8, Number 1 (2014), 107–112 doi:10.7153/jmi-08-06

REFINEMENTS OF THE HERON AND HEINZ MEANS INEQUALITIES FOR MATRICES

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(*Communicated by J. Pečarić*)

Abstract. This article aims to present some unitarily invariant norms inequalities involving Heron and Heinz means for matrices. We give some refinements for the results presented by R. Kaur and M. Singh in [Math. Ineq. Appl., 16 (2013) 93–99].

1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n, i.e., a norm with the property that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. A matrix $A^* \in M_{n,m}$ is called conjugate transpose of $A \in M_{m,n}$. Two classes of such norms, a class of Ky Fan k-norm and a class of Schatten p-norm are especially important. These two classes are defined respectively as

$$\|A\|_{(k)} = \textstyle\sum_{j=1}^{k} s_j(A), \quad k = 1, 2, \ldots, n,$$

and

$$\|A\|_p = (\textstyle\sum_{j=1}^{n} s_j^p(A))^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

where $s_j(A)(j = 1, 2, \ldots, n)$ are singular values of a matrix A with $s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant s_n(A)$, which are the eigenvalues of positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

R. Kaur and M. Singh [4] have proved that for $A, B, X \in M_n$, such that A, B are positive definite, then for any unitarily invariant norm $\|.\|$, $1/4 \leqslant \nu \leqslant 3/4$ and $\alpha \in [1/2, \infty)$, the following inequality holds

$$\frac{1}{2}\|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leqslant \left\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right\|. \tag{1.1}$$

Mathematics subject classification (2010): 47A30, 15A60.
Keywords and phrases: Unitarily invariant norms, Positive definite matrices, Convex function, Heron means, Heinz means.

Paper JMI-08-06

For any unitarily invariant norm $\|.\|$, R. Kaur and M. Singh, also proved the following result in [4]

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \leqslant \frac{1}{2}\left\|A^{\frac{2}{3}}XB^{\frac{1}{3}}+A^{\frac{1}{3}}XB^{\frac{2}{3}}\right\| \leqslant \frac{1}{2+t}\left\|AX+XB+tA^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|, \tag{1.2}$$

where $A,B,X \in M_n$, A,B are positive definite and $-2<t\leqslant 2$.

Obviously, for $A,B,X \in M_n$, such that A,B are positive definite, then for any unitarily invariant norm $\|.\|$, $1/4 \leqslant v \leqslant 3/4$ and $\alpha \in [1/2,\infty)$, the following inequalities hold

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leqslant \frac{1}{2}\|A^vXB^{1-v}+A^{1-v}XB^v\| \leqslant \left\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\Big(\frac{AX+XB}{2}\Big)\right\|, \tag{1.3}$$

for above first inequality (see [1]).

Set

$$g(v)=\left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\|,$$

and

$$f(\alpha)=\left\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\Big(\frac{AX+XB}{2}\Big)\right\|.$$

Then, the inequalities (1.1), (1.2), (1.3), can be simply rewritten as

$$g(v)\leqslant f(\alpha), \tag{1.4}$$

$$g\Big(\frac{1}{2}\Big)\leqslant g\Big(\frac{2}{3}\Big)\leqslant f\Big(\frac{2}{2+t}\Big), \tag{1.5}$$

and

$$g\Big(\frac{1}{2}\Big)\leqslant g(v)\leqslant f(\alpha), \tag{1.6}$$

respectively.

In Section 2, we give the refinements of the inequalities (1.4), the second inequality in (1.5), $g(\frac{1}{2})\leqslant f(\alpha)$ and $g(\frac{1}{2})\leqslant f(\frac{2}{2+t})$, respectively.

2. Main results

First, we give a refinement of the inequality (1.4). The function $g(v)$ is a continuous convex function on $[0,1]$ and attains its minimum at $v=\frac{1}{2}$ (see [2, p. 265]). We utilize the convexity of the function $g(v)$ to obtain unitarily invariant norms inequality that leads to a refinement of the inequality (1.4).

To do this, we need the following Lemma on convex function [3, 5].

LEMMA 2.1. *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2). Then for $x_1 \leqslant x \leqslant x_2$, we have*

$$f(x) \leqslant \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2-x_1}.$$

THEOREM 2.2. *Let $A,B,X \in M_n$, with A,B positive definite. Then for any unitarily invariant norm $\|.\|$, $1/4 \leqslant v \leqslant 3/4$ and $\alpha \in [1/2,\infty)$,*

$$g(v) \leqslant (4r_0-1)g\left(\frac{1}{2}\right) + 2(1-2r_0)f(\alpha), \tag{2.1}$$

where $g(v) = \|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\|$, $f(\alpha) = \|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha(\frac{AX+XB}{2})\|$ *and* $r_0 = \min[v,1-v]$.

Proof. For $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, since $g(v)$ is a convex function then by Lemma 2.1, we have

$$g(v) \leqslant \frac{g(\frac{1}{2})-g(\frac{1}{4})}{\frac{1}{2}-\frac{1}{4}}v - \frac{\frac{1}{4}g(\frac{1}{2})-\frac{1}{2}g(\frac{1}{4})}{\frac{1}{2}-\frac{1}{4}},$$

i.e.,

$$g(v) \leqslant 2(1-2v)g\left(\frac{1}{4}\right) + (4v-1)g\left(\frac{1}{2}\right). \tag{2.2}$$

By (1.4) and (2.2), we have

$$g(v) \leqslant (4v-1)g\left(\frac{1}{2}\right) + 2(1-2v)f(\alpha).$$

So,

$$g(v) \leqslant (4r_0-1)g\left(\frac{1}{2}\right) + 2(1-2r_0)f(\alpha).$$

Similarly, for $\frac{1}{2} \leqslant v \leqslant \frac{3}{4}$, we have

$$g(v) \leqslant \frac{g(\frac{3}{4})-g(\frac{1}{2})}{\frac{3}{4}-\frac{1}{2}}v - \frac{\frac{1}{2}g(\frac{3}{4})-\frac{3}{4}g(\frac{1}{2})}{\frac{3}{4}-\frac{1}{2}},$$

i.e.,

$$g(v) \leqslant (4v-2)g\left(\frac{3}{4}\right) + (3-4v)g\left(\frac{1}{2}\right). \tag{2.3}$$

By (1.4) and (2.3), we have

$$g(v) \leqslant (3-4v)g\left(\frac{1}{2}\right) + (4v-2)f(\alpha).$$

Which is equivalent to the following inequality,

$$g(v) \leqslant (4r_0-1)g\left(\frac{1}{2}\right) + 2(1-2r_0)f(\alpha).$$

The proof is completed. □

REMARK 2.3. We give a comparison between the upper bounds in (1.4) and (2.1).

$$\begin{aligned} f(\alpha)-(4r_0-1)g\Big(\frac{1}{2}\Big)-2(1-2r_0)f(\alpha) &= (4r_0-1)f(\alpha)-(4r_0-1)g\Big(\frac{1}{2}\Big) \\ &\geqslant (4r_0-1)f(\alpha)-(4r_0-1)f(\alpha) \\ &= 0. \end{aligned}$$

Our following result is a refinement of the second inequality in (1.5).

COROLLARY 2.4. *Let $A,B,X\in M_n$, with A,B positive definite. Then for any unitarily invariant norm $\|.\|$, and $-2<t\leqslant 2$, we have*

$$g\Big(\frac{2}{3}\Big)\leqslant (4r_0-1)g\Big(\frac{1}{2}\Big)+2(1-2r_0)f\Big(\frac{2}{2+t}\Big),$$

where $g(\frac{2}{3})=\frac{1}{2}\|A^{\frac{2}{3}}XB^{\frac{1}{3}}+A^{\frac{1}{3}}XB^{\frac{2}{3}}\|$, $f(\frac{2}{2+t})=\frac{1}{2+t}\|AX+XB+tA^{\frac{1}{2}}XB^{\frac{1}{2}}\|$, $g(\frac{1}{2})=\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|$ *and* $r_0=\min[v,1-v]$.

Proof. By taking $v=\frac{2}{3}$ and $\alpha=\frac{2}{2+t}$ in Theorem 2.2, we get the desired result. □

Now, we give a refinement of the inequality $g(\frac{1}{2})\leqslant f(\alpha)$ in (1.6).

THEOREM 2.5. *Let $A,B,X\in M_n$, with A,B positive definite. Then for any unitarily invariant norm $\|.\|$, $1/4\leqslant v\leqslant 3/4$ and $\alpha\in[1/2,\infty)$,*

$$g\Big(\frac{1}{2}\Big)+2\left(2\int_{\frac{1}{4}}^{\frac{3}{4}}g(v)dv-g\Big(\frac{1}{2}\Big)\right)\leqslant f(\alpha), \tag{2.4}$$

where $g(v)=\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\|$ *and* $f(\alpha)=\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha(\frac{AX+XB}{2})\|$.

Proof. For $\frac{1}{4}\leqslant v\leqslant\frac{1}{2}$, from Theorem 2.2, we have

$$g(v)\leqslant(4v-1)g\Big(\frac{1}{2}\Big)+2(1-2v)f(\alpha).$$

Thus

$$\int_{\frac{1}{4}}^{\frac{1}{2}}g(v)dv\leqslant g\Big(\frac{1}{2}\Big)\int_{\frac{1}{4}}^{\frac{1}{2}}(4v-1)dv+2f(\alpha)\int_{\frac{1}{4}}^{\frac{1}{2}}(1-2v)dv,$$

which implies that

$$\int_{\frac{1}{4}}^{\frac{1}{2}}g(v)dv\leqslant\frac{1}{8}g\Big(\frac{1}{2}\Big)+\frac{1}{8}f(\alpha). \tag{2.5}$$

For $\frac{1}{2}\leqslant v\leqslant\frac{3}{4}$, from Theorem 2.2, we have

$$g(v)\leqslant(3-4v)g\Big(\frac{1}{2}\Big)+2(2v-1)f(\alpha).$$

Thus

$$\int_{\frac{1}{2}}^{\frac{3}{4}} g(v)dv \leqslant g\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^{\frac{3}{4}} (3-4v)dv + 2f(\alpha) \int_{\frac{1}{2}}^{\frac{3}{4}} (2v-1)dv,$$

which implies that

$$\int_{\frac{1}{2}}^{\frac{3}{4}} g(v)dv \leqslant \frac{1}{8} g\left(\frac{1}{2}\right) + \frac{1}{8} f(\alpha). \tag{2.6}$$

By (2.5) and (2.6), we have

$$4\int_{\frac{1}{4}}^{\frac{3}{4}} g(v)dv \leqslant g\left(\frac{1}{2}\right) + f(\alpha).$$

So,

$$g\left(\frac{1}{2}\right) + 2\left(2\int_{\frac{1}{4}}^{\frac{3}{4}} g(v)dv - g\left(\frac{1}{2}\right)\right) \leqslant f(\alpha).$$

The proof is completed. □

REMARK 2.6. Obviously, $\left(2\int_{\frac{1}{4}}^{\frac{3}{4}} g(v)dv - g(\frac{1}{2})\right) \geqslant 0$, so, Theorem 2.5 is a refinement of the inequality $g(\frac{1}{2}) \leqslant f(\alpha)$ in (1.6).

Our following result is a refinement of the inequality $g(\frac{1}{2}) \leqslant f(\frac{2}{2+t})$ in (1.5).

COROLLARY 2.7. *Let $A,B,X \in M_n$, with A,B positive definite. Then for any unitarily invariant norm $\|.\|$, and $-2 < t \leqslant 2$, we have*

$$g\left(\frac{1}{2}\right) + 2\left(2\int_{\frac{1}{4}}^{\frac{3}{4}} g\left(\frac{2}{3}\right) dv - g\left(\frac{1}{2}\right)\right) \leqslant f\left(\frac{2}{2+t}\right),$$

where $g(\frac{2}{3}) = \frac{1}{2}\|A^{\frac{2}{3}}XB^{\frac{1}{3}} + A^{\frac{1}{3}}XB^{\frac{2}{3}}\|$, $f(\frac{2}{2+t}) = \frac{1}{2+t}\|AX + XB + tA^{\frac{1}{2}}XB^{\frac{1}{2}}\|$ *and* $g(\frac{1}{2}) = \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|$.

Proof. By taking $v = \frac{2}{3}$ and $\alpha = \frac{2}{2+t}$ in Theorem 2.5, we get the desired result. □

Acknowledgements. The authors would like to thank the referees for their valuable comments and suggestions for revising the manuscript. This work was supported by the National Natural Science Foundation of China (No. 11171361).

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(Received October 20, 2012)

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