ON $k$–QUASI–PARANORMAL OPERATORS

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Abstract. For a positive integer $k$, an operator $T \in B(H)$ is called $k$-quasi-paranormal if
\[ \|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\| \]
for all $x \in H$, which is a common generalization of paranormal and quasi-paranormal. In this paper, firstly we prove some inequalities of this class of operators; secondly we give a necessary and sufficient condition for $T$ to be $k$-quasi-paranormal. Using these results, we prove that: (1) if $\|T_n+1\| = \|T\|_n+1$ for some positive integer $n \geq k$, then a $k$-quasi-paranormal operator $T$ is normaloid; (2) if $E$ is the Riesz idempotent for an isolated point $\lambda_0$ of the spectrum of a $k$-quasi-paranormal operator $T$, then (i) if $\lambda_0 \neq 0$, then $E = \ker(T-\lambda_0)$; (ii) if $\lambda_0 = 0$, then $E = \ker(T^{k+1})$.

1. Introduction

Throughout this paper let $H$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on $H$. An operator is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in H$. In order to discuss the relations between paranormal and $p$-hyponormal and log-hyponormal operators, Furuta, Ito and Yamazaki [4] introduced class A operators defined by $|T^2| - |T|^2 \geq 0$, and they showed that class A is a subclass of paranormal and contains $p$-hyponormal and log-hyponormal operators.

Let $T \in B(H)$ and $\lambda_0$ be an isolated point of $\sigma(T)$. Here $\sigma(T)$ denotes the spectrum of $T$. Then there exists a small enough positive number $r > 0$ such that
\[ \{ \lambda \in C : |\lambda - \lambda_0| \leq r \} \cap \sigma(T) = \{ \lambda_0 \}. \]
Let
\[ E = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = r} (\lambda - T)^{-1} d\lambda. \]

$E$ is called the Riesz idempotent with respect to $\lambda_0$. Stampfl [10] proved that if $T$ is hyponormal(i.e., operators such that $T^*T - TT^* \geq 0$), then
\[ E \text{ is self-adjoint and } E = \ker(T - \lambda_0) = \ker((T - \lambda_0)^*). \]
After that many authors extended this result to many other classes of operators. Chô and Tanahashi [1] proved that (1.1) holds if $T$ is either $p$-hyponormal or log-hyponormal. In the case $\lambda_0 \neq 0$, the result was further shown by Tanahashi and Uchiyama [11] to hold for $p$-quasihyponormal operators, by Tanahashi, Uchiyama and Chô [12] to hold for $(p, k)$-quasihyponormal operators and by Uchiyama and Tanahashi [14, 15] for class $A$ and paranormal operators.

In this paper, we shall study the Riesz idempotent with respect to an isolated point of the spectrum of $k$-quasi-paranormal operators.

**DEFINITION 1.1.** $T \in B(H)$ is called $k$-quasi-paranormal if for a positive integer $k$

$$\|T^{k+1}x\|^2 \leq \|T^{k+2}x\|\|T^kx\|$$

for all $x \in H$.

When $k = 1$, $T$ is called quasi-paranormal operators. quasi-paranormal and $k$-quasi-paranormal operators have been studied in [6, 9, 18].

It is clear that

the class of paranormal operators $\subseteq$ the class of quasi-paranormal operators $\subseteq$ the class of $k$-quasi-paranormal operators $\subseteq$ the class of $(k+1)$-quasi-paranormal operators.

(1.3)

We show that the inclusion relations in (1.3) are strict, by an example which appeared in [5, 8].

**EXAMPLE 1.2.** Given a bounded sequence of positive numbers $\{\alpha_i\}_{i=0}^{\infty}$. Let $T$ be the unilateral weighted shift operator on $l^2$ with the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ by $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$, that is,

$$T = \begin{pmatrix} 0 & & & \\ \alpha_0 & 0 & & \\ & \alpha_1 & 0 & \\ & & \alpha_2 & 0 \\ & & & \ddots \ddots \end{pmatrix}.$$

Straightforward calculations show that $T$ is a $k$-quasi-paranormal operator if and only if $\alpha_k \leq \alpha_{k+1} \leq \alpha_{k+2} \leq \cdots$. So if $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \cdots$ and $\alpha_k > \alpha_{k+1}$, then $T$ is a $(k+1)$-quasi-paranormal operator, but not a $k$-quasi-paranormal operator.

In this paper, firstly we prove some inequalities of this class of operators; secondly we give a necessary and sufficient condition for $T$ to be $k$-quasi-paranormal. Using these results, we prove that: (1) if $\|T^{n+1}\| = \|T\|^{n+1}$ for some positive integer $n \geq k$, then a $k$-quasi-paranormal operator $T$ is normaloid; (2) if $E$ is the Riesz idempotent for an isolated point $\lambda_0$ of the spectrum of a $k$-quasi-paranormal operator $T$, then (i) if $\lambda_0 \neq 0$, then $E H = \text{ker}(T - \lambda_0)$; (ii) if $\lambda_0 = 0$, then $E H = \text{ker}(T^{k+1})$. 
2. Results

It is well known that $T$ is paranormal if and only if $T^*T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$. Similarly, we have the following result.

**Theorem 2.1.** (see [9]) Let $T \in B(\mathcal{H})$. Then $T$ is $k$-quasi-paranormal if and only if

$$T^*k(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0$$

for all $\lambda > 0$.

**Proof.** (1.2) is equivalent to the following (2.2):

$$\langle T^{*(k+2)}T^{k+2}x, x \rangle^\frac{1}{2} \langle T^kT^kx, x \rangle^\frac{1}{2} \geq \langle T^{*(k+1)}T^{k+1}x, x \rangle$$

for all $x \in \mathcal{H}$. By generalized arithmetic-geometric mean inequality, we have

$$\langle T^{*(k+2)}T^{k+2}x, x \rangle^\frac{1}{2} \langle T^kT^kx, x \rangle^\frac{1}{2} = \left\{ \lambda^{-1} \langle T^{*(k+2)}T^{k+2}x, x \rangle \right\}^\frac{1}{2} \left\{ \lambda \langle T^kT^kx, x \rangle \right\}^\frac{1}{2}$$

$$\leq \frac{1}{2} \lambda^{-1} \langle T^{*(k+2)}T^{k+2}x, x \rangle + \frac{1}{2} \lambda \langle T^kT^kx, x \rangle$$

holds for all $x \in \mathcal{H}$ and $\lambda > 0$, so that (2.2) implies the following (2.3):

$$\frac{1}{2} \lambda^{-1} \langle T^{*(k+2)}T^{k+2}x, x \rangle + \frac{1}{2} \lambda \langle T^kT^kx, x \rangle \geq \langle T^{*(k+1)}T^{k+1}x, x \rangle$$

for all $x \in \mathcal{H}$ and $\lambda > 0$. Conversely, (2.2) follows from (2.3) by putting $\lambda = \left\{ \frac{\langle T^{*(k+2)}T^{k+2}x, x \rangle}{\langle T^{*k}T^kx, x \rangle} \right\}^\frac{1}{2} > 0$ in case $\langle T^{*(k+2)}T^{k+2}x, x \rangle \neq 0$, and letting $\lambda \to +0$ in case $\langle T^{*(k+2)}T^{k+2}x, x \rangle = 0$. Hence (2.2) is equivalent to (2.3). Consequently, the proof of Theorem 2.1 is complete since (2.3) is equivalent to (2.1).

$T \in B(\mathcal{H})$ is called a $k$-quasi-class A operator for a positive integer $k$ if $T^*k(|T^2| - |T|^2)T^k \geq 0$, which contains class A and quasi-class A, see [5, 8, 13]. In the following we give the relations between $k$-quasi-paranormal and $k$-quasi-class A operators.

**Theorem 2.2.** Let $T$ be a $k$-quasi-class A operator for a positive integer $k$. Then $T$ is a $k$-quasi-paranormal operator.

**Proof.** Suppose that $T$ is $k$-quasi-class A operator. Then $T^*k(|T^2| - |T|^2)T^k \geq 0$. Let $x \in \mathcal{H}$.

Then

$$\|T^{k+1}x\|^2 = \langle T^{*k}|T|^2T^kx, x \rangle$$

$$\leq \langle T^{*k}|T|^2T^kx, x \rangle$$

$$\leq \|\frac{\langle T^kT^kx \rangle}{\|T^kx\|} \|T^kx\|$$

$$= \|T^{k+2}x\| \|T^kx\|.$$
hence \( T \) is a \( k \)-quasi-paranormal operator. \( \square \)

**Remark.** We give an example which is \( k \)-quasi-paranormal, but not \( k \)-quasi-class A.

**Example 2.3.** Let \( T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in B(l_2 \oplus l_2) \). Then \( T \) is \( k \)-quasi-paranormal, but not \( k \)-quasi-class A.

By simple calculation we have that
\[
T^k |T^2| T^k = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T^k |T|^2 T^k = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Hence \( T \) not \( k \)-quasi-class A. However,
\[
T^* T^2 - 2 \lambda T^* T + \lambda^2 = \begin{pmatrix} 2 - 4\lambda + \lambda^2 & 0 \\ 0 & 0 \end{pmatrix},
\]
we have
\[
T^k (T^* T^2 - 2 \lambda T^* T + \lambda^2) T^k = \begin{pmatrix} 2(1 - \lambda)^2 & 0 \\ 0 & 0 \end{pmatrix} \geq 0
\]
for all \( \lambda > 0 \). Therefore \( T \) is \( k \)-quasi-paranormal.

**Theorem 2.4.** Let \( T \in B(\mathcal{H}) \) be a \( k \)-quasi-paranormal operator for a positive integer \( k \). Then the following assertions hold.

1. \( \|T^{n+2}\| \geq \|T^{n+1}\|^2 \) for all positive integers \( n \geq k \).
2. If \( T^n = 0 \) for some positive integer \( n \geq k \), then \( T^{k+1} = 0 \).
3. \( \|T^{n+1}\| \leq \|T^n\| r(T) \) for all positive integers \( n \geq k \), where \( r(T) \) denotes the spectral radius of \( T \).

**Proof.** The proof is similar to that of [5, Theorem 2.2]. (1) Since that \( k \)-quasi-paranormal operators are \( (k+1) \)-quasi-paranormal operators, we only need prove the case \( n = k \). It is clear by the definition of \( k \)-quasi-paranormal operators.

(2) If \( n = k, k+1 \), it is obvious that \( T^{k+1} = 0 \). If \( T^{k+2} = 0 \), then \( T^{k+1} = 0 \) by (1).

The rest of the proof is similar.

(3) We only need to prove the case \( n = k \), that is,
\[
\|T^{k+1}\| \leq \|T^k\| r(T).
\]
If \( T^n = 0 \) for some \( n \geq k \), then \( T^{k+1} = 0 \) by (2) and in this case \( r(T) = (r(T^{k+1}))^{k+1} = 0 \). Hence (3) is clear. Therefore we may assume \( T^n \neq 0 \) for all \( n \geq k \). Then
\[
\frac{\|T^{k+1}\|}{\|T^k\|} \leq \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \leq \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \leq \cdots \leq \frac{\|T^{mk}\|}{\|T^{mk-1}\|}
\]
by (1), and we have
\[
\left( \frac{\|T^{k+1}\|}{\|T^k\|} \right)^{mk-k} \leq \frac{\|T^{k+1}\|}{\|T^k\|} \times \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \times \cdots \times \frac{\|T^{mk}\|}{\|T^{mk-1}\|} = \frac{\|T^{mk}\|}{\|T^k\|}.
\]
Hence
\[
\left( \frac{\|T^{k+1}\|}{\|T^k\|} \right)^{k-k/m} \leq \frac{\|T^{mk}\|_{1/m}}{\|T^k\|_{1/m}}.
\]
By letting \( m \to \infty \), we have
\[
\|T^{k+1}\|^k \leq \|T^k\|^k (r(T))^k,
\]
that is,
\[
\|T^{k+1}\| \leq \|T^k\| r(T).
\]

It is well known that a paranormal operator is normaloid, that is, \( \|T^n\| = \|T\|^n \) for all \( n \in \mathbb{N} \) (equivalently \( \|T\| = r(T) \)). A nonzero nilpotent operator \( T \) (satisfying \( T^k = 0 \)) is \( k \)-quasi-paranormal but not normaloid. By Theorem 2.4, we give a sufficient condition of a \( k \)-quasi-paranormal operator to be normaloid.

**Corollary 2.5.** Let \( T \in B(\mathcal{H}) \) be a \( k \)-quasi-paranormal operator for a positive integer \( k \). If \( \|T^{n+1}\| = \|T^n\| \|T\| \) for some positive integer \( n \geq k \), then \( T \) is normaloid. In particularly if \( \|T^{n+1}\| = \|T\|^{n+1} \) for some positive integer \( n \geq k \), then \( T \) is normaloid.

**Proof.** It is clear by (3) of Theorem 2.4. \( \square \)

In [9], S. Mecheri studied the matrix representation of \( k \)-quasi-paranormal operator with respect to the direct sum of \( \text{ran}(T^k) \) and its orthogonal complement. In the following we give an equivalent condition for \( T \) to be \( k \)-quasi-paranormal.

**Theorem 2.6.** Suppose that \( T^k \) dose not have dense range. Then \( T \in B(\mathcal{H}) \) is a \( k \)-quasi-paranormal operator for a positive integer \( k \) if and only if \( T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \) on \( \mathcal{H} = \text{ran}(T^k) \oplus \ker T^{*k} \), where \( T_1 \) is a paranormal operator on \( \text{ran}(T^k) \) and \( T_3^k = 0 \), furthermore \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

**Proof.** We first prove the necessary. The necessary has been proved in [9]. We give a proof here for completeness. Suppose that \( T \in B(\mathcal{H}) \) is a \( k \)-quasi-paranormal operator for a positive integer \( k \). Since that \( T^k \) dose not have dense range, we can represent \( T \) as the upper triangular matrix
\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}
\]
on \( \mathcal{H} = \text{ran}(T^k) \oplus \ker T^{*k} \). We shall prove that \( T_1 \) is a paranormal operator on \( \text{ran}(T^k) \) and \( T_3^k = 0 \). Since \( T \) is a \( k \)-quasi-paranormal operator, it follows from Theorems 2.1 that
\[
T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0
\]
for all \( \lambda > 0 \). Therefore
\[
\langle (T^{*2}T^2 - 2\lambda T^*T + \lambda^2)x, x \rangle = \langle (T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)x, x \rangle \geq 0
\]
for all $\lambda > 0$ and for all $x \in \text{ran}(T^k)$. Hence

$$T_1^*T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$$

for all $\lambda > 0$. So we have that $T_1$ is a paranormal operator on $\text{ran}(T^k)$. Let $P$ be the orthogonal projection of $\mathcal{H}$ onto $\text{ran}(T^k)$. For any $x = (x_1, x_2) \in \mathcal{H}$,

$$\langle T_3^k x_2, x_2 \rangle = \langle T^k (I - P)x, (I - P)x \rangle = \langle (I - P)x, T^*k (I - P)x \rangle = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup \mathcal{G} = \sigma(T_1) \cup \sigma(T_3)$, where $\mathcal{G}$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by [7, Corollary 7], and $\sigma(T_1) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Next prove the sufficiency.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\text{ran}(T^k) \oplus \ker T^*k}$, where $T_1$ is a paranormal operator on $\text{ran}(T^k)$ and $T_3^k = 0$. Then we have

$$T^*k(T^*T^2 - 2\lambda T^*T + \lambda^2)T^k = \left(\sum_{i=0}^{k-1} T_1^iT_2T_3^{k-1-i} \right)^* \left(\begin{array}{cc} T_1^* \quad 0 \\ 0 \quad 0 \end{array}\right).$$

$$\times \left(\begin{array}{ccc} T_1^*T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 & T_1^*T_1T_2 + T_1^*T_2T_3 \\ T_2^*T_1T_2^2 + T_3^*T_2T_3^2 - 2\lambda T_2T_3 & |T_1T_2 + T_2T_3|^2 + |T_3|^2 - 2\lambda (T_2^*T_2 + T_3^*T_3) + \lambda^2 \end{array}\right)$$

$$\times \left(\begin{array}{c} T_1^k \quad \sum_{i=0}^{k-1} T_1^iT_2T_3^{k-1-i} \\ 0 \quad 0 \end{array}\right).$$

where $D = \left(\sum_{i=0}^{k-1} T_1^iT_2T_3^{k-1-i} \right)^* \left(\begin{array}{cc} T_1^*T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \\ T_1^*T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \end{array}\right) \sum_{i=0}^{k-1} T_1^iT_2T_3^{k-1-i}$. Let $\lambda > 0$ be arbitrary and $v = x \oplus y$ be a vector in $\mathcal{H} = \overline{\text{ran}(T^k) \oplus \ker T^*k}$, where $x \in \overline{\text{ran}(T^k)}$.
and $y \in \ker T^{*k}$. Then

$$
\langle T^{*k}(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)T^k v, v \rangle \\
= \langle T^{*2k} T^{*2} T^{-2} - 2\lambda T^{*T} T + \lambda^2 \rangle T^k i, x \rangle \\
+ \langle T^{*k} (T^{*2} T^{-2} - 2\lambda T^{*T} T + \lambda^2) \sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i} y, x \rangle \\
+ \langle (\sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i})^* (T^{*2} T^2 - 2\lambda T^{*T} T + \lambda^2) T^{k} x, y \rangle \\
+ \langle (\sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i}) (T^{*2} T^2 - 2\lambda T^{*T} T + \lambda^2) \sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i} y, y \rangle \\
= \langle (T^{*2} T^2 - 2\lambda T^{*T} T + \lambda^2) T^k x + \sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i} y, T^{k} x + \sum_{i=0}^{k-1} T^{i} T^{*} T^{\lambda-1-i} y \rangle.
$$

Since $T_1$ is a paranormal operator, we have that $T^{*2} T^2 - 2\lambda T^{*T} T + \lambda^2 \geq 0$ for all $\lambda > 0$. Therefore

$$
\langle T^{*k}(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)T^k v, v \rangle \geq 0
$$

for all $v \in \mathcal{H}$ and for all $\lambda > 0$. Hence

$$
T^{*k}(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)T^k \geq 0
$$

for $\lambda > 0$. So we have that $T$ is a $k$-quasi-paranormal operator for a positive integer $k$ by Theorem 2.1. □

**Remark.** In the proof of Theorem 2.6, let $P$ be the orthogonal projection of $\mathcal{H}$ onto $\text{ran}(T^k)$. Then

$$
T^{*k}(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)T^k \geq 0 \\
\iff P(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)P \geq 0 \\
\iff T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2 \geq 0.
$$

It is well known that if $T$ is a paranormal operator and $\mathcal{V}$ is a closed invariant subspace of $T$, then $T|\mathcal{V}$ is also paranormal. We shall give a similar result for a $k$-quasi-paranormal operator by Theorem 2.6.

**Corollary 2.7.** Suppose that $T \in B(\mathcal{H})$ is a $k$-quasi-paranormal operator for a positive integer $k$ and $\mathcal{V}$ is its closed invariant subspace. Then the restriction $T|\mathcal{V}$ of $T$ to $\mathcal{V}$ is also a $k$-quasi-paranormal operator.

**Proof.** Let $T = \begin{pmatrix} T_1 & T_2 \\
0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \mathcal{V} \oplus (\mathcal{V})^\perp$. Since $T$ is a $k$-quasi-paranormal operator for a positive integer $k$, it follows from Theorem 2.1 that

$$
T^{*k}(T^{*2}T^2 - 2\lambda T^{*T}T + \lambda^2)T^k \geq 0
$$
for all \( \lambda > 0 \). By the proof Theorem 2.6, we have

\[
\begin{pmatrix}
T_1^*k(T_1^*T_1^2 - 2\lambda T_1^* T_1 + \lambda^2)T_1^k \\
\left( \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \right) (T_1^* T_1^2 - 2\lambda T_1^* T_1 + \lambda^2) T_1^k \\
D
\end{pmatrix}
\]

\( \geq 0, \)

where \( D = (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i}) (T_1^* T_1^2 - 2\lambda T_1^* T_1 + \lambda^2) \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \). Recall that

\[
\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0 \text{ if and only if } X, Z \geq 0 \text{ and } Y = X^* WZ^+ \text{ for some contraction } W. \]

So we have that

\[
T_1^*k(T_1^* T_1^2 - 2\lambda T_1^* T_1 + \lambda^2) T_1^k \geq 0
\]

for all \( \lambda > 0 \). Therefore \( T_1|^\mathcal{H} = T_1 \) is also a \( k \)-quasi-paranormal operator. \( \square \)

In [18], Yuan studied the Riesz idempotent with respect to an isolated point of the spectrum of \( k \)-quasi-paranormal operators by Bishop’s property (\( \beta \)). M. Chô and T. Yamazaki proved that class A operators have property \( \beta \) in [2] Theorem 3.1; A. Uchiyama and K. Tanahashi proved that paranormal operators have property \( \beta \) in [16] Corollary 3.6. Unfortunately there are some mistakes in the proof of these theorems, see detail in [3]. So the Bishop’s property (\( \beta \)) of the class operators such as class A, paranormal and \( k \)-quasi-paranormal operators is still an open problem. In the following we have similar result by Theorem 2.4 without Bishop’s property (\( \beta \)).

**Theorem 2.8.** Suppose \( T \in B(\mathcal{H}) \) is a \( k \)-quasi-paranormal operator for a positive integer \( k \). Let \( \lambda_0 \) be an isolated point of \( \sigma(T) \) and \( E \) the Riesz idempotent for \( \lambda_0 \). Then the following assertions hold:

(i) If \( \lambda_0 \neq 0 \), then \( E \mathcal{H} = \ker(T - \lambda_0) \).

(ii) If \( \lambda_0 = 0 \), then \( E \mathcal{H} = \ker(T^{k+1}) \).

**Proof.** Suppose that \( T \) is a \( k \)-quasi-paranormal operator. (i) If the range \( T^k \mathcal{H} \) is dense, then \( T \) is a paranormal operator. Theorem holds by Theorem 3.7 in [17]. Therefore we may assume that \( \overline{\text{ran}T^k} \neq \mathcal{H} \). Let \( T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \) on \( \mathcal{H} = \overline{\text{ran}(T^k)} \oplus \ker(T^*) \).

By Theorem 2.6 we know that \( T_1 \) is a paranormal operator and \( \sigma(T) = \sigma(T_1) \cup \{0\} \). If \( \lambda_0 \neq 0 \) is an isolated point of \( \sigma(T) \), then \( \lambda_0 \) is an isolated point of \( \sigma(T_1) \). Therefore \( \lambda_0 \) is a simple pole of the resolvent of \( T_1 \) and \( T_1 \) can be written by \( T_1 = \begin{pmatrix} \lambda^* & 0 \\ 0 & T' \end{pmatrix} \) on \( \overline{\text{ran}(T^k)} = \ker(T_1 - \lambda_0) \oplus \overline{\text{ran}(T_1 - \lambda_0)} \). Hence we have \( T - \lambda_0 = \begin{pmatrix} \lambda^* & 0 \\ 0 & T - \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & C \end{pmatrix} \) on \( \mathcal{H} = \ker(T_1 - \lambda_0) \oplus \overline{\text{ran}(T_1 - \lambda_0)} \oplus \ker(T^*), \)

where \( C = \begin{pmatrix} T' - \lambda_0 & T_2 \\ 0 & T_3 - \lambda_0 \end{pmatrix} \).
Since $C$ is an invertible operator on $\overline{\text{ran}(T_1 - \lambda_0)} \oplus \ker(T^{*k})$, it can be easily shown that $p(T - \lambda_0) = q(T - \lambda_0) = 1$. Therefore $\lambda_0$ is a simple pole of the resolvent of $T$. Since $E$ is the Riesz idempotent of $T$ with respect to $\lambda_0$, we have $E \mathcal{H} = \ker(T - \lambda_0)$.

(ii) Since $\ker(T^{k+1}) \subset E \mathcal{H}$ always holds, it suffices to prove $E \mathcal{H} \subset \ker(T^{k+1})$. It is known that $E \mathcal{H}$ is an invariant subspace of $T$ and $\sigma(T|_{E \mathcal{H}}) = \{0\}$. Hence $T|_{E \mathcal{H}}$ is also a $k$-quasi-paranormal operator by Corollary 2.7 and

$$\|(T|_{E \mathcal{H}})^{k+1}\| \leq \|(T|_{E \mathcal{H}})^k\| \rho(T|_{E \mathcal{H}}) = 0$$

by (3) of Theorem 2.4. Hence

$$(T|_{E \mathcal{H}})^{k+1} = T^{k+1}|_{E \mathcal{H}} = 0.$$ 

This implies $E \mathcal{H} \subset \ker(T^{k+1})$. \qed

An operator $T$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$.

**Corollary 2.9.** Let $T \in B(\mathcal{H})$ be a $k$-quasi-paranormal operator for a positive integer $k$. Then $T$ is isoloid.

**Proof.** Let $\lambda \in \sigma(T)$ be an isolated point. If $\lambda \neq 0$, by (1) of Theorem 2.8, $\ker(T - \lambda) = E \mathcal{H} \neq \{0\}$ for $E \neq 0$. Therefore $\lambda$ is an eigenvalue of $T$. If $\lambda = 0$, by (2) of Theorem 2.8, $\ker(T^{k+1}) = E \mathcal{H} \neq \{0\}$ for $E \neq 0$. So we have $\ker(T) \neq \{0\}$. Therefore $0$ is an eigenvalue of $T$. This completes the proof. \qed

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