Journal of
Mathematical
Inequalities
Volume 8, Number 1 (2014), 123–152 doi:10.7153/jmi-08-08

ON THE BOUNDEDNESS OF MAXIMAL AND POTENTIAL OPERATORS IN VARIABLE EXPONENT AMALGAM SPACES

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(*Communicated by S. Samko*)

Abstract. Two–weight estimates for maximal and fractional integral operators in variable exponent amalgam spaces $(L^{p(\cdot)}, l^q)$ are established under the log– Hölder continuity condition on the exponent $p(\cdot)$. Some of the derived results are new even for constant p.

1. Introduction

Our purpose is to derive necessary and sufficient conditions on a weight pair governing the two–weight inequality for the maximal and fractional integral operators in variable exponent amalgam spaces (VEAS) $(L^{p(\cdot)}, l^q)$ under the log–Hölder continuity condition on the exponent $p(\cdot)$. The derived results are new even for constant p in the case of potential operators defined on $\mathbb{R}$. The derived criteria are of various types.

The boundedness for maximal and fractional integral operators in unweighted and weighted variable exponent Lebesgue spaces defined on Euclidean spaces was investigated by many authors (see, e.g., the papers [11], [34], [15], [12], [6], [9], [24], [25], [20], [21], [22], [28], [29], [14], [8] etc). It should be emphasized that in the last two papers a complete characterization of the one–weight inequality for the Hardy–Littlewood maximal operator is given under the Muckhenhoupt–type conditions. We refer also to the monograph [13] for related topics.

Apart from interesting theoretical considerations, the study of variable exponent spaces was motivated by a proposed application to modeling electrorheological fluids (see, [32]), to image restoration (see e.g. [1]), etc.

The paper consists of three sections. In Section 2 we recall some well–known facts about variable exponent Lebesgue spaces and VEAS; also we prove some lemmas and propositions needed to prove the main results. In Section 3 we give weight characterizations for maximal and fractional integral operators to be bounded in VEAS.

Finally, we mention that throughout the paper constants (often different constants in the same series of inequalities) will mainly be denoted by c or C; by the symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$; the relation $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 a \leqslant b \leqslant c_2 a$.

Mathematics subject classification (2010): 42B25, 46E30.
Keywords and phrases: Variable exponent amalgam spaces, maximal operator, potentials, boundedness.

2. Preliminaries

2.1. Variable exponent Lebesgue spaces

Let E be a measurable set in $\mathbb{R}$ with positive measure. We denote:

$$p_-(E) := \inf_E p, \qquad p_+(E) := \sup_E p$$

for a measurable function p on E. Suppose that $1 < p_-(E) \leqslant p_+(E) < \infty$. Denote by ρ a weight function on E. We say that a measurable function f on E belongs to $L^{p(\cdot)}_\rho(E)$ (or to $L^{p(x)}_\rho(E)$) if

$$S_{p(\cdot),\rho}(f) = \int_E \left|f(x)\rho(x)\right|^{p(x)} dx < \infty.$$

It is a Banach space with respect to the norm (see e.g., [26], [33], [37])

$$\|f\|_{L^{p(\cdot)}_\rho(E)} = \inf\left\{\lambda > 0 : S_{p(\cdot),\rho}\left(f/\lambda\right) \leqslant 1\right\}.$$

If $\rho \equiv const$, then we use the symbol $L^{p(\cdot)}(E)$ (resp. $S_{p(\cdot)}$) instead of $L^{p(\cdot)}_\rho(E)$ (resp. $S_{p(\cdot),\rho}$). It is clear that $\|f\|_{L^{p(\cdot)}_\rho(E)} = \|f(\cdot)\rho(\cdot)\|_{L^{p(\cdot)}(E)}$.

In the sequel we will denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of all integers and the set of positive integers, respectively.

Let us recall some well–known facts regarding $L^{p(x)}$ spaces.

PROPOSITION A. ([26], [37], [33]) *Let E be a measurable subset of $\mathbb{R}$. Then*

(i)
$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \leqslant S_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \leqslant 1;$$

$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \leqslant S_{p(\cdot)}(f\chi_E) \leqslant \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \geqslant 1;$$

(ii) *Hölder's inequality*

$$\left|\int_E f(x)g(x)dx\right| \leqslant \left(\frac{1}{p_-(E)} + \frac{1}{(p_+(E))'}\right) \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$, $g \in L^{p'(\cdot)}(E)$.

PROPOSITION B. ([33], [26], [37]) *Let $1 \leqslant r(x) \leqslant p(x)$ and let E be a bounded subset of $\mathbb{R}$. Then the following inequality*

$$\|f\|_{L^{r(\cdot)}(E)} \leqslant (|E|+1)\|f\|_{L^{p(\cdot)}(E)}$$

holds.

DEFINITION 2.1. We say that p satisfies the weak Lipschitz (log-Hölder continuity) condition on $E \subset \mathbb{R}$ $(p \in WL(E))$, if there is a positive constant A such that for all x and y in E with $0 < |x-y| < 1/2$ the inequality

$$|p(x) - p(y)| \leqslant A/(-\ln|x-y|)$$

holds.

The next statement gives another characterization of the weak Lipschitz condition.

LEMMA A. ([11]) *Let I be an interval in $\mathbb{R}$. Then $p \in WL(I)$ if and only if there exists a positive constant c such that*

$$|J|^{p_-(J)-p_+(J)} \leqslant c$$

for all intervals $J \subseteq I$ with $|J| > 0$. Moreover, the constant c does not depend on I.

LEMMA B. (see, e.g. [4]) *Let $1 < q < \bar{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\left\{\sum_{n\in\mathbb{Z}} (|a_n|u_n)^q\right\}^{1/q} \leqslant C\left\{\sum_{n\in\mathbb{Z}} (|a_n|v_n)^{\bar{q}}\right\}^{1/\bar{q}}$$

holds for all sequences $\{a_n\}$ of real numbers.

(ii) $\left\{\sum_{n\in\mathbb{Z}} (u_n v_n{}^{-1})^s\right\}^{1/s} < \infty.$

2.2. Amalgam spaces

Let u be a weight function on $\mathbb{R}$ and let f be a measurable function on $\mathbb{R}$. Let us denote

$$\|f\|_{(L_u^{p(\cdot)}(\mathbb{R}),l^q)} := \left(\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)}(\cdot)f(\cdot)\|^q_{L_u^{p(\cdot)}(\mathbb{R})}\right)^{1/q}.$$

We define the weighted variable exponent amalgam space by

$$(L_u^{p(\cdot)}(\mathbb{R}),l^q) = \{f : \|f\|_{(L_u^{p(\cdot)}(\mathbb{R}),l^q)} < \infty\}.$$

If $u \equiv \text{const}$, then $(L_u^{p(\cdot)}(\mathbb{R}),l^q)$ is denoted by $(L^{p(\cdot)}(\mathbb{R}),l^q)$.

Let $p \equiv p_c \equiv \text{const}$ and $u \equiv \text{const}$. Then we have the usual amalgam (see [38]), which were introduced by N. Wiener (see [40], [41]) in connection with the development of the theory of generalized harmonic analysis.

Some properties of variable exponent amalgam space can be derived in the same way as for usual amalgams $(L_u^p(\mathbb{R}),l^q)$, where p is constant.

THEOREM A. *Let p be a measurable function on $\mathbb{R}$ with $1 < p(\cdot) < \infty$ and q is constant with $1 < q < \infty$. The variable exponent amalgam space $(L^{p(\cdot)}(\mathbb{R}), l^q)$ is a Banach space whose dual space is $(L^{p(\cdot)}(\mathbb{R}), l^q)^* = (L^{p'(\cdot)}(\mathbb{R}), l^{q'})$. Further, Hölder's inequality holds in the following form*

$$\left|\int_{\mathbb{R}} f(t)g(t)dt\right| \leqslant \|f\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)}\|g\|_{(L^{p'(\cdot)}(\mathbb{R}),l^{q'})}.$$

Proof. Since $L^{p(\cdot)}(\mathbb{R})$ is a Banach space and $\left(L^{p(\cdot)}(\mathbb{R})\right)^* = L^{p'(\cdot)}(\mathbb{R})$ (see [26]), from general arguments (see [10], [19], [16], [38]) we have the desired result. $\square$

The next statement for more general amalgam (X, l^q), where X is a Banach space, can be found in [38].

THEOREM B. *Let p be measurable function on $\mathbb{R}$ and $1 \leqslant q_1 \leqslant q_2$, then*

$$(L^{p(\cdot)}(\mathbb{R}), l^{q_1}) \subset (L^{p(\cdot)}(\mathbb{R}), l^{q_2}).$$

Other structural properties of amalgams are investigated e.g., in [16] and [38].

DEFINITION 2.2. Let J be a bounded interval in $\mathbb{R}$. We say that a measure μ satisfies the doubling condition on J $(\mu \in DC(J))$ if there is a positive constant c such that for all $x \in J$ and all r, $0 < r < |J|$, the inequality

$$\mu\big((x-2r, x+2r)\cap J\big) \leqslant c\mu\big((x-r, x+r)\cap J\big)$$

holds.

For a weight function u, we sometimes denote:

$$u(E) := \int_E u(x)dx, \;\; E \subseteq \mathbb{R}.$$

LEMMA C. ([17], [21]) *Let J be a finite interval and let μ be a doubling measure on J. Suppose that p is an exponent defined on J satisfying the conditions $1 \leqslant p_-(J) \leqslant p(x) \leqslant p_+(J) < \infty$ and $p \in WL(J)$. Then there is a positive constant C depending only on doubling constant d such that for all subintervals I of J,*

$$(\mu(I))^{p_-(I)-p_+(I)} \leqslant C.$$

Let J be an interval in $\mathbb{R}$, $J \subseteq \mathbb{R}$ and let

$$(M_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|dy, \quad x \in J,$$

where $x \in J$ and α is a constant satisfying the condition $0 \leqslant \alpha < 1$.

When $\alpha = 0$, then we have the Hardy–Littlewood maximal operator. In this case we denote $M_\alpha^{(J)}$ by $M^{(J)}$.

The next statement is a solution of the one–weight problem for the Hardy–Littlewood maximal operator (see [8]). We formulate the result for a bounded interval.

PROPOSITION 2.1. *The operator $M^{(J)}$ is bounded in $L_w^{p(\cdot)}(J)$ if and only if $w \in A_{p(\cdot)}(J)$, i.e.*

$$\sup_{I\subseteq J} |I|^{-1} \|w\chi_I\|_{L^{p(\cdot)}} \|w^{-1}\chi_I\|_{L^{p'(\cdot)}} < \infty$$

provided that $1 < p_-(J) \leqslant p(\cdot) \leqslant p_+(J) < \infty$ and $p \in WL(J)$.

Now we formulate Sawyer [35] type results for maximal operators in variable exponent Lebesgue spaces.

The next statements (Propositions 2.2- 2.3 and Corollary 2.1) are taken from [21].

PROPOSITION 2.2. *Let an exponent p be defined on a finite interval J and let $1 < p_-(J) \leqslant p(\cdot) \leqslant p_+(J) < \infty$. Suppose that v and w are weight functions on J and that $d\nu(x) = w(x)^{-p'(x)}dx$ belongs to $DC(J)$. Suppose also that $0 \leqslant \alpha < 1$ and that $p \in WL(J)$. Then the inequality*

$$\|v(\cdot)M_\alpha^{(J)} f\|_{L^{p(\cdot)}(J)} \leqslant c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds, if and only if there exists a positive constant c such that for all intervals I, $I \subset J$,

$$\int_I (v(x))^{p(x)} (M_\alpha^{(J)}(w(\cdot)^{-p'(\cdot)}\chi_{I(\cdot)}))^{p(x)} dx \leqslant c \int_I w^{-p'(x)} dx < \infty.$$

COROLLARY 2.1. *Let J be a bounded interval and let $1 < p_-(J) \leqslant p(\cdot) \leqslant p_+(J) < \infty$. Suppose that $0 \leqslant \alpha < 1$. Assume that $p \in WL(J)$. Then the inequality*

$$\big\|v(\cdot)\big(M_\alpha^{(J)} f\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \leqslant c\|f\|_{L^{p(\cdot)}(J)} \quad (\textit{Trace inequality})$$

holds if and only if

$$\sup_{I, I\subset J} \frac{1}{|I|} \int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx < \infty,$$

where the supremum is taken over all subintervals I of J.

PROPOSITION 2.3. *Let $0 \leqslant \alpha < 1$, $1 < p_-(\mathbb{R}) \leqslant p(\cdot) \leqslant p_+(\mathbb{R}) < \infty$, and let $p \in WL(\mathbb{R})$. Suppose that there is a positive number a such that $w^{-p'(\cdot)}(\cdot) \in DC([-a,a])$ and $p \equiv p_c \equiv$const outside $[-a,a]$. Then the inequality*

$$\|vM_\alpha^{(\mathbb{R})} f\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant \|wf\|_{L^{p(\cdot)}(\mathbb{R})},$$

holds if and only if there is a positive constant c such that for all bounded intervals $I \subset \mathbb{R}$,

$$\|vM_\alpha^{(\mathbb{R})}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

To formulate the next statement we need the following definition.

DEFINITION 2.3. Let μ be a measure on $\mathbb{R}$. We say that μ satisfies the reverse doubling condition on $\mathbb{R}$ $(\mu \in RD(\mathbb{R}))$ if there is a constant $b > 1$ such that

$$\mu(x-2r, x+2r) \geqslant b\mu(x-r, x+r).$$

It is well-known that the reverse doubling condition implies the doubling condition.

PROPOSITION 2.4. ([20]) *Suppose that* $p = const$*;* $1 < p < q_-(\mathbb{R}) \leqslant q_+(\mathbb{R}) < \infty$*;* $0 < \alpha < 1$*. Assume that* $w^{-p'} \in RD(\mathbb{R})$*. Then the inequality*

$$\|vM_\alpha f\|_{L^{q(\cdot)}(\mathbb{R})} \leqslant c\|wf\|_{L^p(\mathbb{R})} \tag{2.1}$$

holds if and only if

$$\sup_{I \subset \mathbb{R}} \|v\chi_I |I|^{\alpha-1}\|_{L^{q(\cdot)}(\mathbb{R})} \|w^{-1}\chi_I\|_{L^{p'}(\mathbb{R})} < \infty. \tag{2.2}$$

Let

$$(I_\alpha f)(x) := \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \;\; x \in \mathbb{R}$$

be the fractional integral operator defined on $\mathbb{R}$, where $0 < \alpha < 1$.

The next statement is a generalization of the result by D. Adams [2] for variable exponent Lebesgue spaces:

PROPOSITION 2.5. ([20]) *Let* s *be a measurable function on* $\mathbb{R}$ *such that* $1 < s_-(\mathbb{R}) \leqslant s_+(\mathbb{R}) < \infty$*. Suppose that* r *and* α *are constants satisfying the conditions:* $1 < r < s_-(\mathbb{R})$*,* $0 < \alpha < 1/r$*. Then the following statements are equivalent:*

(i) I_α is bounded from $L^r(\mathbb{R})$ to $L^{s(\cdot)}_v(\mathbb{R})$;

(ii)

$$\sup_{I; I \subset \mathbb{R}} \|\chi_I\|_{L^{s(\cdot)}_v(\mathbb{R})} |I|^{\alpha - 1/r} < \infty,$$

where the supremum is taken over all bounded intervals I in $\mathbb{R}$.

Let

$$\big(\mathscr{I}_\alpha(\{g_k\})\big)_n = \sum_{k \in \mathbb{Z}, k \neq n} \frac{g_k}{|n-k|^{1-\alpha}}\,, \; n \in \mathbb{Z}$$

$$\big(\mathscr{R}_\alpha(\{g_k\})\big)_n = \sum_{k=-\infty}^{n} \frac{g_k}{(n-k+1)^{1-\alpha}}\,, \; n \in \mathbb{Z},$$

$$\big(\mathscr{W}_\alpha(\{g_k\})\big)_n = \sum_{k=n}^{\infty} \frac{g_k}{(k-n+1)^{1-\alpha}}\,, \; n \in \mathbb{Z},$$

be discrete fractional integral operators, where $0 < \alpha < 1$.

It is easy to check that

$$\frac{1}{2}\Big(\big(\mathscr{R}_\alpha(\{g_k\})\big)_{n-1}+\big(\mathscr{W}_\alpha(\{g_k\})\big)_{n+1}\Big)\leqslant\big(\mathscr{I}_\alpha(\{g_k\})\big)_n$$
$$=\big(\mathscr{R}_\alpha(\{g_k\})\big)_{n-1}+\big(\mathscr{W}_\alpha(\{g_k\})\big)_{n+1}.$$

Let $\{u_n\}_{n\in\mathbb{Z}}$ be a positive (weight) sequence. In the sequel by $l^p_{u_n}(\mathbb{Z})$, $1<p<\infty$, will denote the class of all sequences $\{g_k\}_{k\in\mathbb{Z}}$ for which

$$\|g_k\|_{l^p_{u_n}(\mathbb{Z})}=\Big(\sum_{k\in\mathbb{Z}}|g_k|^p u_k\Big)^{1/p}<\infty.$$

If u_k is a constant sequence, then we denote $l^p_{u_k}(\mathbb{Z})$ by $l^p(\mathbb{Z})$.

Sometimes we use the symbol $T(\{g_k\})(n)$ instead of $T(\{g_k\})_n$ for a discrete operator T.

Let $(X,\mathscr{U},\mu)$ and $(Y,\mathscr{B},\nu)$ be measure spaces with ν being σ- finite. Suppose that $k(x,y)$ is a non–negative real–valued $\mathscr{U}\times\mathscr{B}$– measurable function and that

$$Kf(y)=\int_X k(x,y)f(x)d\mu(x)$$

is the kernel operator.

Denote:

$$e_\lambda(x):=\{y\in Y: k(x,y)>\lambda\},\ \ e_\lambda(y):=\{x\in X: k(x,y)>\lambda\},$$

where λ is a positive number;

$$M_r(\mu)(y):=\sup_{\lambda>0}\lambda^r\mu\big(e_\lambda(y)\big);\ \ M_s(\nu)(x):=\sup_{\lambda>0}\lambda^s\nu\big(e_\lambda(x)\big),$$

where r and s are real numbers.

To prove the statements regarding fractional integrals we use the following statement which is a corollary of part (ii) of Theorem A in [2].

THEOREM C. *Suppose that* $1<p<q<\infty$, $\frac{s}{q}=\frac{r}{p}+1-r$, *where* $r>0$. *If* $M_r(\mu)(y)\leqslant A<\infty$ *for all* $y\in Y$*;* $M_s(\nu)(x)\leqslant B<\infty$ *for all* $x\in X$*, then the operator* K *is bounded from* $L^p(X,\mu)$ *to* $L^q(Y,\nu)$*, where* $L^p(X,\mu)$ $L^q(Y,\nu)$ *are Lebesgue spaces defined with respect to the measures* μ *and* ν*, respectively.*

PROPOSITION 2.6. *Suppose that* p*,* q *and* α *are constants satisfying the conditions:* $1<p<q<\infty$, $0<\alpha<1/p$. *Then the following statements are equivalent:*

(i) $\mathscr{R}_\alpha$ is bounded from $l^p(\mathbb{Z})$ to $l^q_{v_k}(\mathbb{Z})$;
(ii) $\mathscr{W}_\alpha$ is bounded from $l^p(\mathbb{Z})$ to $l^q_{v_k}(\mathbb{Z})$;
(iii) $\mathscr{I}_\alpha$ is bounded from $l^p(\mathbb{Z})$ to $l^q_{v_k}(\mathbb{Z})$
(iv)

$$B:=\sup_{m\in\mathbb{Z},j\in\mathbb{N}}\Big(\sum_{k=m}^{m+j}v_k\Big)^{1/q}(j+1)^{\alpha-1/p}<\infty.$$

Proof. $(iv) \Rightarrow (i)$. Suppose that $X = Y = \mathbb{Z}$, μ is the counting measure on $\mathbb{Z}$ and that $d\nu(n) = v_n d\mu(n)$, where $\{v_n\}_{n\in\mathbb{Z}}$ is the weight sequence. In our case the kernel operator is given by

$$\{\mathscr{R}_\alpha\{g_m\}\}_n = \sum_{m=-\infty}^{\infty} k(m,n)g_m, \ \ n \in \mathbb{Z},$$

where

$$k(m,n) = \chi_{\{m\in\mathbb{Z}:m\leqslant n\}}(n-m+1)^{\alpha-1}.$$

Let $r = \frac{1}{1-\alpha}$ and let $\frac{s}{q} = \frac{r}{p} + 1 - r$. That is $s = \frac{q(\alpha-1/p)}{\alpha-1} > 0$. We have

$$\begin{aligned}\sup_{n\in\mathbb{Z}} M_r(\mu)(n) &= \sup_{\lambda\leqslant 1, n\in\mathbb{Z}} \lambda^r \mu\{m\in\mathbb{Z} : m \leqslant n; (n-m+1)^{\alpha-1} > \lambda\}\\ &= \sup_{\lambda\geqslant 1, n\in\mathbb{Z}} \lambda^{r(\alpha-1)} \mu\{m\in\mathbb{Z} : m\leqslant n; n-m+1 < \lambda\}\\ &\leqslant \sup_{k\in\mathbb{N}, n\in\mathbb{Z}} k^{-1} \sum_{m=n-k}^{n} 1 \leqslant c.\end{aligned}$$

Further,

$$\begin{aligned}\sup_{m\in\mathbb{Z}} M_s(\nu)(m) &= \sup_{\lambda\leqslant 1, m\in\mathbb{Z}} \lambda^s \nu\{n\in\mathbb{Z} : m\leqslant n; (n-m+1)^{\alpha-1} > \lambda\}\\ &= \sup_{\lambda\geqslant 1, m\in\mathbb{Z}} \lambda^{s(\alpha-1)} \nu\{n\in\mathbb{Z} : m\leqslant n; n-m+1<\lambda\}\\ &\leqslant \sup_{k\in\mathbb{N}, m\in\mathbb{Z}} k^{s(\alpha-1)} \sum_{n=m}^{m+k} v_n \leqslant cB^q.\end{aligned}$$

$(i) \Rightarrow (iv)$. Let

$$(\beta^{(m)})_k = \begin{cases} 1 \text{ if } m-j<k\leqslant m;\\ 0 \text{ otherwise,}\end{cases}$$

where m, j are positive integers such that $j \leqslant m$. Then we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} v_n \left(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} &\geqslant \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q}\\ &\geqslant c\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^\alpha.\end{aligned}$$

Therefore, by the boundedness of $\mathscr{R}_\alpha$ we conclude that

$$\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha-1/p} \leqslant c, \quad 1\leqslant j\leqslant m.$$

$(i) \Rightarrow (ii)$. Let

$$(\beta^{(m)})_k = \begin{cases} 1 \text{ if } m-j<k\leqslant m; \\ 0 \text{ otherwise,} \end{cases}$$

where $m\in\mathbb{Z}$ and $j\in\mathbb{Z}$. Then we have

$$\left(\sum_{n\in\mathbb{Z}} v_n \left(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} \geqslant \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q}$$
$$\geqslant c\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha}.$$

Therefore, by the boundedness of $\mathscr{R}_\alpha$ we conclude that

$$\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha-1/p}\leqslant c, \quad m\in\mathbb{Z},\ \ j\in\mathbb{Z}.$$

The remaining parts $(ii)\Rightarrow(iv)$ and $(iii)\Rightarrow(iv)$ follows similarly; therefore we omit proofs. □

The next statement gives criteria guaranteeing the trace inequality for the discrete potential operators in the diagonal case, i.e., when $p=q$. Criteria are of Maz'ya-Verbitsky [27] type.

PROPOSITION 2.7. *Let* $1<p<\infty$ *and let* $0<\alpha<1/p$.
(i) The inequality

$$\sum_{i=-\infty}^{+\infty} \left(\mathscr{R}_\alpha g_j\right)_i^p v_i \leqslant c \sum_{i=-\infty}^{+\infty} g_i^p \tag{2.3}$$

holds for all non-negative sequences $\{g_i\}_i$ if and only if $\{\mathscr{W}_\alpha v_i\}_i<\infty$ for all $i\in\mathbb{Z}$ and

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_j]^{p'}\right\}_i \leqslant c\left\{\mathscr{W}_\alpha v_i\right\}_i. \tag{2.4}$$

(ii) The inequality

$$\sum_{i=-\infty}^{+\infty} \left(\mathscr{W}_\alpha g_j\right)_i^p v_i \leqslant c \sum_{i=-\infty}^{+\infty} g_i^p \tag{2.5}$$

holds for all non-negative sequences $\{g_i\}_i$ if and only if $\{\mathscr{R}_\alpha v_i\}_i<\infty$ for all $i\in\mathbb{Z}$ and

$$\left\{\mathscr{R}_\alpha[\mathscr{R}_\alpha v_j]^{p'}\right\}_i \leqslant c\left\{\mathscr{R}_\alpha v_i\right\}_i. \tag{2.6}$$

To prove Proposition 2.7 we need some auxiliary statements.

PROPOSITION C. *Let* $1<p<\infty$, *and let* $0<\alpha<1/p$. *If* $\mathscr{R}_\alpha$ *is bounded from* $l^p(\mathbb{N})$ *to* $l^p_{v_i}(\mathbb{N})$ *then there exist a positive constant* c *such that*

$$\sum_{i=m}^{m+h} v_i \leqslant c h^{1-\alpha p} \tag{2.7}$$

holds for all $m\in\mathbb{Z}$ *and* $h\in\mathbb{N}$.

Proposition C follows just in the same way as in the proof of the implication $(i) \Rightarrow (iv)$ of Proposition 2.6; therefore it is omitted.

We will prove the first part of Proposition 2.7. The second part follows analogously.

Proof of (i) of Proposition 2.7. Let us first show that, from (2.3) it follows that $\{\mathscr{W}_\alpha v_k\}_k < \infty$ for all $k \in \mathbb{Z}$. By the duality arguments (2.3) is equivalent to the inequality

$$\sum_{i=1}^{\infty} \left(\mathscr{W}_\alpha g_j\right)_i^{p'} \leqslant c \sum_{i=1}^{\infty} g_i^{p'} v_i^{1-p'}. \tag{2.8}$$

Let $v_i^{(1)} = v_i \chi_{\{i: m \leqslant i < m+2h\}}$ and $v_i^{(2)} = v_i \chi_{\{i: i<m \text{ or } i \geqslant m+2h\}}$, where $m \in \mathbb{Z}$ and $h \in \mathbb{N}$.

Note that for $k \geqslant m+2h-1$ and $m \leqslant i \leqslant m+h$, we have that $k-m+1 \leqslant 2(k-i+1)$. Further, by using (2.7), we arrive to the estimates:

$$\begin{aligned}
\{\mathscr{W}_\alpha v_j^{(2)}\}_i &\leqslant \sum_{k=m+2h-1}^{\infty} v_k (k-i+1)^{\alpha-1} \leqslant c \sum_{k=m+h}^{\infty} v_k (k-m+1)^{\alpha-1} \\
&\leqslant c \sum_{k=m+h}^{\infty} v_k \Big(\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \Big) \leqslant c \sum_{j=h+1}^{\infty} j^{\alpha-2} \Big(\sum_{k=m}^{j+m-1} v_k \Big) \\
&\leqslant c \sum_{j=h+1}^{\infty} j^{\alpha-2} j^{1-\alpha p} < \infty.
\end{aligned}$$

Therefore $\big(\mathscr{W}_\alpha v_j^{(2)}\big)_i < \infty$. The fact that $\big(\mathscr{W}_\alpha v_j^{(1)}\big)_i < \infty$ is obvious. Thus, $\big(\mathscr{W}_\alpha v_j\big)_i < \infty$ for every $i \in \mathbb{Z}$ because m and h are taken arbitrarily.

Now we prove that (2.3) yields (2.4). For this we need the next lemmas.

LEMMA D. *Let* $0 < \alpha < 1$. *Then there are positive constants* $c_\alpha^{(1)}$ *and* $c_\alpha^{(2)}$ *depending only on* α *such that for all* $m \in \mathbb{Z}$ *the inequality*

$$(\mathscr{W}_\alpha \beta_m)_m \leqslant c_\alpha^{(1)} \sum_{j=1}^{\infty} j^{\alpha-2} \Big(\sum_{k=m}^{m+j-1} \beta_k \Big) \leqslant c_\alpha^{(2)} (\mathscr{W}_\alpha \beta_m)_m$$

holds, where $\beta_m \geqslant 0$.

Proof. The proof follows easily if we observe that there are positive constants $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ independent of k and m such that

$$\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \leqslant b_\alpha^{(1)} (k-m+1)^{\alpha-1} \leqslant b_\alpha^{(2)} \sum_{j=k-m+1}^{\infty} j^{\alpha-2}.$$

It remains to change the order of summation. $\square$

COROLLARY A. *Let* $0<\alpha<1$, $\beta_m \geqslant 0$. *Then there are positive constants* $c_\alpha^{(1)}$ *and* $c_\alpha^{(2)}$ *such that for all* $m\in\mathbb{Z}$ *the inequality*

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha\beta_m]^{p'}\right\}_m \leqslant c_\alpha^{(1)}\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+j-1}\{\mathscr{W}_\alpha\beta_m\}^{p'}\Big) \leqslant c_\alpha^{(2)}\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha\beta_m]^{p'}\right\}_m$$

holds.

Let $v_i^{(1)}$ and $v_i^{(2)}$ be defined as above. Then by using (2.8) we have that

$$\sum_{i=m}^{m+h}\Big(\mathscr{W}_\alpha v_j^{(1)}\Big)_i^{p'} \leqslant c\sum_{i=m}^{m+h} v_i.$$

Thus, by Corollary A we conclude that

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(1)}]^{p'}\right\}_i \leqslant c\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=i}^{i+2(j-1)} v_k\Big) \leqslant c\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i]\right\}_i.$$

For the estimate of $\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(2)}]^{p'}\right\}_i$ we need some auxiliary statements.

LEMMA E. *Let* $0<\alpha<1$. *Then there is a positive constant* c *such that for all natural numbers* m,k *and an integer* j *satisfying the condition* $m\leqslant k\leqslant m+j-1$, *the inequality*

$$\left\{\mathscr{W}_\alpha v_j^{(2)}\right\}_k \leqslant c\sum_{s=j}^{\infty} s^{\alpha-2}\Big(\sum_{t=m}^{m+s-1} v_t\Big)$$

holds.

Proof. We recall that $v_k^{(2)} = v_k\chi_{\{k:\, k<m \text{ or } k\geqslant m+2j\}}$. Using the arguments of the proof of Lemma D and the fact that

$$\Big(\mathscr{W}_\alpha v_j^{(2)}\Big)_k = \sum_{s=m+2j}^{\infty} v_s(s-k+1)^{\alpha-1}$$

we have

$$\begin{aligned}\Big(\mathscr{W}_\alpha v_j^{(2)}\Big)_k &\leqslant c\sum_{s=m+2j}^{\infty} v_s(s-m+1)^{\alpha-1}\\ &\leqslant c\sum_{s=m+2j}^{\infty} v_s\sum_{t=s-m+1}^{\infty} t^{\alpha-2} \leqslant c\sum_{t=j}^{\infty} t^{\alpha-2}\Big(\sum_{s=m}^{m+t-1} v_s\Big). \quad \square\end{aligned}$$

LEMMA F. *Let* $0<\alpha<1$. *Then there is a positive constant* c *such that for all* $m\in\mathbb{Z}$,

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(2)}]^{p'}\right\}_m \leqslant c\sum_{t=1}^{\infty} t^{\alpha-1}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\Big(\sum_{j=m}^{m+s-1} v_j\Big)\Big)^{p'}.$$

Proof. Using Lemma E in Corollary A we have that

$$\begin{aligned}
\Big\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m &\leqslant c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{k=m}^{m+t-1}\{\mathscr{W}_\alpha v_k\}^{p'}\Big)\\
&\leqslant c\sum_{t=1}^{\infty} t^{\alpha-2}\sum_{k=m}^{m+t-1}\Big(\sum_{s=t}^{\infty}s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1}v_\varepsilon\Big)^{p'}\\
&\qquad\text{(the inner sum does not depend on } k)\\
&= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty}s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1}v_\varepsilon\Big)^{p'}\Big(\sum_{k=m}^{m+t-1}1\Big)\\
&= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty}s^{\alpha-2}\sum_{\varepsilon=m}^{m+s-1}v_\varepsilon\Big)^{p'}. \qquad \square
\end{aligned}$$

LEMMA G. *Let* $0<\alpha<1$. *Then there is a positive constant* c *such that for all* $m\in\mathbb{Z}$,

$$\Big\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m \leqslant c\sum_{t=1}^{\infty}t^{\alpha}\Big(\sum_{s=t}^{\infty}s^{\alpha-2}\sum_{\varepsilon=m}^{m+s+1}v_\varepsilon\Big)^{p'-1}\Big(t^{\alpha-2}\sum_{j=m}^{m+t+1}v_j\Big).$$

Proof. We will deduce the discrete case from the continuous case. Let $v(x)=v_j$, $j\leqslant x<j+1$. Then $\int_j^{j+1}v(x)dx=v_j$. Hence, by using lemmas proved above and integration by parts, we find that

$$\begin{aligned}
\Big\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m &\leqslant c\sum_{n=1}^{\infty}n^{\alpha-1}\Big(\sum_{j=n}^{\infty}j^{\alpha-2}\Big(\sum_{k=m}^{m+2j}v_k\Big)\Big)^{p'}\\
&\leqslant c\sum_{n=1}^{\infty}\int_n^{n+1}x^{\alpha-1}\Big(\sum_{i=2n}^{\infty}\int_i^{i+1}y^{\alpha-2}\Big(\sum_{k=m}^{m+y}v_k\Big)dy\Big)^{p'}dx\\
&\leqslant c\int_1^{\infty}x^{\alpha-1}\Big(\int_x^{\infty}y^{\alpha-2}\Big(\sum_{k=m}^{m+y}v_k\Big)dy\Big)^{p'}dx\\
&= c\Big[\frac{x^\alpha}{\alpha}\Big(\int_x^{\infty}\cdots\Big)^{p'}\Big|_1^{\infty}+\int_1^{\infty}x^{\alpha}\Big(\int_x^{\infty}\cdots\Big)^{p'-1}x^{\alpha-2}\Big(\sum_{k=m}^{m+x}v_k\Big)dx\Big]\\
&\leqslant c\int_1^{\infty}x^{\alpha}\Big(\int_x^{\infty}\cdots\Big)^{p'-1}x^{\alpha-2}\Big(\sum_{k=m}^{m+x}v_k\Big)dx\\
&= c\sum_{n=1}^{\infty}\int_n^{n+1}x^{\alpha}\Big(\int_x^{\infty}\cdots\Big)^{p'-1}x^{\alpha-2}\Big(\sum_{k=m}^{m+n+1}v_k\Big)dx\\
&\leqslant c\sum_{n=1}^{\infty}n^{\alpha}\Big(\int_n^{\infty}\cdots\Big)^{p'-1}n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1}v_k\Big)
\end{aligned}$$

$$\leqslant c\sum_{n=1}^{\infty} n^{\alpha}\Big(\sum_{k=n}^{\infty}\int_{k}^{k+1} k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)dy\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big)$$

$$= c\sum_{n=1}^{\infty} n^{\alpha}\Big(\sum_{k=n}^{\infty} k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big). \qquad \square$$

Now necessity of Proposition 2.7 follows easily because of Proposition C. Indeed, by using Proposition C we have that

$$\Big\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_j^{(2)}]^{p'}\Big\}_m \leqslant c\sum_{n=1}^{\infty} n^{\alpha}\Big(\sum_{k=n}^{\infty} k^{\alpha-2}(k+2)^{1-\alpha p}\Big)^{p'-1}\Big(n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\Big)$$

$$\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k \leqslant c\Big\{\mathscr{W}_\alpha v_m\Big\}_m .$$

In the last inequality we used Lemma D, in particular, the right-hand side inequality.

Necessity of Proposition 2.7 is proved.

Now we prove sufficiency of Proposition 2.7. We need some auxiliary statements.

LEMMA H. *Let $1<p<\infty$ and $0<\alpha<1$. Then there exists a positive constant c such that for all non-negative sequences $\{g_i\}_{i\in\mathbb{Z}}$ and all $i\in\mathbb{Z}$, the following inequality holds*

$$\{\mathscr{R}_\alpha g_k\}_i^p \leqslant c\{\mathscr{R}_\alpha[\mathscr{R}_\alpha g_k]_j^{p-1} g_m\}_i. \tag{2.9}$$

Proof. First we assume that $\{V_\alpha g_i\}_i := \{\mathscr{R}_\alpha[\mathscr{R}_\alpha g_k]^{p-1} g_j\}_i$ and

$$\{V_\alpha g_j\}_i \leqslant \{\mathscr{R}_\alpha g_j\}_i^p .$$

Otherwise (2.9) is obvious for $c=1$. Now let us assume that $1<p\leqslant 2$. Then we have

$$\{\mathscr{R}_\alpha g_k\}_i^p = \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}$$

$$\leqslant \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{k}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}$$

$$+\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} =: I_i^{(1)}+I_i^{(2)}.$$

It is obvious that if $j\leqslant k\leqslant i$, then $k-j+1\leqslant i-j+1$. Consequently,

$$I_i^{(1)} \leqslant \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{k}(k-j+1)^{\alpha-1}g_j\Big)^{p-1} = \{V_\alpha g_i\}_i.$$

Now we use Hölder's inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and measure $d\mu(k) = (i-k+1)^{\alpha-1} g_k \, d\mu_c(k)$ (here μ_c is the counting measure on $\mathbb{Z}$). We have

$$
\begin{aligned}
I_i^{(2)} &\leqslant \Big(\sum_{k=-\infty}^{i} (i-k+1)^{\alpha-1} g_k \Big)^{2-p} \Big(\sum_{k=-\infty}^{i} \Big(\sum_{j=k}^{i} (i-j+1)^{\alpha-1} g_j \Big) (i-k+1)^{\alpha-1} g_k \Big)^{p-1} \\
&= \{\mathscr{R}_\alpha g_i\}_i^{2-p} (J_i)^{p-1},
\end{aligned}
$$

where

$$
J_i \equiv \sum_{k=-\infty}^{i} \Big(\sum_{j=k}^{i} (i-j+1)^{\alpha-1} g_j \Big) (i-k+1)^{\alpha-1} g_k .
$$

Using Fubini's Theorem we have

$$
J_i = \sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \Big(\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k \Big).
$$

Further, it is obvious that the following simple inequality

$$
\begin{aligned}
\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k &\leqslant \Big(\sum_{k=-\infty}^{j} (i-k+1)^{\alpha-1} g_k \Big)^{p-1} \Big\{ \mathscr{R}_\alpha g_i \Big\}_i^{2-p} \\
&\leqslant \{\mathscr{R}_\alpha g_j\}_j^{p-1} \{\mathscr{R}_\alpha g_i\}_i^{2-p}
\end{aligned}
$$

holds, where $j \leqslant i$. Taking into account the last estimate, we obtain

$$
J_i \leqslant \Big(\sum_{j=-\infty}^{i} (i-j+1)^{\alpha-1} g_j \{\mathscr{R}_\alpha g_j\}_j^{p-1} \Big) \Big\{ \mathscr{R}_\alpha g_i \Big\}_i^{2-p} = \{V_\alpha g_i\}_i \{\mathscr{R}_\alpha g_i\}_i^{2-p}.
$$

Thus,

$$
I_i^{(2)} \leqslant \{\mathscr{R}_\alpha g_i\}_i^{2-p} \{\mathscr{R}_\alpha g_i\}_i^{(2-p)(p-1)} \{V_\alpha g_i\}_i^{p-1} = \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)} \{V_\alpha g_i\}_i^{p-1}.
$$

Combining the estimate for $I^{(1)}$ and $I^{(2)}$ we derive

$$
\{\mathscr{R}_\alpha g_i\}_i^{p} \leqslant \{V_\alpha g_i\}_i + \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)} \{V_\alpha g_i\}_i^{p-1}.
$$

As we have assumed that $\{V_\alpha g_i\}_i \leqslant \{\mathscr{R}_\alpha g_i\}_i^{p}$, we obtain

$$
\{V_\alpha g_i\}_i = \{V_\alpha g_i\}_i^{2-p} \{V_\alpha g_i\}_i^{p-1} \leqslant \{V_\alpha g_i\}_i^{p-1} \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)}.
$$

Hence

$$
\begin{aligned}
\{\mathscr{R}_\alpha g_i\}_i^{p} &\leqslant \{V_\alpha g_i\}_i^{p-1} \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)} + \{V_\alpha g_i\}_i^{p-1} \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)} \\
&= 2 \{V_\alpha g_i\}_i^{p-1} \{\mathscr{R}_\alpha g_i\}_i^{p(2-p)}.
\end{aligned}
$$

Applying the fact $\left(\mathscr{R}_\alpha g_j\right)_i < \infty$ we find that

$$\{\mathscr{R}_\alpha g_i\}_i^p \leqslant 2^{\frac{1}{p-1}} \{V_\alpha g_i\}_i.$$

Now we shall deal with the case $p > 2$. Let us assume again that

$$\{V_\alpha g_j\}_i \leqslant \{\mathscr{R}_\alpha g_j\}_i^p.$$

Since $p > 2$ we have

$$\begin{aligned}
\{\mathscr{R}_\alpha g_i\}_i^p &= \sum_{k=1}^{i}(i-k+1)^{\alpha-1} g_k \Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1} g_j\Big)^{p-1} \\
&\leqslant 2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1} g_k \Big(\sum_{j=1}^{k}(i-j+1)^{\alpha-1} g_j\Big)^{p-1} \\
&\quad +2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1} g_k \Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j\Big)^{p-1} \\
&=: 2^{p-1} I_i^{(1)} + 2^{p-1} I_i^{(2)}.
\end{aligned}$$

It is clear that if $j \leqslant k \leqslant i$, then $(i-j+1)^{\alpha-1} \leqslant (k-j+1)^{\alpha-1}$. Therefore $I_i^{(1)} \leqslant \{V_\alpha g_i\}_i$. Now we estimate $I_i^{(2)}$. We obtain

$$\begin{aligned}
\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j\Big)^{p-1} &= \Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j\Big)^{p-2}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j\Big) \\
&\leqslant \Big\{\mathscr{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j.
\end{aligned}$$

Using Fubini's theorem and the last estimate we have

$$\begin{aligned}
I_i^{(2)} &\leqslant \Big\{\mathscr{R}_\alpha g_i\Big\}_i^{p-2}\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1} g_k \sum_{j=k}^{i}(i-j+1)^{\alpha-1} g_j \\
&= \Big\{\mathscr{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1} g_j \sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1} g_k \\
&\leqslant \Big\{\mathscr{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1} g_j \sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1} g_k.
\end{aligned}$$

Due to Hölder's inequality with respect to the exponents $\left\{p-1, \frac{p-1}{p-2}\right\}$ and the measure $d\mu(j) = (i-j+1)^{\alpha-1} g_j d\mu_c(j)$ (μ_c is the counting measure on $\mathbb{Z}$) we derive

$$\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k$$
$$\leqslant\Big(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{\frac{p-2}{p-1}}\Big(\sum_{j=-\infty}^{i}\Big(\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k\Big)^{p-1}(i-j+1)^{\alpha-1}g_j\Big)^{\frac{1}{p-1}}$$
$$=\{\mathscr{R}_\alpha g_i\}_i^{\frac{p-2}{p-1}}\{V_\alpha g_i\}_i^{\frac{1}{p-1}}.$$

Combining these estimates we obtain

$$\{\mathscr{R}_\alpha g_i\}_i^p\leqslant 2^{p-1}\{V_\alpha g_i\}_i+2^{p-1}\{\mathscr{R}_\alpha g_i\}_i^{\frac{p(p-2)}{p-1}}\{V_\alpha g_i\}_i^{\frac{1}{p-1}}.$$

By virtue of the inequality $\{V_\alpha g_i j\}_i\leqslant\{\mathscr{R}_\alpha g_j\}_i^p$ it follows that

$$\{V_\alpha g_j\}_i=\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{V_\alpha g_j\}_i^{\frac{p-2}{p-1}}\leqslant\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathscr{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Hence

$$\begin{aligned}\{\mathscr{R}_\alpha g_j\}_i^p&\leqslant 2^{p-1}\Big(\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathscr{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}+\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathscr{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}\Big)\\&=2^p\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathscr{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.\end{aligned}$$

Further, from the last estimate we conclude that

$$\{\mathscr{R}_\alpha g_j\}_i^p\leqslant 2^{p(p-1)}\{V_\alpha g_j\}_i,$$

where $2<p<\infty$. $\square$

LEMMA I. *Let $1<p<\infty$, $0<\alpha<1$ and v_i be a sequence of positive numbers on $\mathbb{Z}$. Let there exist a constant $c>0$ such that the inequality*

$$\|\mathscr{R}_\alpha\{g_i\}\|_{l^p_{v_i^{(1)}}(\mathbb{Z})}\leqslant c_1\|g_i\|_{l^p(\mathbb{Z})},\ \ \{v_i^{(1)}\}_i=\{\mathscr{W}_\alpha v_i\}_i^{p'}$$

holds for all sequences $g_i\in l^p(\mathbb{Z})$. Then

$$\|\mathscr{R}_\alpha\{g_i\}\|_{l^p_{v_i}(\mathbb{Z})}\leqslant c_2\|g_i\|_{l^p(\mathbb{Z})},\ \ g_i\in l^p(\mathbb{Z}),$$

where $c_2=c_1^{1/p'}c^{1/p}$.

Proof. Let $g_i\geqslant 0$. Using Lemma H, Fubini's theorem and Hölder's inequality we derive the following chain of inequalities:

$$\begin{aligned}\sum_{k\in\mathbb{Z}}^{\infty}\{\mathscr{R}_\alpha g_k\}_k^p v_k&\leqslant c\sum_{k\in\mathbb{Z}}^{\infty}\sum_{i=-\infty}^{k}\{\mathscr{R}_\alpha g_j\}_i^{p-1}g_i(k-i+1)^{\alpha-1}v_k\\&=c\sum_{i\in\mathbb{Z}}^{\infty}\{\mathscr{R}_\alpha g_j\}_i^{p-1}g_i\{\mathscr{R}_\alpha v_j\}_i\leqslant c\Big(\sum_{i=1}^{\infty}g_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty}\{\mathscr{R}_\alpha g_j\}_i^p v_i^{(1)}\Big)^{1/p'}\end{aligned}$$

$$= c\,\|g_i\|_{l^p(\mathbb{Z})}\,\|\mathscr{R}_\alpha g_i\|^{p-1}_{l^p_{v_i^{(1)}}(\mathbb{Z})} \leqslant c_1^{p-1} c\,\|g_i\|_{l^p(\mathbb{Z})}\,\|g_i\|^{p-1}_{l^p(\mathbb{Z})}$$

$$= c_1^{p-1} c\,\|g_i\|^p_{l^p(\mathbb{Z})}.$$

Hence,

$$\left\|\mathscr{R}_\alpha g_j\right\|_{l^p_{v_i}(\mathbb{Z})} \leqslant c_1^{1/p'} c^{1/p}\left\|g_j\right\|_{l^p(\mathbb{Z})}. \quad \square$$

LEMMA J. *Let* $0<\alpha<1$ *and* $1<p<\infty$. *Suppose that* $\{\mathscr{W}_\alpha v_i\}_i<\infty$ *and*

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha v_i]^{p'}\right\}_i \leqslant c\left\{\mathscr{W}_\alpha v_i\right\}_i$$

for all $i\in\mathbb{Z}$. *Then we have*

$$\|\mathscr{R}_\alpha\{g_i\}\|_{l^p_{v_i^{(1)}}(\mathbb{N})} \leqslant c\,\|g_i\|_{l^p(\mathbb{Z})},\quad g_i\in l^p(\mathbb{Z}), \tag{2.10}$$

where $\{v_i^{(1)}\}_i=\{\mathscr{W}_\alpha v_i\}_i^{p'}$.

Proof. Let $g_i\geqslant 0$ and let g_i be supported on the set $E_{m,l}:=\{i: l\leqslant i\leqslant m\}$, where $m,\ l\ \in\mathbb{Z}$. Let $t_{i,j}^{(n)}=\chi_{\{j:j\leqslant i\}}\min\{(i-j+1)^{\alpha-1},n\}$, $n\in\mathbb{Z}$. Then using Lemma H (which is true also for the kernel $t_{i,j}^{(n)}$), Fubini's theorem and Hölder's inequality we obtain the following chain of inequalities:

$$\begin{aligned}
\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)} &\leqslant c\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1} g_j\Big) v_i^{(1)}\\
&\leqslant c\sum_{j=-\infty}^{\infty} g_j\Big(\sum_{k=-\infty}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1}\Big(\sum_{i=j}^{\infty} t_{i,j}^{(n)} v_i^{(1)}\Big)\\
&\leqslant c\,\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=-\infty}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathscr{R}_\alpha[\mathscr{R}_\alpha v_j]^{p'}\Big\}_j^{p'}\Big)^{1/p'}\\
&\leqslant c\,\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathscr{R}_\alpha v_j\Big\}_j^{p'}\Big)^{1/p'}.
\end{aligned}$$

Since $\sum_{k=1}^{j} t_{j,k}^{(n)} g_k<\infty$ and $\{\mathscr{W} v_j\}_j<\infty$ for all j, therefore we have that

$$\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)}\Big)^{1/p} \leqslant c\,\|g_i\|_{l^p(\mathbb{N})}.$$

Passing now by to the limits as m and n to $+\infty$, and by l to $-\infty$ we derive (2.10). $\square$

Combining these lemmas we have also sufficiency of Proposition 2.7. Proposition 2.7 is completely proved.

The next lemma will also be useful for us:

LEMMA K. *Let $1 < r,s < \infty$ and let g_n be a non-negative sequence. Suppose that u_n be a positive sequence on $\mathbb{Z}$.*

(i) The following two inequalities are equivalent

$$\left(\sum_{n\in\mathbb{Z}}\left[\sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}g_m\right]^r u_n\right)^{1/r} \leqslant c_1\|g_k\|_{l^s(\mathbb{Z})}$$

and

$$\left(\sum_{n\in\mathbb{Z}}(\mathscr{R}_\alpha g_k)_n]^r u_{n+1}\right)^{1/r} \leqslant c_1\|g_k\|_{l^s(\mathbb{Z})},$$

where the positive constant c_1 does depend on g_k;

(ii) The following two inequalities are equivalent

$$\left(\sum_{n\in\mathbb{Z}}\left[\sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}g_m\right]^r u_n\right)^{1/r} \leqslant c_2\|g_k\|_{l^s(\mathbb{Z})}$$

and

$$\left(\sum_{n\in\mathbb{Z}}(\mathscr{W}_\alpha g_k)_n]^r u_{n-3}\right)^{1/r} \leqslant c_2\|g_k\|_{l^s(\mathbb{Z})},$$

where again the positive constant c_2 does depend on g_k.

3. Boundedness on VEAS

This section is devoted to the boundedness of maximal operators in VEAS.

3.1. General operators in VEAS

We begin this subsection by the following definition:

DEFINITION 3.1. ([4]) Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}$. Define a sequence of local operators

$$(T_n f)(x) := T(f\chi_{(n-1,n+2)})(x), \ x\in(n-1,n+2), \quad n\in\mathbb{Z}.$$

Let us assume that there is a discrete operator T^d satisfying the following conditions:

(i) There exists a positive constant c such that for all non-negative functions f, all $n\in\mathbb{Z}$ and all $x\in(n,n+1)$, the inequality

$$T(f\chi_{(-\infty,n-1)}+f\chi_{(n+2,\infty)})(x) \leqslant cT^d\left(\int_{m-1}^{m} f\right)(n)$$

holds.

(ii) There is $c > 0$ such that for all sequences $\{a_k\}$ of non-negative real numbers and $n \in \mathbb{Z}$, the inequality

$$T^d(\{a_k\})(n) \leqslant cTf(y)$$

holds for all $y \in (n, n+1)$ and all non-negative f, where $\int_{m-1}^{m} f =: a_m$, $m \in \mathbb{Z}$. It is also assumed that T satisfies the conditions

$$Tf = T|f|, \qquad T(\lambda f) = |\lambda| Tf, \qquad T(f+g) \leqslant Tf + Tg, \qquad Tf \leqslant Tg \ \text{ if} f \leqslant g.$$

We will say that an operator T satisfying all the above– mentioned conditions is admissible on $\mathbb{R}$.

For example, Hardy operators, Hardy -Littlewood maximal operators, fractional integral operators, fractional maximal operators are admissible on $\mathbb{R}$ (see [4]). C. Lebrun, H. Heinig and S. Hofmann [7] established two weighted criteria for the Hardy transform $(\mathscr{H} f)(x) = \int_{-\infty}^{x} f(t)dt$ in amalgam spaces defined on $\mathbb{R}$ (see also [30], [18] for related topics). In [7] the authors derived one–weighted inequality for the Hardy-Littlewood maximal operator. Y. Rakotondratsimba [31] characterized two–weighted inequalities for the Hardy–Littlewood and fractional maximal operators and fractional integrals in amalgam spaces defined on $\mathbb{R}$. In the paper [3] the two–weight problem for generalized Hardy–type kernel operators including the fractional integrals of order greater than one (without singularity) was solved. Finally we mention that criteria for the boundedness of the weighted kernel operator $K_v f(x) = v(x) \int\limits_{-\infty}^{x} k(x,y) f(y) dy$ from $(L^{\bar{p}(\cdot)}, l^{\bar{q}})$ to $(L^{p(\cdot)}, l^{q})$ were derived in the recent paper [23]. In that paper the authors studied also the compactness problem for K_v in VEAS.

General type results for admissible operators read as follows:

THEOREM D. ([4]) *Let $1 < p, \bar{p}, q, \bar{q} < \infty$, and let v and w be weight functions on $\mathbb{R}$. Suppose that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^p(\mathbb{R}), l^q)} \leqslant c\|wf\|_{(L^{\bar{p}}(\mathbb{R}), l^{\bar{q}})}$$

holds for all measurable f if and only if

(i) T^d is bounded from $l^{\bar{q}}(\{w_n\})$ to $l^q(\{v_n\})$, where $w_n := \left(\int_{n-1}^{n} w^{-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}$, $v_n := \left(\int_{n}^{n+1} v\right)^{\frac{q}{p}}$.

(ii) (a) $\sup\limits_{n \in \mathbb{Z}} \|T_n\|_{[L_w^{\bar{p}}(n-1,n+2) \to L_v^p(n-1,n+2)]} < \infty$ for $1 < \bar{q} \leqslant q < \infty$.

(b) $\|T_n\|_{[L_w^{\bar{p}}(n-1,n+2) \to L_v^p(n-1,n+2)]} \in l^s$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ for $1 < q < \bar{q} < \infty$.

Let $X(\mathbb{R})$ be a Banach function space defined with respect to the Lebesgue measure on $\mathbb{R}$ (see [5], Chapter 1 for the definition and basic properties of a Banach function space). We establish the statement similar to Theorem D for amalgam spaces defined with respect to a Banach function space i.e., in the amalgam spaces, where instead of the $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R})}$ norm is taken Banach function norm $\|\cdot\|_{X(\mathbb{R})}$. This general amalgam space will be denoted by $(X(\mathbb{R}),l^q)$. Associate space of $X(\mathbb{R})$ is denoted by $X'(\mathbb{R})$. In a Banach function spaces Hölder's inequality holds ([5], P. 9):

$$\int |fg| \leqslant \|f\|_X \|g\|_{X'}, \quad f \in X, \ g \in X'. \tag{3.1}$$

Let, as before, T be an operator defined on a set of measurable functions on $\mathbb{R}$ and let $T_{v,w}$ be an operator defined by

$$T_{v,w}f = vT(wf),$$

where v and w are a.e. positive functions on $\mathbb{R}$.

THEOREM 3.1. *Let $X(\mathbb{R})$ and $Y(\mathbb{R})$ be Banach function spaces. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$. Suppose that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} \leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})} \tag{3.2}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}(\{\bar{w}_n\})$ to $l^q(\{\bar{v}_n\})$ where $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{X(\mathbb{R})}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{Y(\mathbb{R})}$.*

(ii) (a) $\sup\limits_{n\in\mathbb{Z}} \|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} < \infty$ *for* $1 < \bar{q} \leqslant q < \infty$.

(b) $\|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (3.2) hold. Then

1) *conditions* (ii) *are satisfied;*

2) *condition* (i) *is satisfied for $w \equiv const$.*

Proof. Let (i) and (ii) hold. We have

$$\|vTf\|_{(Y(\mathbb{R}),l^q)} \leqslant c\left\{\sum_{n\in\mathbb{Z}} \|T[wf(\chi_{(-\infty,n-1)} + \chi_{(n+2,\infty)})]v(\cdot)\|^q_{Y(n,n+1)}\right\}^{1/q}$$
$$+c\left\{\sum_{n\in\mathbb{Z}} \|vT_n(fw)\|^q_{Y(n,n+1)}\right\}^{1/q} =: S_1 + S_2.$$

Let $a_m := \int_{m-1}^m fw$. By the hypothesis and Hölder's inequality (see (3.1)) we have that

$$S_1 \leqslant c\left\{\sum_{n\in\mathbb{Z}} (T^d(\{a_m\})(n))^q \|\chi_{(n,n+1)}v\|^q_{Y(n,n+1)}\right\}^{1/q}$$
$$\leqslant c\left\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \|\chi_{(n-1,n)}w\|^{-\bar{q}}_{X'(n-1,n)}\right\}^{1/\bar{q}} \leqslant c\|f\|_{(X(\mathbb{R}),l^q)}.$$

Let us estimate S_2. Suppose that $1<\bar{q}\leqslant q<\infty$. Since the operators $(T_n)_{v,w}$ are uniformly bounded we find that

$$\begin{aligned} S_2 &\leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^q_{X(n-1,n+2)}\Big\}^{1/q}\leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^{\bar{q}}_{X(n-1,n+2)}\Big\}^{1/\bar{q}}\\ &\leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})}. \end{aligned}$$

If $1<q<\bar{q}<\infty$, then by using Hölder's inequality (see (3.1)) we find that

$$\begin{aligned} S_2 &\leqslant c\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^q_{[X(n-1,n+2)\to Y(n-1,n+2)]}\|\chi_{(n-1,n+2)}f\|^q_{X(\mathbb{R})}\Big\}^{1/q}\\ &\leqslant c\bigg[\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^{\frac{q\bar{q}}{\bar{q}-q}}\Big\}^{\frac{\bar{q}-q}{q}}\Big\{\sum_{n\in\mathbb{Z}}\|\chi_{(n-1,n+2)}f\|^{\bar{q}}_{X(\mathbb{R})}\Big\}^{\frac{q}{\bar{q}}}\bigg]^{1/q}\leqslant c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})}. \end{aligned}$$

Conversely, let (3.2) holds. Suppose that $n\in\mathbb{Z}$ and f is a non-negative function supported in $(n-1,n+2)$. Then

$$\|f\|_{(X(\mathbb{R}),l^{\bar{q}})}\leqslant 3\|f\chi_{(n-1,n+2)}\|_{(X(\mathbb{R}))}.$$

On the other hand,

$$\begin{aligned} \|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} &\geqslant \|v\chi_{(n-1,n+2)}T(fw)\|_{Y(\mathbb{R})}\\ &\geqslant \|vT_n(fw)\|_{Y(\mathbb{R})}. \end{aligned}$$

By the two–weight inequality we conclude that (a) of (ii) holds. Let us now show that if $1<q<\bar{q}<\infty$, then (b) of (ii) is satisfied.

Since $\|(T_n)_{v,w}\|_{[X(\mathbb{R})\to Y(\mathbb{R})]}=\sup\limits_{\{f:\|f\|_{X(\mathbb{R})}=1\}}\|vT_n(fw)\|_{Y(\mathbb{R})}$ we have that for each n, there exists a non-negative measurable function f_n, with the support in $(n-1,n+2)$ and with $\|\chi_{(n-1,n+2)}f_n\|_{X(\mathbb{R})}=1$, such that $\|(T_n)_{v,w}\|_{X(\mathbb{R})\to Y(\mathbb{R})}<\|vT_n(f_nw)\|_{Y(\mathbb{R})}+\frac{1}{2^{|n|}}$. So it is sufficient to prove that $\|vT_n(f_nw)\|_{X(\mathbb{R})}\in l^s$.

Let $\{a_n\}$ be a sequence of non-negative real numbers and $f=\sum\limits_n a_nf_n$. For each $n\in\mathbb{Z}$, $f(x)>a_nf_n(x)$ and then $v(x)T(fw)(x)\geqslant a_nv(x)T_n(f_nw)(x)$ for all $x\in(n-1,n+2)$.

Thus,

$$\begin{aligned} \|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} &\geqslant \Big\{\sum_{n\in\mathbb{Z}}ca_n^q\|\chi_{(n-1,n+2)}vT_n(fw)\|^q_{Y(\mathbb{R})}\Big\}^{1/q}\\ &= c\Big\{\sum_{n\in\mathbb{Z}}a_n^q\|vT_n(f_nw)\|^q_{Y(\mathbb{R})}\Big\}^{1/q}. \end{aligned}$$

Hence, the two– weight inequality yields that

$$\Big\{\sum_{n\in\mathbb{Z}} a_n^q \|vT_n(f_n w)\|_{Y(\mathbb{R})}^q\Big\}^{1/q} \leqslant c\Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n-1,n+2)} f\|_{X(\mathbb{R})}^{\bar{q}}\Big\}^{1/\bar{q}}$$
$$\leqslant c\Big\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \|\chi_{(n-1,n+2)} f_n\|_{X(\mathbb{R})}^{\bar{q}}\Big\}^{1/\bar{q}} = c\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\}.$$

Finally, by Lemma B we see that (b) of (ii) holds.

Now let us prove that (i) holds when $w \equiv \text{const}$. If $\{a_m\}$ is a sequence of non-negative real numbers and if $f := \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)}$, then $\int_{m-1}^{m} f = a_m$, and $\|\chi_{(n,n+1)} f\|_{X(\mathbb{R})}^{\bar{q}}$ $= a_n^{\bar{q}} \|\chi_{(n,n+1)}\|_{X(\mathbb{R})}^{\bar{q}} = a_n^{\bar{q}}$. By the properties of T we have,

$$\|vTf\|_{(Y(\mathbb{R}),l^q)} = \Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vTf\|_{Y(\mathbb{R})}^q\Big\}^{1/q}$$
$$\geqslant \Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vT^d\Big(\int_{m-1}^{m} f\Big)\|_{Y(\mathbb{R})}^q\Big\}^{1/q}$$
$$\geqslant c\Big\{\sum_{n\in\mathbb{Z}} T^d(a_m)^q(n) \|\chi_{(n,n+1)} v\|_{Y(\mathbb{R})}^q\Big\}^{1/q} = \|\bar{v}_n T^d\{a_m(n)\}\|_{l^q}.$$

Applying the two-weight inequality we have that

$$\|\bar{v}_n T^d\{a_m(n)\}\|_{l^q} \leqslant c\Big\{\sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} f\|_{X(\mathbb{R})}^{\bar{q}}\Big\}^{1/\bar{q}} = c\Big\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\Big\}^{1/\bar{q}} = \|a_n\|_{l^{\bar{q}}}.$$

Hence (i) holds. $\square$

Theorem 3.1 implies the following statement:

THEOREM 3.2. *Let $\bar{p}(\cdot)$, $p(\cdot)$ be measurable functions on $\mathbb{R}$ satisfying $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$, $1 < \bar{p}_-(\mathbb{R}) \leqslant \bar{p}_+(\mathbb{R}) < \infty$. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$. Suppose that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \leqslant c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})} \tag{3.3}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}(\{\bar{w}_n\})$ to $l^q(\{\bar{v}_n\})$ where $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|_{L^{\bar{p}'(\cdot)}}^{-\bar{q}}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|_{L^{p(\cdot)}}^q$.*

(ii) (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{[L_w^{\bar{p}(\cdot)}(n-1,n+2)\to L_v^{p(\cdot)}(n-1,n+2)]} < \infty$ *for* $1 < \bar{q} \leqslant q < \infty$.

(b) $\|T_n\|_{[L_w^{\bar{p}(\cdot)}(n-1,n+2)\to L_v^{p(\cdot)}(n-1,n+2)]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (3.3) *hold. Then*

1) *conditions* (ii) *are satisfied;*

2) *condition* (i) *is satisfied for* $w \equiv const$ *or for* p *and* $\bar{p}$ *being constant outside some large interval* $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.

Proof. Proof follows from Theorem 3.1. We only need to show that if (3.3) holds, then condition (i) is satisfied for p and $\bar{p}$ being constant outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.

Suppose now that w is a general weight and there is a positive integer m_0 such that p, $\bar{p}$ are constants outside $[-m_0, m_0]$. Taking

$$f(x) = \sum_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)}(x) \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1} w^{-\bar{p}'(x)}(x)$$

it is easy to see that $\int_{m-1}^{m} f = a_m$. Moreover, by Proposition A and the fact that

$$\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \leqslant \int_{-m_0}^{m_0} w^{-\bar{p}'(y)}(y)dy < \infty, \ \ [m-1,m] \subset [-m_0, m_0],$$

we have for $m \leqslant m_0 + 1$,

$$\begin{aligned} \|\chi_{(m-1,m)} f w\|_{L^{\bar{p}(\cdot)}} &= a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1} \|\chi_{(m-1,m)} w^{1-\bar{p}'(\cdot))}\|_{L^{\bar{p}(\cdot)}} \\ &\leqslant c a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1/\bar{p}_+\left([m-1,m)\right)}, \end{aligned}$$

where the positive constant c depends on m_0. Since

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \geqslant C \|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q},$$

using again Proposition A we find that

$$\begin{aligned} \|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q} &\leqslant C \left[\sum_m \|\chi_{(m-1,m)} f w\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(\mathbb{R})} \right]^{1/\bar{q}} \\ &\leqslant c \left[\sum_m a_m^{\bar{q}} \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-\bar{q}/\bar{p}_+([m-1,m))} \right]^{1/\bar{q}} \\ &= \|a_m \bar{w}_m\|_{l^{\bar{q}}}. \quad \square \end{aligned}$$

3.2. Maximal operators in amalgams $(L^{p(\cdot)}(\mathbb{R}), l^q)$

In this section we establish criteria for the boundedness of maximal operators in variable exponent amalgam spaces.

Recall the E. Sawyer [35] result for the discrete fractional maximal operator

$$M_\alpha^d(\{a_n\})(j) = \sup_{r\leqslant j\leqslant k} \frac{1}{(k-r+1)^{1-\alpha}} \sum_{i=r}^{k} |a_i|, \ \ 0<\alpha<1.$$

which is a consequence of more general result regarding two-weight criteria for maximal operators defined on spaces of homogeneous type (see [36]).

THEOREM E. *Let r and s be constants satisfying the condition $1<r\leqslant s<\infty$ and let α_n, β_n be positive sequences on $\mathbb{Z}$. Then the two–weight inequality*

$$\Big(\sum_{n\in\mathbb{Z}} \big(M^d(\{a_n\})\big)_n^s \alpha_n\Big)^{1/s} \leqslant c\Big(\sum_{n\in\mathbb{Z}} |a_n|^r \beta_n\Big)^{1/r},$$

holds if and only if there is a positive constant c such that for all $r,k\in\mathbb{Z}$ with $r\leqslant k$,

$$\Big(\sum_{j=r}^{k} \big(M^d(\{\beta_n^{1-r'}\}\chi_{[r,k]})\big)^s(j)\alpha_j\Big)^{1/s} \leqslant c\Big(\sum_{j=r}^{k} \beta_n^{1-r'}\Big)^{1/r}.$$

COROLLARY B. *Let $1<r\leqslant s<\infty$ and let α_n be a positive sequences on $\mathbb{Z}$. Then the weighted inequality*

$$\Big(\sum_{n\in\mathbb{Z}} \big(M_\alpha^d(\{a_n\})\big)_n^s \alpha_n\Big)^{1/s} \leqslant c\Big(\sum_{n\in\mathbb{Z}} |a_n|^r\Big)^{1/r} \tag{3.4}$$

holds if and only if

$$\sup_{k,r\in\mathbb{Z},r<k} \Big(\sum_{j=r}^{k} \alpha_j\Big)^{1/s} (k-r+1)^{\alpha-1/r} \leqslant c, \tag{3.5}$$

where the positive constant c is independent of $\{a_n\}$.

THEOREM F. ([39]) *Let s and r be constants satisfying the condition $1<s<r<\infty$ and let α_n be a positive sequence on $\mathbb{Z}$. We set $h_j := \sup_{r\leqslant i\leqslant k} \frac{1}{(k-r+1)^{1-\alpha r}} \sum_{i=r}^{k} \alpha_j$. Then the inequality (3.4) holds if and only if $\{h_j\} \in l_{\alpha_j}^{\frac{s}{r-s}}$.*

Now we formulate our result regarding variable exponent amalgam spaces.

THEOREM 3.3. *Let p be continuous function defined on $\mathbb{R}$ satisfying the conditions $1<p_-(\mathbb{R})\leqslant p(x)\leqslant p_+(\mathbb{R})<\infty$. Suppose that $p\in WL(\mathbb{R})$. If*

(a) $w\in A_{p(\cdot)}([n-1,n+2))$ uniformly with respect to n;

(b) the pair of discrete weights $(\{\bar{w}_n\},\{\bar{v}_n\})$ satisfies the condition: there is a positive constant c such that for all $r,k\in\mathbb{Z}$ with $r\leqslant k$,

$$\sum_{j=r}^{k}\big(M^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\,\bar{v}_j\leqslant c\sum_{j=r}^{k}\bar{w}_j^{1-q'}, \tag{3.6}$$

where

$$\bar{w}_n:=\|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-q}_{L^{p'(\cdot)}(\mathbb{R})},\ \bar{v}_n:=\|\chi_{(n,n+1)}(\cdot)w(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Then $M^{(\mathbb{R})}$ is bounded in $(L^{p(\cdot)}_w(\mathbb{R}),l^q)$.

Conversely, let $M^{(\mathbb{R})}$ be bounded in $(L^{p(\cdot)}_w(\mathbb{R}),l^q)$. Then (a) holds. If, in addition, there is a large positive integer m_0 such that p is constant outside $[-m_0,m_0]$, then condition (b) is also satisfied.

Proof. Observe that the Hardy–Littlewood maximal operator $M^{(\mathbb{R})}$ is admissible (see [31]) and associated discrete operator is given by

$$M^d(\{a_n\})(j)=\sup_{r\leqslant j\leqslant k}\frac{1}{k-r+1}\sum_{i=r}^{k}|a_i|.$$

Also, $(M^{(\mathbb{R})}f)_n=(M^{([n-1,n+2)}f)(x)$, $x\in[n-1,n+2)$.

Now by Theorems E, 3.2 and Proposition 2.1 we have the desired result. □

THEOREM 3.4. *Let p be a continuous function defined on $\mathbb{R}$ satisfying the condition $1<p_-(\mathbb{R})\leqslant p_+(\mathbb{R})<\infty$. Let $0\leqslant\alpha<1$. Suppose that v, w are weight functions on $\mathbb{R}$ and that $d\nu(x):=w(x)^{-p'(x)}dx$ belongs to $DC([n-1,n+2))$ uniformly with respect to n. Suppose also that $p\in WL(\mathbb{R})$. Then the operator $M_\alpha^{(\mathbb{R})}$ is bounded from $(L^{p(\cdot)}_w(\mathbb{R}),l^q)$ to $(L^{p(\cdot)}_v(\mathbb{R}),l^q)$ if*

(i) *there is a positive constant c such that for all n and all intervals $I\subseteq[n-1,n+2)$ the inequality*

$$\int_I(v(x))^{p(x)}M_\alpha^{[n-1,n+2)}\big(w(\cdot)^{-p'(\cdot)}\chi_I(\cdot)\big)\big)^{p(x)}dx\leqslant c\int_I w^{-p'(x)}dx<\infty$$

holds*;*

(ii) there is a positive constant c such that for all $r,k\in\mathbb{Z}$ with $r\leqslant k$,

$$\sum_{j=r}^{k}\big((M_\alpha)^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\,\bar{v}_j\leqslant c\sum_{j=r}^{k}\bar{w}_j^{1-q'}, \tag{3.7}$$

where

$$\bar{w}_n:=\|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}(\mathbb{R})},\quad \bar{v}_n:=\|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Conversely, let $M_\alpha^{(\mathbb{R})}$ be bounded from $(L^{p(\cdot)}_{w(\cdot)}(\mathbb{R}), l^q)$ to $(L^{p(\cdot)}_{v(\cdot)}(\mathbb{R}), l^q)$. Then (i) holds. If, in addition, there is a large positive integer m_0 such that p is constant outside $[-m_0, m_0]$, then condition (ii) is also satisfied.

Proof. It is known (see [31]) that the operator $M_\alpha^{\mathbb{R}}$ is admissible and that its discrete analog is M_α^d.

By Proposition 2.2 and Theorems E, 3.2 we have the desired result. □

THEOREM 3.5. *Let p be a continuous function defined on $\mathbb{R}$ satisfying the condition $1 < p_-(\mathbb{R}) \leqslant p(x) \leqslant p_+(\mathbb{R}) < \infty$. Assumed that $0 < \alpha < 1$. Suppose that v is a weight function on $\mathbb{R}$. Suppose also that $p \in WL(\mathbb{R})$. Then the operator $M_\alpha^{(\mathbb{R})}$ is bounded from $(L^{p(\cdot)}(\mathbb{R}), l^{\bar{q}})$ to $(L^{p(\cdot)}_v(\mathbb{R}), l^q)$ if and only if*

(i) in the case $1 < \bar{q} \leqslant q < \infty$,

$$\sup_{\substack{n\in\mathbb{Z} \\ I\subset(n-1,n+2)}} \frac{1}{|I|}\int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx < \infty$$

and

$$\sup_{k,r\in\mathbb{Z}, r<k} \left(\sum_{j=r}^{k} \bar{v}_j\right)(k-r+1)^{\alpha q-1} \leqslant c, \tag{3.8}$$

where $\bar{v}_n = \|\chi_{[n,n+1)} v\|^q_{L^{p(\cdot)}(\mathbb{R})}$;

(ii) in the case $1 < q < \bar{q} < \infty$, $\{J_n\} \in l^s$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$, and $\{H_j\} \in l_{v_j}^{\frac{q}{\bar{q}-q}}$, where

$$J_n := \sup_{\substack{n\in\mathbb{Z} \\ I\subset(n-1,n+2)}} \frac{1}{|I|}\int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx,$$

$$H_j := \sup_{r\leqslant i\leqslant k} \frac{1}{(k-r+1)^{1-\alpha\bar{q}}} \sum_{i=r}^{k} \bar{v}_j, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot) v(\cdot)\|^q_{L^{p(\cdot)}(\mathbb{R})}.$$

Proof. Part (i) follows in the same way as Theorem 3.4 was proved. We observe that in this case we use Corollary B. The proof of Part (ii) is similar by applying Theorems 3.2, F and Corollary 2.1. □

THEOREM 3.6. *Let p be a measurable function on $\mathbb{R}$ such that $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. Let $\bar{p}$, q, $\bar{q}$ and α be constants satisfying the condition $1 < \bar{p} < p_-$, $1 < \bar{q} \leqslant q < \infty$, $0 < \alpha < 1$. Suppose that $w^{-\bar{p}'} \in RD(\mathbb{R})$. Then the M_α is bounded from $(L^{\bar{p}}_w(\mathbb{R}), l^{\bar{q}})$ to $(L^{p(\cdot)}_v(\mathbb{R}), l^q)$ if and only if*

(i)

$$\sup_{\substack{n\in\mathbb{Z} \\ I\subset[n-1,n+2)}} \|v\chi_I |I|^{\alpha-1}\|_{L^{p(\cdot)}(\mathbb{R})} \|w^{-1}\chi_I\|_{L^{\bar{p}'}(\mathbb{R})} < \infty. \tag{3.9}$$

(ii)

$$\left(\sum_{j=r}^{k}\left(M^d(\{\bar{w}_n^{1-\bar{q}'}\}\chi_{[r,k]})\right)^q(j)\bar{v}_j\right)^{1/q} \leqslant c\left(\sum_{j=r}^{k}\bar{w}_j^{1-\bar{q}'}\right)^{1/\bar{q}}, \tag{3.10}$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|_{L^{\bar{p}'}(\mathbb{R})}^{-\bar{q}}, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}^{q}.$$

Theorem 3.6 is a direct consequence of Proposition 2.4 and Theorems E, 3.2.

3.3. Fractional integrals: trace inequality

In this subsection we discuss trace inequality criteria for the fractional integrals operators I_α, R_α and W_α in weighted VEAS defined on $\mathbb{R}$. In particular, we show that the following statement holds.

LEMMA L. (see the proof of Theorem 3.1 in [31]) *The following equivalences hold:*

$$(I_\alpha f\chi_{(-\infty,n-1)})(x) \approx \sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}\mathscr{G}(m); \tag{3.11}$$

$$(I_\alpha f\chi_{(n+2,\infty)})(x) \approx \sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}\mathscr{G}(m) \tag{3.12}$$

where $x \in [n, n+1)$ *and* $\mathscr{G}(m) = \int_{m-1}^{m} f(y)dy$.

THEOREM 3.7. *Let* p *be a measurable function on* $\mathbb{R}$ *such that* $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. *Let* $\bar{p}$, q, $\bar{q}$ *and* α *be constants satisfying the condition* $1 < \bar{p} < p_-(\mathbb{R})$, $1 < \bar{q} < q < \infty$, $0 < \alpha < \min\{1/\bar{p}, 1/\bar{q}\}$. *Then the following statements are equivalent:*

(i) I_α is bounded from $(L^{\bar{p}}(\mathbb{R}), l^{\bar{q}})$ to $(L_v^{p(\cdot)}(\mathbb{R}), l^q)$;

(ii) (a)

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L_v^{p(\cdot)}(I)}|I|^{\alpha-1/\bar{p}} < \infty; \tag{3.13}$$

(b)

$$\sup_{m\in\mathbb{Z}, j\in\mathbb{N}}\left(\sum_{k=m}^{m+j}\bar{v}_k\right)^{1/q}(j+1)^{\alpha-1/\bar{q}} < \infty, \tag{3.14}$$

where $\bar{v}_n := \|\chi_{[n,n+1)}(\cdot)\|_{L_v^{p(\cdot)}(\mathbb{R})}^q$.

THEOREM 3.8. *Let* p *be a measurable function on* $\mathbb{R}$ *such that* $1 < p_-(\mathbb{R}) \leqslant p_+(\mathbb{R}) < \infty$. *Let* $\bar{p}$, q *and* α *be constants satisfying the condition* $1 < \bar{p} < p_-(\mathbb{R})$, $1 < q < \infty$, $0 < \alpha < \min\{1/\bar{p}, 1/q\}$. *Then the following statements are equivalent:*

(i) I_α is bounded from $(L^{\bar{p}}(\mathbb{R}), l^q)$ to $L_v^{p(\cdot)}(\mathbb{R})), l^q)$;

(ii) (a)

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L^{p(\cdot)}_v(I)} |I|^{\alpha-1/\bar{p}} < \infty;$$

(b) $\{\mathscr{W}_\alpha \bar{v}_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that

$$\left\{\mathscr{W}_\alpha[\mathscr{W}_\alpha(\bar{v}_j)]^{q'}\right\}_k \leqslant c\left\{\mathscr{W}_\alpha(\bar{v}_j)\right\}_k \tag{3.15}$$

for all $k \in \mathbb{Z}$, where $\bar{v}_n$ is the same as in Theorem 3.7;

$\{\mathscr{R}_\alpha \bar{v}_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that

$$\left\{\mathscr{R}_\alpha[\mathscr{R}_\alpha(\bar{v}_j)]^{q'}\right\}_k \leqslant c\left\{\mathscr{R}_\alpha(\bar{v}_j)\right\}_k \tag{3.16}$$

for all $k \in \mathbb{Z}$, where $\bar{v}_n$ is defined in Theorem 3.7.

Proof of Theorem 3.7. First observe that

$$(I_\alpha)_n f(x) = \int_{n-1}^{n+2} \frac{f(t)}{|x-t|^{1-\alpha}} dt, \ \ x \in [n-1, n+2).$$

Due to Proposition 2.5, uniform boundedness of $(I_\alpha)_n$ is equivalent to (3.13). Further, it is easy to check that condition (3.14) is equivalent to each of the following two conditions:

$$\sup_{m\in\mathbb{Z}, j\in\mathbb{N}} \left(\sum_{k=m}^{m+j} \bar{v}_k^{(i)}\right)^{1/q} (j+1)^{\alpha-1/\bar{q}} < \infty, \quad i = 1, 2, \tag{3.17}$$

where $\bar{v}_k^{(1)} = \bar{v}_{k+1}$, $\bar{v}_k^{(2)} = \bar{v}_{k-3}$.

Since (see [31])

$$(I_\alpha)^d(\{a_j\})(n) \approx \sum_{k=-\infty}^{n-1} \frac{a_k}{(k-n)^{1-\alpha}} + \sum_{k=n+3}^{+\infty} \frac{a_k}{(k-n+1)^{1-\alpha}}, \tag{3.18}$$

by Theorem 3.2, Lemma L, Lemma K and Proposition 2.6 we have the desired result. $\square$

Proof of Theorem 3.8. Follows similarly by applying Proposition 2.5, Proposition 2.7, Lemma L, Lemma K and Theorem 3.2. $\square$

Acknowledgements.

The first author was supported by the Shota Rustaveli National Science Foundation grant (Contract numbers: D/13-23 and 31/47).

The part of this work is carried out at Abdus Salam School of Mathematical Sciences, GC University, Lahore. The authors are thankful to the Higher Education Commission, Pakistan for the financial support.

The first author expresses his gratitude to Professor V. Kokilashvili regarding the discussions about the problem studied in the work.

The authors are also grateful to the referees for useful remarks that overall improved the paper.

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(Received January 23, 2013)

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